ADDENDUM TO "ON SOME ALGEBRAIC PROPERTIES OF THE BESSEL POLYNOMIALS"(1)

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In [2] it has been shown that the Galois groups $G_n$ of the irreducible (over the rational field) Bessel polynomials $z_n(x) = \sum_{v=0}^{n} (n+v)!/[2^{v}(n-v)!v!]^{-1}x^{n-v}$ are the symmetric groups on $n$ symbols $S_n$, except, possibly, for $n=9, 11$ and 12. It also was shown that if $G_9$ contains an element of order 5 and if $G_{11}$ and $G_{12}$ each contain elements of order 7, then also $G_9, G_{11}$ and $G_{12}$ are the symmetric groups $S_n$ with $n=9, 11$ and 12, respectively.

In order to prove the existence of elements of orders 5 and 7 respectively, one may use a classical theorem of Dedekind (see [3, p. 445]), the relevant part of which is the following:

If (*) $z_n(x) = f_1(x)f_2(2)^{n} \cdots f_r(x)$ (mod $p$) ($p =$ rational prime)

with $f_i(x)$ (1 $\leq$ $i$ $\leq$ $r$) irreducible and incongruent mod $p$, then the Galois group of $z_n(x)$ contains at least one permutation of $r$ cycles, each cycle corresponding to one of the factors in (*) and of the same order as the degree of the polynomial $f_i(x)$ to which it corresponds.

The following factorizations were obtained for $z_n(x)$ ($n=9, 11, 12$):

$z_9(x) = (x+3)(x^2+12x^2-13x+3)(x^5+x^4-5x^3-x^2+7x+13)$ (mod 29)

$z_{11}(x) = (x^4+27x^3+14x^2+76x+23)$

$(x^7+39x^6+35x^5+121x^4+148x^3+60x^2+25x+53)$ (mod 149)

$z_{12}(x) = (x^5+76x^4+65x^3+63x^2+3x+29x+15)$

$(x^5+2x^5+7x^5+28x^4+30x^3+38x^2+15x+3)$ (mod 89).

The first factorization was done by Dr. M. Newman and Mr. K. Kloss on the IBM 704 of the National Bureau of Standards in October 1961, the last two by Professor J. D. Brillhart and R. Stauduhar, using an algorithm of E. Berlekamp, on the CDC 6400 at the University of Arizona, in February 1969. (Required computer time: 4 and 6 seconds, resp.)

For $z_n(x)$, $n=9, 11$ or 12, denote by $P_n$ the permutation whose existence is guaranteed by Dedekind’s theorem. Then $P_9=(a, b, c)(d, e, f, g, h)$ (one symbol is left invariant) and $Q=P_9^5=(d, g, e, h, f) \in G_9$ and is of order 5; hence, $G_9 \cong S_9$.

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Similarly, \( P_{11} = (a, b, c, d)(e, f, g, h, i, j, k) \) and \( Q' = P_{11}^1 = (e, i, f, j, g, k, h) \in G_{11} \) and is of order 7, so that \( G_{11} \cong S_{11} \). Finally, \( P_{12} = (a, b, c, d, e)(f, g, h, i, j, k, l) \) and \( Q' = P_{12}^2 = (f, k, i, g, j, l, h) \in G_{12} \) and is of order 7, so that \( G_{12} \cong S_{12} \). The details may be found in [1] and [2]. In conclusion and with previous notations one obtains, therefore the following

**THEOREM.** The Galois groups \( G_n \) of the irreducible Bessel polynomials \( z_n(x) \) satisfy \( G_n \cong S_n \).

The following conjecture is proven in [2] for special values of \( n \) (e.g., for \( n \leq 400, n = p^m, n = k \cdot p - 1 \ (k < p) \), etc.):

**CONJECTURE A.** All Bessel polynomials are irreducible over the rational field.

For general \( n \) the problem is still open. By the present Theorem, the truth of Conjecture A immediately implies

**CONJECTURE B.** For all \( n \), \( G_n \cong S_n \).

**BIBLIOGRAPHY**


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