LOCAL AND GLOBAL SUBORDINATION THEOREMS FOR VECTOR-VALUED ANALYTIC FUNCTIONS

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Introduction. We extend the basic result on the subordination of analytic functions to analytic functions with values in a Banach space. The fundamental form of this theorem is as follows [2, p. 421]: Let f and g be two analytic (scalar-valued) functions defined on the unit disc D with g univalent and the range of f contained in the range of g. Then f and g are analytically related, i.e. there is an analytic function ω mapping D into itself with $f(z)=g(\omega(z))$ for all z in D. This result extends to vector-valued analytic functions (for definitions see [1]) but there are subtleties: (1) the proof for scalar-valued functions consists of noting that $\omega(z)=g^{-1}(f(z))$ is analytic; for vector-valued functions not only is the proper notion of analyticity for g^{-1} unavailable but, as we show below, g^{-1} need not even be continuous, and (2) the result for scalar-valued functions is true for arbitrary domains (usually stated for D only to avoid superfluous generality) while for vector-valued functions we cannot allow punctures in the domain of g.

The basic theorem on subordination has as a simple consequence the interesting result that if f and g are nonconstant analytic (scalar-valued) functions on D with intersecting ranges then they are locally analytically related, i.e. there is a neighborhood V in D and an analytic function ω mapping V into D with $f(z)=g(\omega(z))$ for z in V. This result also extends to vector-valued analytic functions but in this case the appropriate hypothesis is that f and g have ranges intersecting in an uncountable set. For f and g scalar-valued, if the ranges of f and g intersect at all, then they intersect in an uncountable set by the open mapping theorem; so, as in the basic theorem, it is the failure of the open mapping theorem for vector-valued analytic functions which lends interest to the generalization and makes the proof more delicate. We will show that for vector-valued f and g their ranges may intersect in a set containing an infinite number of accumulation points and yet not be locally analytically related.

I. Global theorem.

THEOREM. *Let X be a complex Banach space and f and g be two analytic functions mapping the unit disc D into X. If g is one-to-one with the range of g containing the range of f, then there is an analytic function ω mapping the disc into itself with $f(z)=g(\omega(z))$ for all z in D.*

Received by the editors June 28, 1968 and, in revised form, January 27, 1969.
[1] Research supported in part by National Science Foundation Grant GP-7500.

Proof. We may suppose that neither f nor g is constant, otherwise the result is immediate. The equation $g(\omega(z))=f(z)$ unambiguously defines ω as a map from D into itself.

Let W be the set of points in D at which ω is analytic; W is open by definition. We want to show that $W=D$. We will first show that $D-W$ is countable.

Assume that $D-W=W'$ is uncountable. We can write $D=\bigcup K_n$, a countable union of compact sets. Then $W'\cap K_p$ is uncountable for some index p and is compact. Note that the function ω maps at most countably many points into a single point a, for if $\omega(K)=a$ with K an uncountable set, then $g(a)=g(\omega(K))=f(K)$ and f is constant on an uncountable set. So, since such a set has an accumulation point, f is constant throughout D by the identity theorem. Hence $\omega(W'\cap K_p)$ is uncountable and so has an uncountable number of accumulation points in D. Since g' can vanish at no more than a countable number of points in D, we can find ω_0 an accumulation point of $\omega(W'\cap K_p)$ with $g'(\omega_0)\neq 0$. Then there are points z_n in $W'\cap K_p$ for which the sequence $\{\omega(z_n)\}$ converges to ω_0. The sequence $\{z_n\}$ lies in a compact set and so by passing to a subsequence we may suppose that it converges to z_0. Since $g(\omega(z_0))=f(z_0)=\lim f(z_n)=\lim g(\omega(z_n))=g(\omega_0)$ and g is one-to-one, we find that $\omega(z_0)=\omega_0$.

Let x^* be a continuous linear functional on X with $0\neq x^*g'(\omega(z_0))=(x^*g)'(\omega(z_0))$. Since the derivative of x^*g does not vanish at $\omega(z_0)$ there is a neighborhood U of $\omega(z_0)$ with x^*g one-to-one on U. Let V be a neighborhood of z_0 such that $x^*f(V)\subseteq x^*g(U)$. We define the analytic function

$$h(z) = (x^*g)^{-1}x^*f(z) \quad \text{for } z \text{ in } V.$$

For large enough n, z_n is in V and $\omega(z_n)$ is in U and so $h(z_n)=(x^*g)^{-1}x^*f(z_n)=(x^*g)^{-1}x^*g(\omega(z_n))=\omega(z_n)$ and it follows that $g(h(z_n))=f(z_n)$. By the identity theorem the analytic functions gh and f must agree on V. The function g is one-to-one and $g(h(z))=f(z)=g(\omega(z))$ so $\omega(z)=h(z)$ is thus analytic on V. Hence z_0 is in W, a contradiction following from our assumption that $W'=D-W$ was uncountable.

We now know that W' is countable and will use this fact to show that it is empty. Assume that W' is not empty. The set W' is relatively closed in D and so is a locally compact Hausdorff space which is countable and then the Baire category theorem [3, p. 85] tells us that W' must have an isolated point y_0. We now proceed to show that ω is analytic at y_0. Now ω is analytic in a deleted neighborhood D_0 of y_0 and is bounded on D_0, since the domain of g is bounded, and thus ω has a removable singularity at y_0; let $\tilde{\omega}$ be ω redefined at y_0 so as to be analytic at y_0. Choosing points $\{y_n\}$ in the deleted neighborhood of y_0 which converge to y_0, we have $\{\tilde{\omega}(y_n)\}$ converging to $\tilde{\omega}(y_0)$. Noting that since $\tilde{\omega}$ is analytic and not constant, $\tilde{\omega}(D_0)$ is a (perhaps) deleted neighborhood of $\tilde{\omega}(y_0)$ and so $\tilde{\omega}(y_0)$ is in the domain of g since the domain of g has no punctures; hence $g(\tilde{\omega}(y_0))=\lim g(\omega(y_n))=\lim f(y_n)=f(y_0)=g(\omega(y_0))$. Because g is one-to-one, $\omega(y_0)=\tilde{\omega}(y_0)$, i.e. ω, alias $\tilde{\omega}$, is analytic at y_0. This is a contradiction since y_0 was not in W. Hence $D=W$. This completes the proof.

Notice that the only properties of the domain of g which were used in the last paragraph of the above proof were that the domain be bounded and unpunctured. Clearly, the theorem holds when the domain of g is conformally equivalent to such a domain, while the domain of f is arbitrary.

To see that some restriction on the domain of g is necessary, consider the following:

EXAMPLE 1. Let $X=C^2$, and

$$g(z) = (z(z-1), z(z-1)^2) \quad \text{on } 0 < |z| < 2,$$
$$f(z) = (z(z-1), z(z-1)^2) \quad \text{on } |z| < 1.$$

Then g is one-to-one, the range of f is contained in the range of g, yet f is not analytically related to g (note that the domain of g is punctured).

Note also that the function f above provides us with an analytic, univalent function whose inverse is not continuous at (0, 0).

II. Local theorem.

THEOREM. *Let f and g be analytic (vector-valued) on the unit disc D. Suppose the intersection of the ranges of f and g is uncountable. Then f is locally analytically related to g. That is, there exists an analytic function $\omega(z)$ defined on a subdomain D^* of D, mapping into D, such that $f(z)=g(\omega(z))$, all $z \in D^*$.*

Proof. By excluding at most a countable discrete set from D we may assume that f' and g' do not vanish on a domain D_1. The domain D_1 may be covered by a union of closed (compact) discs K_n contained in D_1. Hence for some K_n there are uncountably many ζ_α in K_n, and for each such ζ_α there is z_α in D_1 with $f(z_\alpha)=g(\zeta_\alpha)$. Hence the corresponding z_α form an uncountable set (since, arguing as in the proof of the global theorem, the set $\{g(\zeta_\alpha)\}$ is uncountable) from which can be extracted an uncountable subset $\{z_{\alpha'}\}$ lying in some compact K_m. The corresponding $\{\zeta_{\alpha'}\}$ in K_n, which is also an uncountable set, contain a convergent sequence $\{\zeta_n\}$ converging to ζ_0 in K_n. From the corresponding z_n in K_m a convergent subsequence can be chosen. Now, after relabelings we have $z_n \to z_0$, $\zeta_n \to \zeta_0$, $f(z_n)=g(\zeta_n)$, and $f'(z_0)\neq 0 \neq g'(\zeta_0)$.

We now argue as in the global theorem. Let x^* be any continuous functional with $x^*g'(\zeta_0)\neq 0$. Then since $x^*f(z_0)=x^*g(\zeta_0)$, the function $\omega(z)=(x^*g)^{-1}(x^*f)(z)$ is analytic on a neighborhood D^* of z_0 and for large n

$$\omega(z_n) = (x^*g)^{-1}x^*f(z_n) = (x^*g)^{-1}x^*g(\zeta_n) = \zeta_n.$$

Thus $g(\omega(z_n))=f(z_n)$, for n large enough. Hence $g\omega\equiv f$ on D^* by the identity theorem.

We present an example to show that the hypothesis in the local theorem cannot be weakened. The basic idea is best seen in the following:

EXAMPLE 2a. The range X is C^2 and

$$g(z) = (z(z-1), z^2(z-1)), \qquad |z| < 1,$$
$$f(z) = \left(z(z-1), z^2(z-1)\cos\frac{2\pi}{1-z}\right), \qquad |z| < 1.$$

Then

$$f(1-1/n) = g(1-1/n) \to (0, 0) = f(0) = g(0).$$

So the intersection of the ranges of f and g contain an accumulation point, yet f is not locally analytically related to g. For if $f(z)=g(\omega(z))$ on some D^* contained in D, then from the first component $\omega(z)=z$ or $\omega(z)=1-z$, contradicting $z^2(z-1)\cos(2\pi/(1-z))=\omega(z)^2(\omega(z)-1)$ on D^*. Building on the above idea, we exhibit f, g with ranges intersecting in a set with infinitely many accumulation points.

EXAMPLE 2b. Construct a sequence of points α_n on the unit circle with $0<\arg\alpha_n<\pi/3$, and with

(i) $\alpha_n \to 1/2+i(\sqrt{3}/2)$,
(ii) $\sum(1-|\alpha_n-1|)<\infty$.

Now for each n choose a sequence $\{z_k^{(n)}\}$ with the properties

(i) $|z_k^{(n)}|<1$, $k=1, 2, 3, \ldots$,
(ii) $z_k^{(n)} \to \alpha_n$, as $k\to\infty$, and
(iii) $\sum_k (1-|z_k^{(n)}|)<2^{-n}$, $n=1, 2, 3, \ldots$.

Then there exists [2, p. 240] a Blaschke product $B(z)$ with zeros at each $z_k^{(n)}$, and each α_n-1.

We define the (C^2-valued) functions f, g as follows:

$$g(z) = (\sin 2\pi z, 0), \quad f(z) = (\sin 2\pi z, B(z)).$$

Then $f(z_k^{(n)})=g(z_k^{(n)}) \to (\sin 2\pi\alpha_n, 0)=f(\alpha_n-1)=g(\alpha_n-1)$. And so the ranges intersect with infinitely many accumulation points.

If $f(z)=g(\omega(z))$, then comparing second coordinates, $B(z)\equiv 0$ on the domain of ω, an obvious contradiction.

REFERENCES

1. N. Dunford and J. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.
2. E. Hille, *Analytic function theory*, Vol. II, Ginn, Boston, 1962.
3. J. Kelly and I. Namioka, *Linear topological spaces*, Van Nostrand, Princeton, N. J., 1963.

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