UNIVALENT FUNCTIONS WITH UNIVALENT DERIVATIVES. II

BY

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1. **Introduction.** Let D denote the open unit disc with center at the origin. It is known that if f and all its derivatives are univalent in D, then f must be an entire function of exponential type [11]. However, to conclude that f is entire, we shall show it is not necessary to suppose that each derivative is univalent in D.

Let ρ_n be the largest number with the property that $f^{(n)}$ is univalent in an open disc about the origin of radius ρ_n. (Note that ρ_n is finite unless $f^{(n)}(z)=az+b$. We shall exclude this possibility by always assuming that f is not a polynomial.) In this paper, we investigate the relation between the growth of $\{\rho_n\}_{n=0}^{\infty}$ and the radius of convergence of f about the origin. In particular, we show that if ρ_n converges to zero slowly enough, then f must still be an entire function. (See the corollary to Theorem 1.) Further, if f is entire, we exhibit relations between the growth of $\{\rho_n\}_{n=0}^{\infty}$ and the order and type of f.

It is well known [12, p. 212], [3] that if f is defined in a disc about the origin of radius ρ by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and if $a_1 \neq 0$, then f is univalent in this disc if

$$\sum_{n=2}^{\infty} n|a_n|\rho^{n-1} \leqq |a_1|. \tag{1.1}$$

It is also well known [6, p. 213], [4, p. 3] that if f is univalent in D, then $a_1 \neq 0$ and

$$|a_2| \leqq 2|a_1|. \tag{1.2}$$

We shall use both these facts in the proofs below. For convenience of notation, we shall sometimes write $f^{(0)}=f$ and $a_n=a(n)$.

2. **Derivatives with varying radii of univalence.** We single out two kinds of functions. Let f be an analytic function defined in $|z|<R$. (We allow $R=\infty$.) We shall say that f has property (A) at N if there is a nonnegative integer, N, such that for $n \geqq N$, $\rho_n > 0$. Note that this implies that if $n \geqq N$, then $a_{n+1} \neq 0$. We shall say that f has property (B) at N if there is a positive integer, N, such that

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$\{|a_{n-1}/a_n|\}_{n=N}^{\infty}$ is a positive and nondecreasing sequence. Note that property (B) implies property (A).

THEOREM 1. *Let f be defined by $f(z)=\sum_{k=0}^{\infty} a_k z^k$. Let R denote the radius of convergence of f and ρ_n the radius of univalence of $f^{(n)}$. Then*

$$\liminf_{n\to\infty} n\rho_n \leqq 4R \tag{2.1}$$

and

$$R \log 2 \leqq \limsup_{n\to\infty} n\rho_n. \tag{2.2}$$

If f has property (A) *at N, then*

$$\liminf_{n\to\infty} [n(\rho_N\rho_{N+1}\cdots\rho_n)^{1/n}] \leqq 4eR. \tag{2.3}$$

If f has property (B), *then*

$$R\log 2 \leqq \liminf_{n\to\infty} n\rho_n \leqq \limsup_{n\to\infty} n\rho_n \leqq 4R. \tag{2.4}$$

Proof. If an infinite number of the ρ_n are zero, then (2.1) is obviously true. So, suppose f has property (A) at N. Let $F_n(z)=f^{(n)}(\rho_n z)$. Using (1.2) on F_n and assuming $n \geqq N$, we have

$$|a_{n+2}| \leqq 4|a_{n+1}|/(n+2)\rho_n. \tag{2.5}$$

Since $\liminf_{n\to\infty} |a_n/a_{n+1}| \leqq R$, (2.1) is established. If f has property (B), then $\lim_{n\to\infty} |a_n/a_{n+1}|=R$ and the right-hand part of (2.4) is established.

Using (2.5), an induction argument shows that if $k \geqq N+2$, then

$$|a_k| \leqq \frac{4^{k-N-1}|a_{N+1}|(N+1)!}{(\rho_N\rho_{N+1}\cdots\rho_{k-2})k!}.$$

So,

$$\frac{1}{R} = \limsup_{k\to\infty} |a_k|^{1/k} \leqq \frac{4e}{\liminf_{k\to\infty} [k(\rho_N\cdots\rho_k)^{1/k}]}.$$

This proves (2.3).

We no longer assume that f has property (A). Let $0<r<R$. Since $\sum |a_n|r^n<\infty$, there is an increasing sequence, $\{n_p\}_{p=1}^{\infty}$, of positive integers such that for $p=1,2,\ldots,$ and $k=2,3,\ldots,$ we have $|a(n_p+1)| \geqq |a(n_p+k)|r^{k-1}$. For $n=1,2,\ldots,$ let $x_n= n(1-2^{-1/(n+2)})$. Then

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{(k+n_p)!\,|a(n_p+k)|r^k[x(n_p)]^k}{(k-1)!\,n_p^k} &\leqq r|a(n_p+1)| \sum_{k=2}^{\infty} \frac{(k+n_p)!}{(k-1)!}\left(\frac{x(n_p)}{n_p}\right)^k \\ &= \frac{(n_p+1)!\,|a(n_p+1)|rx(n_p)}{n_p}.\end{aligned} \tag{2.6}$$

Define F_p in D by

$$F_p(z) = f^{(n_p)}(rzx(n_p)/n_p).$$

From (1.1) and (2.6), it follows that F_p is univalent in D. Hence, $rx(n_p)/n_p \leqq \rho(n_p)$. Since $\lim_{n\to\infty} x_n = \log 2$, (2.2) is proved.

Now assume that f has property (B) at N. If $n \geqq N$, let $r_n = |a_{n-1}/a_n|$. Then for $k=2, 3, \ldots$, we have $|a_{n+1}| \geqq |a_{n+k}| r_n^{k-1}$. Using the preceding argument, it follows that if $n \geqq N$, then

$$r_n x_n \leqq n\rho_n. \tag{2.7}$$

Since $\lim_{n\to\infty} r_n x_n = R \log 2$, (2.4) follows and the entire theorem is proved.

COROLLARY. *If* $\lim_{n\to\infty} n\rho_n = \infty$, *then* f *is a transcendental entire function. If* f *is a transcendental entire function, then* $\limsup_{n\to\infty} n\rho_n = \infty$. *If* f *is a transcendental entire function with property* (B), *then* $\lim_{n\to\infty} n\rho_n = \infty$.

The converse of the first part of this corollary is false. In fact, if $\{b_n\}_{n=0}^{\infty}$ is any sequence of positive numbers such that $\liminf_{n\to\infty} b_n > 0$, then there is an entire function, f, such that each $\rho_n > 0$ but $\liminf_{n\to\infty} b_n\rho_n = 0$. For instance, consider the following: If $n=0, 1, \ldots$, let $a_{2n} = 1/((2n)!)^{1/2}$ and $a_{2n+1} = 1/b_{2n}((2n+1)!)^{1/2}$. (If $b_{2n}=0$, let $a_{2n+1}=0$. There can only be a finite number of these.) Define f_1 by $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$. Then f_1 is certainly entire, and for large n, (2.5) becomes $b_{2n}\rho_{2n} \leqq 4/(2n+2)^{1/2}$.

The converse of the second part of the corollary is also false. Let $n_1 = 2$. Suppose that $p \geqq 1$ and that n_p has been chosen. Let n_{p+1} be an integer such that $n_{p+1} > n_p$ and if $j \geqq n_{p+1} + 1 - n_p$, then

$$j^2(j+n_p)^{n_p+1} \leqq n_p^{(j-1)/2}.$$

If $n = n_p$ for some p, let $a_{n+1} = 1$. Otherwise, let $a_n = 0$. Define f_2 by $f_2(z) = \sum_{j=1}^{\infty} a_j z^j$. We use (1.1) to show that $f_2^{(n_p)}$ is univalent in a disc about the origin of radius $1/(n_p)^{1/2}$:

$$\sum_{j=2}^{\infty} j \left| \frac{(j+n_p)!\,a(j+n_p)}{j!} \right| \frac{1}{n_p^{(j-1)/2}} \leqq \sum_{j=n_{p+1}+1-n_p}^{\infty} \frac{(j+n_p)!}{(j-1)!\,n_p^{(j-1)/2}}$$
$$< \sum_{j=n_{p+1}+1-n_p}^{\infty} \frac{(j+n_p)^{n_p+1}}{n_p^{(j-1)/2}} \leqq \sum_{j=2}^{\infty} \frac{1}{j^2} < (n_p+1)!.$$

Hence, for f_2, $\limsup_{n\to\infty} n\rho_n \geqq \limsup_{p\to\infty} n_p/(n_p)^{1/2} = \infty$, but the radius of convergence of f_2 about the origin is 1.

3. **Entire functions and univalent derivatives.** Next, we obtain relations between the radii of univalence of the derivatives of an entire function, f, and the order and type of f. In this and the following section, we shall let Λ be the order and λ be the lower order of f. If $0 < \Lambda < \infty$, we let T be the type and t be the lower type of f.

Several results connecting the order and type of an entire function with the sequence, $\{\rho_n\}_{n=1}^{\infty}$, already exist. Boas has shown [1] that if f is a transcendental entire function of exponential type less than $\log 2$, then there is a subsequence,

$\{\rho(n_p)\}_{p=1}^{\infty}$, such that $\rho(n_p)\geqq 1$ for all p. (Levinson [5] supplied a second proof of this.) Boas also pointed out [1] that if the order of f is less than one, or if f is of order one but minimal type, then $\limsup_{n\to\infty} \rho_n=\infty$. Pólya [7, p. 18] has stated that

$$\liminf_{n\to\infty}\frac{\log \rho_n}{\log n} \leqq \frac{1-\Lambda}{\Lambda} \leqq \limsup_{n\to\infty}\frac{\log \rho_n}{\log n}.$$

We shall improve these results and establish several more as well.

LEMMA. *Let f be defined on $|z|<R$ by $f(z)=\sum_{k=0}^{\infty} a_k z^k$. (We allow $R=\infty$.) Let $\nu(r)$ denote its central index. For $n=1, 2, \ldots$, let $x_n=n(1-2^{-1/(n+2)})$. Then if $R>r>0$ and $\nu(r)\geqq 2$,*

$$rx(\nu(r)-1) < (\nu(r)-1)\rho(\nu(r)-1). \tag{3.1}$$

Proof. Let $0<r<R$. Then $|a(\nu(r))|>|a(\nu(r)+k)|r^k$ for $k=1, 2, \ldots$. Suppose $n=\nu(r)-1$. Using the same argument that proved (2.2), it follows that $rx_n<n\rho_n$.

THEOREM 2. *Let f be a transcendental entire function defined by $f(z)=\sum_{n=0}^{\infty} a_n z^n$. Let $\delta=\liminf_{r\to\infty} \nu(r)/r$. Then*

$$\liminf_{n\to\infty}\frac{\log(\max\{1, n\rho_n\})}{\log n} \leqq \frac{1}{\Lambda}, \tag{3.2}$$

$$\frac{1-\lambda}{\lambda} \leqq \limsup_{n\to\infty}\frac{\log \rho_n}{\log n}, \tag{3.3}$$

and

$$\frac{\log 2}{\delta} \leqq \limsup_{n\to\infty} \rho_n. \tag{3.4}$$

Suppose that $0<\Lambda<\infty$ and that f has property (A) at N. In this case

$$e^{\Lambda-1}\liminf_{n\to\infty} n^{\Lambda-1}\rho_n^{\Lambda} \leqq \liminf_{n\to\infty}\,(\rho_N\rho_{N+1}\cdots\rho_n)^{\Lambda/n}(n^{\Lambda-1}) \leqq \frac{4^{\Lambda}e^{\Lambda-1}}{\Lambda T}. \tag{3.5}$$

In any case,

$$\liminf_{n\to\infty} n^{\Lambda-1}\rho_n^{\Lambda} \leqq \frac{4^{\Lambda}}{\Lambda T}. \tag{3.6}$$

Proof. To prove (3.2), we may assume that $n\rho_n\geqq 1$ for all n and that $\Lambda>0$. It is known [8] that

$$\frac{1}{\Lambda} \geqq \liminf_{n\to\infty}\frac{\log|a_n/a_{n+1}|}{\log n}.$$

From (2.5), we have that

$$\log n\rho_n < \log 4+\log|a_{n+1}/a_{n+2}|.$$

Hence,

$$\frac{1}{\Lambda} \geqq \liminf_{n\to\infty}\frac{\log n}{\log(n+1)}\left(\frac{\log n\rho_n}{\log n}-\frac{\log 4}{\log n}\right) = \liminf_{n\to\infty}\frac{\log n\rho_n}{\log n}.$$

It has been shown [13] that

$$\frac{1}{\lambda} = \limsup_{r\to\infty} \frac{\log r}{\log v(r)}.$$

Since f is entire, $\lim_{r\to\infty} v(r)=\infty$. So, from (3.1), for all large r, we have

$$\frac{\log r}{\log v(r)} \leqq \frac{\log \rho(v(r)-1)}{\log (v(r)-1)}+1+\frac{\log x(v(r)-1)}{\log v(r)}$$

Hence,

$$\frac{1}{\lambda} \leqq \limsup_{r\to\infty} \frac{\log \rho(v(r)-1)}{\log (v(r)-1)}+1$$

$$\leqq 1+\limsup_{n\to\infty} \frac{\log \rho_n}{\log n}.$$

This establishes (3.3).

Using (3.1) again, for large r we have

$$\frac{r}{v(r)} < \frac{\rho(v(r)-1)}{x(v(r)-1)}.$$

Therefore,

$$\frac{1}{\delta} \leqq \limsup_{r\to\infty} \frac{\rho(v(r)-1)}{x(v(r)-1)} \leqq \frac{\limsup_{n\to\infty} \rho_n}{\log 2}.$$

This proves (3.4).

Now suppose that f has property (A) at N. It is known [2, p. 11] that

$$e\Lambda T = \limsup_{n\to\infty} n|a_n|^{\Lambda/n}. \tag{3.7}$$

But

$$\limsup_{n\to\infty} n|a_n|^{\Lambda/n} = \limsup_{n\to\infty} n \prod_{k=N+2}^{n} \left|\frac{a_k}{a_{k-1}}\right|^{\Lambda/n}.$$

From (2.5),

$$e\Lambda T \leqq \limsup_{n\to\infty} n \prod_{k=N+2}^{n} \left(\frac{4}{k\rho_{k-2}}\right)^{\Lambda/n}$$

$$= (4e)^{\Lambda} \limsup_{n\to\infty} \frac{n^{1-\Lambda}}{(\rho_N\rho_{N+1}\cdots\rho_n)^{\Lambda/n}}.$$

Since the left-hand inequality of (3.5) is always true, (3.5) is proved. If an infinite number of the ρ_n are zero, then (3.6) is trivial. If only a finite number of the ρ_n are zero, then (3.6) becomes (3.5). This establishes the theorem.

Suppose that $0<\Lambda<\infty$, and let $\delta'=\liminf_{r\to\infty} v(r)/r^{\Lambda}$. In [9], it was shown that $\delta'\leqq \Lambda t\leqq \Lambda T$. Hence, if $\Lambda=1$, (3.4) can be written as

$$\frac{\log 2}{t} \leqq \frac{\log 2}{\delta} \leqq \limsup_{n\to\infty} \rho_n. \tag{3.8}$$

Further, if $\lambda<1$, then $\delta=0$, and either (3.3) or (3.4) imply that $\limsup_{n\to\infty} \rho_n=\infty$. This, together with (3.8), yield improvements on the results of Boas. We note that there are transcendental entire functions of order one such that $\delta=0$ and $T>0$. For example, let

$$\psi(z) = \sum_{n=1}^{\infty} \frac{z^{p_n}}{p_n!}$$

where $p_1=3$ and if $n>1$, $p_n=[p_{n-1}\log p_{n-1}]$. Then $\Lambda=T=1$, but $\delta=0$. We summarize these results in a corollary.

COROLLARY 1. *If* $\lambda<1$, *then* $\limsup_{n\to\infty} \rho_n=\infty$. *If* $\Lambda=1$, *then*

$$\frac{\log 2}{t} \leqq \frac{\log 2}{\delta} \leqq \limsup_{n\to\infty} \rho_n.$$

In particular, if $\delta=0$, *then* $\limsup_{n\to\infty} \rho_n=\infty$.

From (3.2) and (3.4), we get a second corollary.

COROLLARY 2. *If* $\Lambda>1$, *then* $\liminf_{n\to\infty} \rho_n=0$. *If* $\Lambda=1$, *then*

$$\frac{4}{T} \geqq \liminf_{n\to\infty} \rho_n.$$

In particular, if $\Lambda=1$ *and* $T=\infty$, *then* $\liminf_{n\to\infty} \rho_n=0$.

If $\Lambda=1$ and $0<T<\infty$, then it may be the case that

$$0 < \liminf_{n\to\infty} \rho_n \leqq \limsup_{n\to\infty} \rho_n < \infty.$$

The function, $\phi(z)=e^z$, is an example of this.

THEOREM 3. *Let f be an entire function with property* (B) *at N. Let*

$$\gamma = \limsup_{r\to\infty} \nu(r)/r.$$

Then

$$\liminf_{n\to\infty} \frac{\log \rho_n}{\log n} = \frac{1-\Lambda}{\Lambda} \leqq \frac{1-\lambda}{\lambda} = \limsup_{n\to\infty} \frac{\log \rho_n}{\log n}, \tag{3.9}$$

$$\frac{\log 2}{\gamma} \leqq \liminf_{n\to\infty} \rho_n \leqq \frac{4}{\gamma}, \tag{3.10}$$

and

$$\frac{\log 2}{\delta} \leqq \limsup_{n\to\infty} \rho_n \leqq \frac{4}{\delta}. \tag{3.11}$$

Suppose $0<\Lambda<\infty$. *Then*

$$(\log 2)^{\Lambda} \leqq \Lambda t e^{1-\Lambda} \limsup_{n\to\infty} (\rho_N\rho_{N+1}\cdots\rho_n)^{\Lambda/n}(n^{\Lambda-1}) \leqq 4^{\Lambda}$$

and

$$(\log 2)^{\Lambda} \leqq \Lambda T e^{1-\Lambda} \liminf_{n\to\infty} (\rho_N\rho_{N+1}\cdots\rho_n)^{\Lambda/n}(n^{\Lambda-1}) \leqq 4^{\Lambda}.$$

Proof. It is known [8] that

$$\liminf_{n\to\infty} \frac{\log |a_n/a_{n+1}|}{\log n} = \frac{1}{\Lambda} \leqq \frac{1}{\lambda} = \limsup_{n\to\infty} \frac{\log |a_n/a_{n+1}|}{\log n}. \tag{3.12}$$

From (2.5) and (2.7), it follows that for $n \geqq N$

$$x_n \left| \frac{a_{n-1}}{a_n} \right| < n\rho_n < 4 \left| \frac{a_{n+1}}{a_{n+2}} \right| . \tag{3.13}$$

Hence, from (3.12) and (3.13), we get (3.9).

Since f has property (B), we have that

$$\gamma = \limsup_{n\to\infty} n \left| \frac{a_n}{a_{n-1}} \right|$$

and

$$\delta = \liminf_{n\to\infty} n \left| \frac{a_n}{a_{n-1}} \right| .$$

From this, (2.5), and (2.7), parts (3.10) and (3.11) can be proved.

Assume now that $0 < \Lambda < \infty$. From [10], it follows that

$$e\Lambda t = \liminf_{n\to\infty} n|a_n|^{\Lambda/n}.$$

We use this, (3.7), and (3.13) to establish the last two parts of the theorem. The proofs are similar to the proof of (3.5), and so are omitted.

COROLLARY. *Let f be an entire function with property* (B).

(i) *If* $\Lambda < 1$, *then* $\lim_{n\to\infty} \rho_n = \infty$.

(ii) *If* $\lambda > 1$, *then* $\lim_{n\to\infty} \rho_n = 0$.

(iii) *If* $\Lambda = 1$, *then* $\lim_{n\to\infty} \rho_n = \infty$ *if and only if* $T = 0$. *If* $\lim_{n\to\infty} \rho_n = 0$, *then* $t = \infty$.

Proof. Parts (i) and (ii) follow from (3.9). From [9], we have that if $\Lambda = 1$, then $\delta \leqq t \leqq T \leqq \gamma \leqq eT$. Part (iii) follows from this, (3.10), and (3.11).

4. **Conclusion.** So far, all of the work has been done with functions defined in discs centered at the origin. However, this work immediately carries over to functions defined in a disc centered at any point in the plane. To be specific, let f be analytic on $\Delta = \{z : |z - z_0| < r\}$ and let $g(z) = f(rz + z_0)$ for $z \in D$. Then $f^{(n)}$ is univalent on $\{z : |z - z_0| < \rho_n \leqq r\}$ if and only if $g^{(n)}$ is univalent on $\{z : |z| < \rho_n/r\}$.

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