DECEPTIVE CONVERGENCE OF FOURIER SERIES ON $SU(2)$

BY

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1. **Introduction and summary.** In this paper G will always denote a compact group that has a faithful finite-dimensional unitary representation. The Haar measure on G will be denoted by μ, and we normalize μ so that $\mu(G)=1$. By a *representation* of G we will mean a finite-dimensional unitary representation of G. Let U be a faithful representation of G and let $V=U\oplus\bar{U}$ be the direct sum of U and its complex conjugate representation. Let $T_0(U)$ be the space of constant functions on G, let $T_1(U)$ be the linear space spanned by $T_0(U)$ and the coordinate functions of V, and for each integer $n\geqq 1$ let $T_n(U)$ be the space spanned by all functions $f_1f_2\cdots f_n$ where $f_i\in T_1(U)$ for $1\leqq i\leqq n$. We will call $T_n(U)$ the *space of U-trigonometric polynomials of degree* $\leqq n$. For each nonnegative integer n let U_n be the orthogonal projection of $L^2(G)$ onto $T_n(U)$. There is a natural way to extend U_n to a projection from $L^1(G)$ onto $T_n(U)$ (see (2.1)) and we will denote this extension by U_n. If f is a function in $L^1(G)$ we will call the sequence $\{U_nf\}$ the *U-Fourier series for f*, and we will call U_nf the nth *partial sum of the U-Fourier series for f*. If $f\in L^2(G,\mu)$, then $U_nf\to f$ in $L^2(G,\mu)$ as $n\to\infty$. If $x\in G$ then the *U-Fourier series for f at x* is defined to be the sequence $\{U_nf(x)\}$.

If $G=\boldsymbol{T}$ is the group of complex numbers of absolute value 1, and U is the 1-dimensional representation of $\boldsymbol{T}$ on $\boldsymbol{C}$ defined by

$$U(e^{it})z = e^{it}z \quad \text{for all } e^{it}\in \boldsymbol{T},\ z\in \boldsymbol{C} \tag{1.1}$$

then the U-trigonometric polynomials of degree $\leqq n$ are ordinary trigonometric polynomials of degree $\leqq n$, and for any $f\in L^1(\boldsymbol{T})$, U_nf is the nth partial sum of the ordinary Fourier series for f.

Let $\{a_n\}$, $\{b_n\}$ be two sequences of complex numbers. We say that $\{a_n\}$ and $\{b_n\}$ *co-converge* if they either both converge to the same limit or both diverge. Let U and W be two faithful representations of the compact group G. We will say that U and W are *series equivalent* if for every f in $L^1(G)$ and every x in G the sequences $\{U_nf(x)\}$ and $\{W_nf(x)\}$ co-converge. Series equivalence is clearly an equivalence relation on the set of faithful representations of G. It is easy to show that any two faithful representations of $\boldsymbol{T}$ are series equivalent (see 2.2). For the group $SU(2)$ of 2×2 unitary matrices with determinant 1 the situation is more complicated. $SU(2)$ has exactly one irreducible representation R^n of dimension n for each

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positive integer n (see [8, p. 137]). It follows from (3.19) and (7.18) that the representations $R^q \oplus R^{p+1}$ where $1 \leqq q \leqq p$ and $p \equiv q \bmod 2$ form a complete set of equivalence class representatives for the relation of series equivalence on the faithful representations of $SU(2)$. If U is a faithful representation of $SU(2)$ which is series equivalent to $R^q \oplus R^{p+1}$ we will say that *U is of type* (q, p).

Let U be a faithful representation of G, let $x \in G$ and let f be a function in $L^1(G)$ which is continuous at x. If the U-Fourier series for f at x converges to a value different from $f(x)$ we will say that the U-Fourier series for *f converges deceptively* at x. It is well known that the ordinary Fourier series of a function in $L^1(T)$ cannot converge deceptively [10, p. 89] and it follows from [5, p. 683] that the R^2-Fourier series of a function in $L^1(SU(2))$ cannot converge deceptively. Let S be a subset of $L^1(G)$. If no function f in S has a point of continuity at which the U-Fourier series for f converges deceptively, we will say that *U-Fourier series are honest for functions in S*.

In this paper we will prove the following results. Let S be the set of all functions in $L^\infty(SU(2))$ which are continuous except on a set of Hausdorff dimension $\leqq 2$, and let T be the set consisting of those functions in S whose set of discontinuities has Hausdorff dimension <2. Let U be a faithful representation of $SU(2)$. Then U-Fourier series are honest for functions in T; U-Fourier series are honest for functions in S if and only if U is of type (1, 1), (2, 2) or (3, 3); U-Fourier series are honest for functions in $L^\infty(SU(2))$ if and only if U is of type (1, 1), (2, 2) or (3, 3); and U-Fourier series are honest for functions in $L^1(SU(2))$ if and only if U is of type (1, 1). With each faithful representation U of $SU(2)$ we can associate a set D_U (the *deception set* of U) with the following two properties.

I. If $x \in SU(2)$ and y is any element of xD_U different from x, then there exists a function f in $L^1(SU(2))$ such that f is analytic except at y and the U-Fourier series for f converges deceptively at x.

II. If $x \in SU(2)$ and f is any function in $L^1(SU(2))$ which is continuous on xD_U then the U-Fourier series for f does not converge deceptively at x.

In (7.1) we give a very explicit description of D_U which shows that if U is of type (q, p) with p odd, then D_U consists of all elements of $SU(2)$ whose eigenvalues are pth roots of unity. The main tool for proving the above results is the technical Theorem 5.63 which gives an explicit formula for the error $f(x) - \lim_{n\to\infty} U_n f(x)$, valid for a fairly large class of functions. Let U be any faithful representation of $SU(2)$, and let f be a function in $L^2(SU(2))$. Then the set of points where the U-Fourier series for f converges deceptively has measure zero. It would be desirable to prove this result for arbitrary functions in $L^1(SU(2))$, but I have been unable to do this.

2. **Series equivalence of the representations of T.** Let U be a faithful representation of the compact group G. Since $T_1(U)$ is the space spanned by the coordinate functions of a representation of G, $T_1(U)$ is clearly left and right translation in-

variant, and hence $T_n(U)$ is a left and right translation invariant subspace of $L^2(G)$ for all n. Also $T_n(U)$ is closed (in fact finite dimensional) in $L^2(G)$, so $T_n(U)$ is a two-sided ideal in $L^2(G)$ by [4, §31F and 39E]. By the structure theory for such ideals [4, §39] we know that $T_n(U)$ contains a unique central idempotent D_n^U with the property that $f \to f * D_n^U$ is the projection of $L^2(G)$ onto $T_n(U)$, i.e.

$$U_n f = f * D_n^U \quad \text{for all } f \in L^2(G). \tag{2.1}$$

(The $*$ here denotes convolution.) We will call the sequence $\{D_n^U\}$ $(n=0, 1, 2, \ldots)$ the *Dirichlet kernel* for U. The right-hand side of (2.1) makes sense for any f in $L^1(G)$ and we use (2.1) to define $U_n f$ for any f in $L^1(G)$.

PROPOSITION 2.2. *Any two faithful representations of* $\boldsymbol{T}$ *are series equivalent.*

Proof. Let W be any faithful representation of $\boldsymbol{T}$, and let U be the representation defined in (1.1). Note that for each nonnegative integer n, $T_n(U)$ is the linear space spanned by $\{e^{ikt} : -n \leqq k \leqq n\}$. If f is a function in $L^1(\boldsymbol{T})$ whose (ordinary) Fourier series is $f \sim \sum c_k e^{ikt}$ then the U-Fourier series for f is $\{\sum_{k=-n}^{n} c_k e^{ikt}\}$, and the W-Fourier series for f is $\{\sum_{k \in A(n)} c_k e^{ikt}\}$ where $A(n) = \{k \in \boldsymbol{Z} : e^{ikt} \in T_n(W)\}$. We will show that W and U are series equivalent.

Let χ be the character of W. We can write $\chi(e^{it}) = \sum a_n e^{int}$ where the a_n are nonnegative integers and all but a finite number of the a_n's are zero. Let N be the largest integer such that $a_N + a_{-N} > 0$. For any positive integer k we have

$$T_k(W) \subseteq T_{Nk}(U). \tag{2.3}$$

Since W is a faithful representation of $\boldsymbol{T}$ it follows from [2, pp. 189–190] that the algebra generated by $\{e^{int} : a_n > 0\} \cup \{e^{-int} : a_n > 0\}$ is the algebra of all trigonometric polynomials, and hence there exists an integer p such that $T_N(U) \subseteq T_p(W)$. Since

$$e^{iNt} T_{pN}(U) + e^{-iNt} T_{pN}(U) = T_{(p+1)N}(U)$$

we can show by induction that

$$T_{N(q+1)}(U) \subseteq T_{p+q}(W), \qquad q \geqq 0. \tag{2.4}$$

Combining (2.3) and (2.4) we obtain

$$T_{Nk-N(p-1)}(U) \subseteq T_k(W) \subseteq T_{Nk}(U), \qquad k \geqq p. \tag{2.5}$$

Let $B(k) = \{n \in Z : e^{int} \in T_{Nk}(U),\ e^{int} \notin T_k(W)\}$. Then $U_{Nk} f - W_k f = \sum_{n \in B(k)} c_n e^{int}$ and it follows from (2.5) that

$$\| U_{Nk} f - W_k f \|_\infty \leqq 2N(p-1) \operatorname{Sup} \{|c_j| : |j| \geqq N(k-p+1)\}$$

whenever $k \geqq p$. Thus by the Riemann-Lebesgue lemma we see that $U_{Nk} f - W_k f$ converges uniformly to 0 on $\boldsymbol{T}$ as $k \to \infty$, and for any $e^{ix} \in \boldsymbol{T}$ we see that $\{U_{Nk} f(e^{ix})\}$ and $\{W_k f(e^{ix})\}$ co-converge. Also the Riemann-Lebesgue lemma shows that

$\{U_{Nk}f(e^{ix})\}$ and $\{U_kf(e^{ix})\}$ co-converge, and it follows that $\{W_kf(e^{ix})\}$ and $\{U_kf(e^{ix})\}$ co-converge. Thus U and W are series equivalent.

3. **Formulas for the Dirichlet kernels.** Let G be a compact group and let $\{\chi_a\}$ ($a \in A$) be the set of all irreducible characters of G. If ϕ is any character of G and $a \in A$, we will say that χ_a is contained in ϕ if $(\chi_a, \phi) > 0$. Here (χ_a, ϕ) denotes the inner product of χ_a and ϕ. Let ϕ, ψ be two characters of G. We will write $\phi \succ \psi$ if every irreducible character contained in ψ is also contained in ϕ, and we will write $\psi \sim \phi$ if $\phi \succ \psi$ and $\psi \succ \phi$. Thus $\phi \sim \psi$ if and only if the representations corresponding to ϕ and ψ are quasi equivalent in the sense of [7, p. 631]. We will use the facts that

$$\phi \prec \psi \Rightarrow \phi+\psi \sim \psi, \tag{3.1}$$

$$\phi \prec \psi \text{ and } \phi_1 \prec \psi_1 \Rightarrow \phi\phi_1 \prec \psi\psi_1, \tag{3.2}$$

and similar obvious properties of the relations $\prec$ and $\sim$ without specifically mentioning them. Let $E(\chi_a)$ be the two-sided ideal in $L^2(G)$ generated by χ_a. Then $\{E(\chi_a) : a \in A\}$ is the set of all minimal two-sided ideals in $L^2(G)$, and the generating idempotent for $E(\chi_a)$ is $d_a\chi_a$ where $d_a = \chi_a(e)$. Let U be a faithful representation of G. Then since $T_n(U)$ is a closed ideal in $L^2(G)$ we have by [4, §39] that

$$T_n(U) = \sum E(\chi_a) \qquad (\chi_a \in T_n(U))$$

and hence the Dirichlet kernel $\{D_n^U\}$ for U is given by

$$D_n^U = \sum d_a\chi_a \qquad (\chi_a \in T_n(U)). \tag{3.3}$$

Let χ_U be the character of U. Then it follows from [5, p. 684] that $\chi_a \in T_n(U)$ if and only if $((1+\chi_U+\bar{\chi}_U)^n, \chi_a) > 0$, and hence (3.3) becomes

$$D_n^U = \sum d_a\chi_a, \qquad \chi_a \prec (1+\chi_U+\bar{\chi}_U)^n. \tag{3.4}$$

Let θ be the function on $SU(2)$ defined by

$$\theta(g) = \arccos\left(\tfrac{1}{2}\operatorname{Trace}(g)\right), \qquad g \in SU(2) \tag{3.5}$$

so that the eigenvalues of an element g of $SU(2)$ are $\exp(\pm i\theta(g))$. For each positive integer n, $SU(2)$ has exactly one irreducible representation R^n of dimension n, and the character χ_n of R^n is

$$\chi_n = \sin n\theta / \sin\theta \tag{3.6}$$

(see [8, p. 151]). Since $\chi_{2n}(x) = 2n$ only if $\theta(x) = 1$, or equivalently only if $x = e$, the characters χ_{2n} are all faithful (we call a character faithful if it is the character of a faithful representation). Note that $\chi_{2n+1}(e) = \chi_{2n+1}(-e)$ so χ_{2n+1} is never faithful. Let U be a faithful representation of $SU(2)$ with character $\chi_U = \sum n_k(U)\chi_k$. Then $\chi_U(-e) = \sum kn_k(U)(-1)^{k+1}$. Since U is faithful $\chi_U(-e) \neq \chi_U(e) = \sum kn_k(U)$, and hence χ_U contains some irreducible character of even dimension. Let p be the greatest integer such that $n_{p+1}(U)>0$ (so $p\geqq 1$) and let q be the greatest integer such that $q\equiv p \bmod 2$ and $n_q(U)>0$ (if no such q exists then p is odd, and in this case we take $q=1$). We will say that U *is of type* (q,p). If q,p are any integers such that $1\leqq q\leqq p$ and $q\equiv p \bmod 2$ then $R^q\oplus R^{p+1}$ is of type (q,p). We will show in (7.18) that this definition of type agrees with the definition given in the introduction.

THEOREM 3.7. *Let U be a faithful representation of $SU(2)$ of type (q,p), and let $\{D_n^U\}$ be the Dirichlet kernel for U. Then for every $n\geqq 3$*

$$D_n^U = \sum_{m=q+1}^{p+1} (-1)^{p+m+1} D_{(n-1)p+m}, \tag{3.8}$$

where

$$D_j = \sum_{k=1}^{j} k\chi_k. \tag{3.9}$$

Proof. By the Clebsch-Gordan formula [8, p. 163] we know

$$\chi_r\chi_s = \sum_{j=1}^{r} \chi_{r+s+1-2j} \qquad \text{if } r\leqq s. \tag{3.10}$$

We define $S(n)$ and $S^*(n)$ by the formulas

$$\begin{aligned} S(n) &= \sum \chi_j \qquad (1\leqq j\leqq n)\\ S^*(n) &= \sum \chi_j \qquad (1\leqq j\leqq n,\, j\equiv n \bmod 2). \end{aligned} \tag{3.11}$$

It then follows from (3.10) that

$$\begin{aligned} \chi_r\cdot S(n) &\sim S(n+r-1) \qquad (r\leqq n)\\ \chi_r\cdot S^*(n) &\sim S^*(n+r-1) \qquad (r\leqq n+1) \end{aligned} \tag{3.12}$$

and that

$$\begin{aligned} \chi_r\cdot S(n) &\prec \chi_s\cdot S(n) \qquad (r\leqq s\leqq n)\\ \chi_r\cdot S^*(n) &\prec \chi_s\cdot S^*(n) \qquad (r\leqq s\leqq n+1,\, r\equiv s \bmod 2). \end{aligned} \tag{3.13}$$

LEMMA 3.14. *Let U be a faithful representation of $SU(2)$ of type (q,p) and if $q=1$ assume that χ_1 is contained in χ_U. Then for every $n\geqq 3$*

$$(\chi_U)^n \sim S((n-1)p+q)+S^*(np+1) \tag{3.15}$$

where $S(n)$ and $S^*(n)$ are as defined in (3.11).

Proof. Using the Clebsch-Gordan formula (3.10) we get that $(\chi_U)^3\succ(\chi_{p+1})^3 \sim S^*(3p+1)$ and $(\chi_U)^3\succ\chi_{p+1}^2\chi_q\sim S^*(2p+q)$ and hence

$$(\chi_U)^3 \succ S^*(3p+1)+S^*(2p+q) \sim S^*(3p+1)+ S(2p+q), \tag{3.16}$$

since p and q have the same parity. Let us say that χ_r is *larger* than χ_s if and only if $r>s$. The largest irreducible character in $(\chi_U)^3$ is the largest character in $(\chi_{p+1})^3$

which is χ_{3p+1}. The largest character in $(\chi_U)^3$ whose dimension has parity different from $3p+1$ is the largest character in $\chi_{p+1}^2\chi_q$ which is χ_{2p+q}. Thus $(\chi_U)^3 \prec S^*(3p+1) + S(2p+q)$. This fact combined with (3.16) proves the lemma for the case $n=3$. Suppose now that we know that (3.15) holds for a given $n \geqq 3$. Then by (3.12), (3.13), (3.1) and (3.2) we have

$$\begin{aligned}(\chi_U)^{n+1} &\sim \chi_U(S((n-1)p+q)+S^*(np+1))\\ &\sim (\chi_q+\chi_{p+1})(S((n-1)p+q)+S^*(np+1))\\ &\sim S(np+q)+S^*(np+q)+S^*((n+1)p+1).\end{aligned}$$

Since $S^*(np+q) \prec S(np+q)$, (3.15) also holds for $(n+1)$, and the lemma is proved.

It follows from Lemma 3.14 that for $n \geqq 3$

$$\begin{aligned}(1+\chi_U)^n &\sim \sum_{j=0}^{n} (\chi_U)^j \sim S((n-1)p+q)+S^*(np+1)\\ &\sim S((n-1)p+q+1)+\sum_{m=1}^{(p-q)/2} \chi_{(n-1)p+q+1+2m}.\end{aligned} \tag{3.17}$$

It is easy to see that (3.17) holds even if $q=1$ and χ_1 is not contained χ_U, since $\chi_1=1$. We have $D_j-D_{j-1}=j\chi_j$ from the definition (3.9). Since χ_U is a real character it follows from (3.4) that

$$D_n^U = \sum k\chi_k, \qquad \chi_k \prec (1+\chi_U)^n,$$

and using this with (3.17) we get

$$\begin{aligned}D_n^U &= D_{(n-1)p+q+1}+\sum_{m=1}^{(p-q)/2} (D_{(n-1)p+q+1+2m}-D_{(n-1)p+q+2m})\\ &= \sum_{m=q+1}^{p+1} (-1)^{m+p+1} D_{(n-1)p+m}\end{aligned} \tag{3.18}$$

for $n \geqq 3$. This completes the proof of Theorem 3.7.

COROLLARY 3.19. *If U is a faithful representation of $SU(2)$ of type (q, p) then U is series equivalent to $R^q \oplus R^{p+1}$.*

COROLLARY 3.20. *If U is a faithful representation of $SU(2)$ of type (q, p) then $\|D_n^U\|_\infty < 2n^3p^3$ for $n \geqq 3$.*

Proof.

$$\begin{aligned}\|D_n^U\|_\infty &= D_n^U(e) \leqq D_{np+1}(e)\\ &= \sum_{j=1}^{np+1} j^2 < 2n^3p^3 \quad \text{for } n \geqq 3.\end{aligned}$$

4. **Some technical lemmas.** In this section all functions denoted by capital Latin letters will be complex valued measurable functions of period 2π defined on $\boldsymbol{R}$. If F is a function we will write

$$\|F\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(t)|\,dt \tag{4.1}$$

whenever the integral on the right exists, and L^1 will denote the space of functions F for which $\|F\|_1$ is finite. If F, G are functions we will write

$$(F, G) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\bar{G}(t)\,dt \tag{4.2}$$

whenever the integral on the right exists. If F has a right-hand (left-hand) limit at a point t_0 we will denote this limit by $F_+(t_0)$ $(F_-(t_0))$.

This section contains a number of technical lemmas that will be used later in the paper. Since most of the lemmas are not interesting in themselves, I would suggest that the reader go to §5 at this point, and refer back to the material in this section as it is used.

For $0<r<1$ define

$$A_r(t) = (1-r^2)(1-2r\cos t+r^2)^{-1} = 1+2\sum_{k=1}^{\infty} r^k \cos kt, \tag{4.3}$$

$$B_r(t) = -r^{-1}\sin tA_r'(t) = 1+\sum_{k=1}^{\infty}((k+1)r^k-(k-1)r^{k-2})\cos kt. \tag{4.4}$$

A_r and B_r are positive even functions, $\|A_r\|_1=\|B_r\|_1=1$ and for any $\delta>0$, A_r and B_r both converge uniformly to 0 on $[\delta, 2\pi-\delta]$ as $r\to 1$ (see [10, pp. 96, 97 and 100]). Hence it follows from [10, p. 86] that if F is any function in L^1 such that $F_+(0)$ and $F_-(0)$ both exist, we have

$$\lim_{r\to 1}(F, A_r) = \lim_{r\to 1}(F, B_r) = \tfrac{1}{2}(F_+(0)+F_-(0)). \tag{4.5}$$

Define

$$\begin{aligned} C_r(t) &= (A_r(t)-r^2B_r(t))(1+r)^{-1} \\ &= (1-r)\left[1+\sum_{k=1}^{\infty}(k+1)r^k\cos kt\right]. \end{aligned} \tag{4.6}$$

LEMMA 4.7. *Let F be a function in L^1 such that F is bounded on a neighborhood of* 0 *and such that the limit* $\lim_{r\to 1}(F, C_r)=L$ *exists. Then* $L=0$.

Proof. Since $(F, C_r)=0$ for any odd function F, we may assume without loss of generality that F is even. Suppose $L\neq 0$. Then we may assume without loss of generality that F is real valued, and $L>0$. Choose δ such that $0<\delta<\pi$ and such that F is bounded on $[-\delta, \delta]$. Let g be the characteristic function of $[-\delta, \delta]$, and let G be the periodic function of period 2π that agrees with Fg on $[-\pi, \pi]$. Then G is bounded, and

$$\lim_{r\to 1}(G, C_r) = \lim_{r\to 1}(F, C_r)+\lim_{r\to 1}\frac{(G-F, A_r)}{1+r}-\lim_{r\to 1}\frac{r^2(G-F, B_r)}{1+r}. \tag{4.8}$$

Since $G-F$ vanishes on $[-\delta, \delta]$ it follows from (4.5) that the last two limits in (4.8) are both zero, and hence

$$\lim_{r\to 1}(G, C_r) = L. \tag{4.9}$$

Write $G\sim a_0+\sum_{k=1}^{\infty} a_k \cos kt$, and define a function h on the interval $[0, 1)$ by

$$h(r) = \sum_{k=0}^{\infty} a_k r^{k+1} = r(G, A_r), \qquad 0 \leqq r < 1.$$

Since $|h(r)| \leqq \|G\|_\infty \|A_r\|_1 = \|G\|_\infty$ for $0\leqq r<1$, we see that h is bounded on $[0, 1)$. Now

$$(G, C_r) = \tfrac{1}{2}(1-r)\left(a_0+\sum_{k=0}^{\infty}(k+1)a_k r^k\right) = \tfrac{1}{2}(1-r)(a_0+h'(r)),$$

and hence by (4.9) we have

$$L = \tfrac{1}{2}\lim_{r\to 1}(1-r)h'(r). \tag{4.10}$$

It follows from (4.10) that there exists a number $\varepsilon>0$ such that $h'(r)>L(1-r)^{-1}$ for $r\geqq 1-\varepsilon$, and hence h is increasing on the interval $(1-\varepsilon, 1)$. By the mean value theorem we know that for any integer $k\geqq 0$ there is a number t_k in the interval $[1-2^{-k}\varepsilon, 1-2^{-k-1}\varepsilon]$ such that

$$\begin{aligned} h(1-2^{-k-1}\varepsilon)-h(1-2^{-k}\varepsilon) &= 2^{-k-1}\varepsilon h'(t_k) \\ &> 2^{-k-1}\varepsilon L(1-t_k)^{-1} \geqq \tfrac{1}{2}L. \end{aligned}$$

Thus

$$h(1-2^{-k-1}\varepsilon)-h(1-\varepsilon) = \sum_{j=0}^{k}(h(1-2^{-j-1}\varepsilon)-h(1-2^{-j}\varepsilon)) \geqq \tfrac{1}{2}(k+1)L.$$

Since $L\neq 0$ this contradicts the boundedness of h, and the lemma follows.

LEMMA 4.11. *Let $\{F_r\}$ $(0<r<1)$ be a family of functions such that $F_r\to 0$ uniformly on $[\delta, 2\pi-\delta]$ for every $\delta>0$, and such that the set $\{\|F_r \sin t\|_\infty : 0<r<1\}$ is bounded. Let $G\in L^1$, and suppose that there is a neighborhood N of 0 such that on N we can write $G=FH$ where $F\in L^1$, H is analytic on N and $H(0)=0$. Then $\lim_{r\to 1}(G, F_r)=0$.*

Proof. Our hypotheses imply that we can write $G=(1-e^{it})K$ where $K\in L^1$. Hence

$$(G, F_r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K(t)(1-e^{it})F_r(t)\, dt. \tag{4.12}$$

It follows from our hypotheses that there exists a constant M and an $\varepsilon>0$ such that $(1-e^{it})F_r(t)\leqq M$ for all t, and all $r>1-\varepsilon$. Thus

$$|K(t)(1-e^{it})F_r(t)| \leqq MK(t)$$

for all t and all $r>1-\varepsilon$. The lemma now follows from the dominated convergence theorem.

LEMMA 4.13. *Let A_r, C_r be as defined in* (4.3), (4.6) *and define*

$$\begin{aligned} D_r(t) &= -(1-r\cos t)A_r'(t)/(1+r) \\ &= 2r(1-r)\sin t(1-r\cos t)(1-2r\cos t+r^2)^{-2}. \end{aligned} \tag{4.14}$$

Let G be a function in L^1, and suppose that there is a neighborhood N of 0 *such that on N we can write $G=FH$ where $F\in L^1$, H is analytic on N, and $H(0)=0$. Then*

$$\lim_{r\to 1}(G, A_r) = \lim_{r\to 1}(G, C_r) = \lim_{r\to 1}(G, D_r) = 0.$$

Proof. Let

$$G_r(t) = (1-r)\sin t(1-2r\cos t+r^2)^{-1}. \tag{4.15}$$

Then we can easily verify that

$$A_r\sin t = G_r[1+r], \tag{4.16}$$

$$C_r\sin t = G_r\left[\frac{(1-r\cos t)^2-r^2\sin^2 t}{(1-r\cos t)^2+r^2\sin^2 t}\right], \tag{4.17}$$

$$D_r\sin t = G_r\left[\frac{2(1-r\cos t)r\sin t}{(1-r\cos t)^2+r^2\sin^2 t}\right]. \tag{4.18}$$

The expressions in square brackets on the right-hand sides of (4.16), (4.17) and (4.18) are all bounded by 2 in absolute value, so

$$\|A_r\sin t\|_\infty \leqq 2\|G_r\|_\infty,\ \|C_r\sin t\|_\infty \leqq 2\|G_r\|_\infty,\ \|D_r\sin t\|_\infty \leqq 2\|G_r\|_\infty. \tag{4.19}$$

In [10, p. 96] it is shown that there exists a constant k such that $A_r(t)\leqq k\delta/(\delta^2+t^2)$ where $\delta=1-r$ and $|t|\leqq\pi$. Thus

$$|G_r(t)| = (1+r)^{-1}|\sin tA_r(t)| \leqq |tA_r(t)| \leqq k\delta t/(\delta^2+t^2) \leqq k$$

for $|t|\leqq\pi$. Lemma 4.14 thus follows from (4.19) and Lemma 4.11.

LEMMA 4.20. *Let F be a function in L^1 such that $F_+(0)$ and $F_-(0)$ both exist, and let D_r be defined by* (4.14). *Then*

$$\lim_{r\to 1}\int_0^\pi F(t)D_r(t)\,dt = F_+(0),$$

$$\lim_{r\to 1}\int_{-\pi}^0 F(t)D_r(t)\,dt = -F_-(0),$$

and hence

$$\lim_{r\to 1}(F, D_r) = \frac{1}{2\pi}(F_+(0)-F_-(0)).$$

Proof. D_r is an odd function that is positive on the interval $(0, \pi)$, and D_r converges uniformly to zero on $[\delta, 2\pi-\delta]$ for any $\delta>0$. A straightforward calculation shows that

$$\int_0^\pi D_r(t)\,dt = \frac{2r}{1+r}+(1-r)\log\left(\frac{1+r}{1-r}\right),$$

and hence $\lim_{r\to 1}\int_0^\pi D_r(t)\,dt=1$. The lemma follows from these facts by a standard kind of argument.

LEMMA 4.21. *Let F be a function which is continuous at* 0, *and such that $F\sin^2 t$ is in L^1. Then* $\lim_{r\to 1}(F\sin t, D_r)=0$, *where D_r is defined by* (4.14).

Proof. Since $F\sin^2 t$ is in L^1 the inner product $(F\sin t, D_r)$ exists for $0<r<1$. Let H be the periodic function of period 2π that agrees with the characteristic function of $[-\pi/2, \pi/2]$ on $[-\pi, \pi]$. Our hypothesis implies that $FH\sin t$ is in L^1 and $FH\sin t$ is continuous at 0, and hence by Lemma 4.20

$$\lim_{r\to 1}(FH\sin t, D_r) = 0. \tag{4.22}$$

Now

$$((1-H)F\sin t, D_r) = 2((1-H)F\sin^2 t, r(1-r)(1-r\cos t)(1-2r\cos t+r^2)^{-2}).$$

Since $(1-H)F\sin^2 t$ is in L^1, and $r(1-r)(1-r\cos t)(1-2r\cos t+r^2)^{-2}$ converges uniformly to zero on the set where $1-H\neq 0$, it follows that $\lim_{r\to 1}((1-H)F\sin t, D_r)=0$. This fact together with (4.22) proves the lemma.

If F is a function and p is a positive integer, define a function $F^{[p]}$ by

$$F^{[p]}(t) = F(pt). \tag{4.23}$$

Then it is easy to verify that

$$(G, F^{[p]}) = \frac{1}{2\pi p}\int_{-\pi}^{\pi}\sum_{j\in J(p)} G((t+2\pi j)/p)\bar{F}(t)\,dt \tag{4.24}$$

where

$$J(p) = \{n\in \boldsymbol{Z} : -[\tfrac{1}{2}p] \leqq n \leqq [\tfrac{1}{2}(p-1)]\}. \tag{4.25}$$

Here $[\frac{1}{2}p]$ is the greatest integer $\leqq \frac{1}{2}p$.

LEMMA 4.26. *Let p be a positive integer and let G be an L^1 function such that for every $j\in J(p)$ either G is bounded near $2\pi j/p$, or else there exists a neighborhood N_j of $2\pi j/p$ such that $G=F_jH_j$ on N_j where $F_j\in L^1$, H_j is analytic on N_j, and $H_j(2\pi j/p)=0$. Suppose that the limit* $\lim_{r\to 1}(G, C_r^{[p]})=L$ *exists, where C_r is defined in* (4.6). *Then* $L=0$.

Proof. Write $J(p)$ as a disjoint union $J(p)=I\cup K$ where for each $j\in I$, G is bounded near $2\pi j/p$, and for each $j\in K$, G has a factorization $G=F_jH_j$ as in the statement of the lemma. Let

$$g(t) = \sum_{j\in I} G((t+2\pi j)/p),\quad k(t) = \sum_{j\in K} G((t+2\pi j)/p). \tag{4.27}$$

Then g is bounded near 0, and there is a neighborhood N of 0 such that $k=fh$ on N where $f\in L^1$, h is analytic on N and $h(0)=0$. It follows from (4.24) that

$$(G, C_r^{[p]}) = (2\pi p)^{-1}\left(\int_{-\pi}^{\pi} g(t)C_r(t)\,dt+\int_{-\pi}^{\pi} k(t)C_r(t)\,dt\right). \tag{4.28}$$

By lemma 4.13 the second integral in (4.28) goes to zero as $r\to 1$, and hence

$$L = \lim_{r\to 1}(G, C_r^{[p]}) = (2\pi p)^{-1}\lim_{r\to 1}\int_{-\pi}^{\pi} g(t)C_r(t)\,dt.$$

It follows from Lemma 4.7 that $L=0$.

LEMMA 4.29. *Let p be a positive integer, and let $J(p)$ be written as a disjoint union $J(p)=I\cup K$. Let G be a function in L^1 such that G has left- and right-hand limits at $2\pi j/p$ for each $j\in I$, and for each $j\in K$ there exists a neighborhood N_j of $2\pi j/p$ such that $G=F_jH_j$ on N_j where $F_j\in L^1$, H_j is analytic on N_j and $H_j(2\pi j/p)=0$. Then if D_r is defined by* (4.14) *we have*

$$\lim_{r\to 1}(G, D_r^{[p]}) = (2\pi p)^{-1}\sum_{j\in I}(G_+(2\pi j/p)-G_-(2\pi j/p)).$$

Proof. Define functions $g(t)$ and $k(t)$ by (4.27). Then

$$g_+(0) = \sum_{j\in I} G_+(2\pi j/p),\quad g_-(0) = \sum_{j\in I} G_-(2\pi j/p), \tag{4.30}$$

and k has a factorization $k=fh$ as in the previous lemma. By (4.24) we have

$$(G, D_r^{[p]}) = (2\pi p)^{-1}\left(\int_{-\pi}^{\pi} g(t)D_r(t)\,dt+\int_{-\pi}^{\pi} k(t)D_r(t)\,dt\right). \tag{4.31}$$

By Lemma 4.13 the second integral in (4.31) goes to zero as $r\to 1$, and by Lemma 4.20

$$\lim_{r\to 1}(2\pi p)^{-1}\int_{-\pi}^{\pi} g(t)D_r(t)\,dt = (2\pi p)^{-1}(g_+(0)-g_-(0)).$$

This result combined with (4.30) and (4.31) proves the lemma.

5. **A formula for** $\lim U_nf(x)$. It t is any real number let $x(t)$ be the element of $SU(2)$ defined by

$$x(t) = \operatorname{diag}(e^{it}, e^{-it}). \tag{5.1}$$

If f is any class function on $SU(2)$ let $[f]$ be the function on $\boldsymbol{R}$ defined by

$$[f](t) = f(x(t)),\qquad t\in \boldsymbol{R}. \tag{5.2}$$

Then $[f]$ is an even periodic function of period 2π, and if F is any even periodic function of period 2π we can write $F=[f]$ for some class function f on $SU(2)$. If f, g are functions on $SU(2)$ we will write

$$(f, g)^* = \int_{SU(2)} f(x)\bar{g}(x)\,d\mu(x) \tag{5.3}$$

whenever the integral in (5.3) exists. If f is a class function on $SU(2)$ then if follows from [8, pp. 163, 386–389] that $[f]\sin^2 t \in L^1[-\pi, \pi]$, and

$$\int_{SU(2)} f(x)\, d\mu(x) = \frac{2}{\pi}\int_0^\pi [f](t)\sin^2 t\, dt. \tag{5.4}$$

Hence if f and g are class functions on $SU(2)$ we have

$$(f, g)^* = 2([f], [g]\sin^2 t). \tag{5.5}$$

Let Q be the projection onto the space of class functions on $SU(2)$,

$$Qf(y) = \int_{SU(2)} f(xyx^{-1})\, d\mu(x). \tag{5.6}$$

If $f \in L^p(SU(2))$ and $g \in L^q(SU(2))$ where $p^{-1}+q^{-1}=1$ then $(f, Qg)^* = (Qf, g)^*$. If g is a class function then

$$(f, g)^* = (f, Qg)^* = (Qf, g)^* = 2([Qf], [g]\sin^2 t). \tag{5.7}$$

LEMMA 5.8. *Let U be a faithful representation of $SU(2)$ of type (q, p), and let f be a function in $L^1(SU(2))$ such that the U-Fourier series for f at e converges to the value L. Then*

$$L = \lim_{r\to 1} \sum_{m=q+1}^{p+1} (-1)^{m+p+1}(f, (1-r)Z_{mpr})^*,$$

where

$$Z_{mpr} = \sum_{n=0}^{\infty} r^n D_{np+m}. \tag{5.9}$$

Here D_j is defined by (3.9).

Proof.

$$L = \lim_{n\to\infty} U_n f(e) = \lim_{n\to\infty} f * D_n^U(e) = \lim_{n\to\infty} (f, D_n^U)^*. \tag{5.10}$$

Since the Abel method of finding the limit of a sequence of numbers is regular we have

$$L = \lim_{r\to 1}(1-r)\sum_{n=0}^{\infty} r^n (f, D_{n+1}^U)^* = \lim_{\to 1}(f, Y_r)^*, \tag{5.11}$$

where

$$Y_r = (1-r)\sum_{n=0}^{\infty} r^n D_{n+1}^U. \tag{5.12}$$

For each r with $0<r<1$ the sum in (5.12) converges absolutely by Corollary 3.20. By formula (3.8) we have

$$Y_r = (1-r)\sum_{m=q+1}^{p+1} (-1)^{m+p+1} Z_{mpr} + \varepsilon(r) \tag{5.13}$$

where Z_{mpr} is defined by (5.9), and $\varepsilon(r)$ is an error term which arises from the fact that (3.8) may not be valid for $n<3$. Since $\|\varepsilon(r)\|_\infty \to 0$ as $r \to 1$ we have $\lim_{r\to 1} (f, \varepsilon(r))^*=0$, and hence the proposition follows from (5.11) and (5.13).

Proposition 5.14. *Let h be a function in $L^1(SU(2))$ that is continuous at e. Let Z_{mpr} be defined by* (5.9), *and let C_r, D_r be defined by* (4.6) *and* (4.14). *If p is odd, then*

$$\begin{aligned}(h, (1-r)Z_{mpr})^* &= 2p([Qh], \cos \tfrac{1}{2}t \sin (m+\tfrac{1}{2})tD_r^{[p]}) \\ &\quad -2p([Qh], \cos \tfrac{1}{2}t \cos (m+\tfrac{1}{2})tC_r^{[p]})+h(e)+E(r),\end{aligned} \tag{5.15}$$

where $\lim_{r\to 1} E(r)=0$. *If p is even then* (5.15) *holds if we make the additional assumption that* $[Qh] \sin t \in L^1[-\pi, \pi]$.

Proof. Since $h \in L^1(SU(2))$ it follows that $Qh \in L^1(SU(2))$ and hence $[Qh] \sin^2 t \in L^1[-\pi, \pi]$. Thus, since $[Qh]$ is continuous at 0, we have $[Qh]G \in L^1[-\pi, \pi]$ for any analytic function G of period 2π such that $G(\pi)=G'(\pi)=0$. Since $\cos \pi/2= \cos (m+\frac{1}{2})\pi = D_r^{[p]}(\pi)=0$, the two inner products indicated on the right-hand side of (5.15) exist. It follows from the definition (3.9) of D_j that

$$Z_{mpr} = \sum_{j=0}^{\infty} r^j \sum_{k=1}^{m} k\chi_k + \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} r^k \sum_{j=m+1+(n-1)p}^{m+pn} j\chi_j.$$

In this formula write $\chi_k=\sin k\theta/\sin \theta$ and we get

$$\begin{aligned}(1-r) \sin \theta Z_{mpr} &= \operatorname{Im} \left(\sum_{k=1}^{m} ke^{ik\theta}+\sum_{n=1}^{\infty} r^n \sum_{j=(m+1)+(n-1)p}^{m+pn} je^{ij\theta}\right) \\ &= -\frac{d}{d\theta}\left(\operatorname{Re}\left(\frac{(1-r)e^{i(m+1)\theta}}{(e^{i\theta}-1)(1-re^{ip\theta})}+\frac{e^{i\theta}}{1-e^{i\theta}}\right)\right).\end{aligned}$$

Since $(d/d\theta)(\operatorname{Re} (e^{i\theta}/(1-e^{i\theta}))=0$, this becomes

$$\sin \theta Z_{mpr} = -\frac{d}{d\theta}\left[\frac{1}{2 \sin \theta} \operatorname{Im} \frac{e^{im\theta}(1+e^{i\theta})}{(1-re^{ip\theta})}\right]. \tag{5.16}$$

Now define functions H_{mpr} on $SU(2)$ by

$$\operatorname{Im} \frac{e^{im\theta}(1+e^{i\theta})}{(1-re^{ip\theta})} = \frac{1}{1-r^2} A_r(p\theta)H_{mpr}, \tag{5.17}$$

where A_r is defined in (4.3). Then it follows from (5.16) that

$$-2(1-r^2) \sin^2 \theta Z_{mpr} = pH_{mpr}A_r'(p\theta)+A_r(p\theta) \sin \theta\, (d/d\theta) (\csc \theta H_{mpr}). \tag{5.18}$$

Using the definition (5.17) we can verify that

$$H_{mpr} = 2 \cos \tfrac{1}{2}\theta[r \sin p\theta \cos (m+\tfrac{1}{2})\theta+(1-r \cos p\theta) \sin (m+\tfrac{1}{2})\theta]. \tag{5.19}$$

By (5.19), (4.4) and (4.14) we have

$$H_{mpr}A_r'(p\theta) = -2 \cos \tfrac{1}{2}\theta[r^2 \cos (m+\tfrac{1}{2})\theta B_r(p\theta)+(1+r) \sin (m+\tfrac{1}{2})\theta D_r(p\theta)]. \tag{5.20}$$

By applying a few trigonometric identities to (5.19) we obtain

$$(5.21)\qquad H_{mpr} = \sin\theta\left[2\sin\frac{p\theta}{2}\frac{\cos\frac{1}{2}(p-1-2m)\theta}{\sin\frac{1}{2}\theta}-(1-r)\frac{\sin(p-m-\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\right],$$

and hence

$$(5.22)\qquad \sin\theta\,(d/d\theta)\,(\csc\theta H_{mpr}) = 2(1-r)M_{mp}+2pN_{mp}+2\sin\tfrac{1}{2}p\theta R_{mp},$$

where

$$(5.23)\qquad M_{mp} = -\tfrac{1}{2}\sin\theta\,(d/d\theta)\,(\sin(p-m-\tfrac{1}{2})\theta\csc\tfrac{1}{2}\theta),$$

$$(5.24)\qquad N_{mp} = \cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}p\theta\cos\tfrac{1}{2}(p-1-2m)\theta,$$

$$(5.25)\qquad R_{mp} = \sin\theta\,(d/d\theta)\,(\cos\tfrac{1}{2}(p-1-2m)\theta\csc\tfrac{1}{2}\theta).$$

From (5.18), (5.20), (5.22) and (4.6) we get

$$(5.26)\qquad (1-r)Z_{mpr} = \sum_{j=1}^{5}\csc^2\theta F_{rmpj},$$

where

$$(5.27)\qquad F_{rmp1} = (r-1)(r+1)^{-1}M_{mp}A_r(p\theta),$$

$$(5.28)\qquad F_{rmp2} = -(r+1)^{-1}\sin\tfrac{1}{2}p\theta R_{mp}A_r(p\theta),$$

$$(5.29)\qquad F_{rmp3} = -p(r+1)^{-1}(N_{mp}-\cos\tfrac{1}{2}\theta\cos(m+\tfrac{1}{2})\theta)A_r(p\theta),$$

$$(5.30)\qquad F_{rmp4} = p\cos\tfrac{1}{2}\theta\sin(m+\tfrac{1}{2})\theta D_r(p\theta),$$

$$(5.31)\qquad F_{rmp5} = -p\cos\tfrac{1}{2}\theta\cos(m+\tfrac{1}{2})\theta C_r(p\theta).$$

LEMMA 5.32. *For any function h in $L^1(SU(2))$,*

$$\lim_{r\to 1}(h, \csc^2\theta F_{rmp1})^* = 0.$$

Proof. By (5.7) we have

$$(5.33)\qquad (h, \csc^2\theta F_{rmp1})^* = 2(r-1)(r+1)^{-1}([Qh][M_{mp}], A_r^{[p]}).$$

We know that $[Qh]\sin^2 t\in L^1[-\pi,\pi]$, and since $[M_{mp}](0)=[M_{mp}]'(0)=[M_{mp}](\pi)=[M_{mp}]'(\pi)=0$ it follows that $[Qh][M_{mp}]\in L^1[-\pi,\pi]$. By (4.24) and (4.25) we have

$$(5.34)\qquad ([Qh][M_{mp}], A_r^{[p]}) = (M^*, A_r)$$

where M^* is the even periodic function of period 2π defined by

$$M^*(t) = p^{-1}\sum_{j\in J(p)}[Qh]((t+2\pi j)/p)[M_{mp}]((t+2\pi j)/p).$$

Let $M^*\sim a_0+\sum_{n=1}^{\infty}a_n\cos nt$ be the Fourier series for M^*. Then by (4.3) we have $(M^*, A_r)=\sum_{n=0}^{\infty}a_nr^n$. The Riemann-Lebesgue lemma tells us that $\lim_{n\to\infty}a_n=0$ and hence

$$(5.35)\qquad \lim_{r\to 1}(1-r)(M^*, A_r) = \lim_{r\to 1}(1-r)\sum_{n=0}^{\infty}a_nr^n = \lim_{n\to\infty}a_n = 0.$$

The lemma follows from (5.33), (5.34) and (5.35).

LEMMA 5.36. *Let h be a function in $L^1(SU(2))$ which is continuous at the identity, and let F_{rmp2} be defined by* (5.28). *Then for any odd p,*

$$\lim_{r\to 1} (h, \csc^2 \theta F_{rmp2})^* = h(e). \tag{5.37}$$

If p is even then (5.37) *holds if we make the additional assumption that* $[Qh] \sin t \in L^1[-\pi, \pi]$.

Proof. By (5.7) we have

$$(h, \csc^2 \theta F_{rmp2})^* = -2(\sin \tfrac{1}{2} pt [R_{mp}][Qh], A_r^{[p]})(r+1)^{-1}. \tag{5.38}$$

We know that $[Qh] \sin^2 t \in L^1[-\pi, \pi]$, and since $[Qh]$ is continuous at 0 and $\sin \frac{1}{2}pt[R_{mp}]$ is an everywhere analytic function that vanishes at π together with its derivative, it follows that $\sin \frac{1}{2}pt [R_{mp}][Qh] \in L^1[-\pi, \pi]$. By (4.24) and (4.25)

$$(\sin \tfrac{1}{2}pt [R_{mp}][Qh], A_r^{[p]}) = p^{-1} \sum_{j\in J(p)} (-1)^j \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin \tfrac{1}{2}t f_{pmj}(t) A_r(t)\, dt, \tag{5.39}$$

where

$$f_{pmj}(t) = [Qh]((t+2\pi j)/p)[R_{mp}]((t+2\pi j)/p).$$

Let

$$I(r, j, p, m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin \tfrac{1}{2}t f_{pmj}(t) A_r(t)\, dt. \tag{5.40}$$

If the interval $[(2j-1)\pi/p, (2j+1)\pi/p]$ does not contain 0 or $\pm\pi$ then $f_{pmj} \in L^1[-\pi, \pi]$ and hence by Lemma 4.13

$$\lim_{r\to 1} I(r, j, p, m) = 0, \qquad j \in J(p),\ j \neq 0, -\tfrac{1}{2}p, \pm\tfrac{1}{2}(p-1). \tag{5.41}$$

Now

$$I(r, 0, p, m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin \tfrac{1}{2}t [Qh](t/p)[R_{mp}](t/p) A_r(t)\, dt, \tag{5.42}$$

and since $\sin \frac{1}{2}t [Qh](t/p)[R_{mp}](t/p)$ is continuous at 0 if we define its value at 0 to be $-p[Qh](0) = -ph(e)$, it follows from (5.42) and (4.5) that

$$\lim_{r\to 1} I(r, 0, p, m) = -ph(e). \tag{5.43}$$

Now suppose p is odd. Then $\pm\frac{1}{2}(p-1) \in J(p)$ but $-\frac{1}{2}p \notin J(p)$. Since $\sin \frac{1}{2}t f_{pmj} \in L^1[-\pi, \pi]$ for all j, and

$$f_{pm, \pm(p-1)/2} = [Qh]((t \pm \pi(p-1))/p)[R_{mp}]((t \pm \pi(p-1))/p)$$

is locally in L^1 near $t=0$, it follows from Lemma 4.13 that

$$\lim_{r\to 1} I(r, \pm\tfrac{1}{2}(p-1), p, m) = 0. \tag{5.44}$$

Equation (5.37) for odd p follows from (5.38)–(5.41) and (5.43) and (5.44). Now suppose p is even, so that $-\frac{1}{2}p \in J(p)$ but $\pm\frac{1}{2}(p-1) \notin J(p)$. Then

$$I(r, -\tfrac{1}{2}p, p, m) = \frac{1}{2\pi}\int_{\pi}^{\pi} \sin \tfrac{1}{2}t[Qh]\left(\frac{t}{p}-\pi\right)[R_{mp}]\left(\frac{t}{p}-\pi\right)A_r(t)\, dt.$$

The assumption that $[Qh] \sin t \in L^1[-\pi, \pi]$ implies that $[Qh](t/p-\pi)[R_{mp}(t/p-\pi)] \in L^1[-\pi, \pi]$, and hence it follows from Lemma 4.13 that $\lim_{r\to 1} I(r, -\frac{1}{2}p, p, m)=0$. This result combined with (5.38)–(5.41) and (5.43) proves (5.37) for even p.

LEMMA 5.45. *Let h be a function in $L^1(SU(2))$ which is continuous at the identity, and let F_{rmp3} be defined by* (5.29). *Then for any odd p, $\lim_{r\to 1}(h, \csc^2 \theta F_{rmp3})^*=0$. If p is even the same result holds if we make the additional assumption that $[Qh] \sin t \in L^1[-\pi, \pi]$.*

Proof. The proof is almost identical with the proof of the previous lemma, so we omit it.

Completion of the proof of Proposition 5.14. It follows from formula (5.7) and equations (5.30), (5.31) that

$$(5.46) \qquad (h, \csc^2 \theta F_{rmp4})^* = 2p([Qh] \cos \tfrac{1}{2}t \sin (m+\tfrac{1}{2})t, D_r^{[p]})$$

$$(5.47) \qquad (h, \csc^2 \theta F_{rmp5})^* = -2p([Qh] \cos \tfrac{1}{2}t \cos (m+\tfrac{1}{2})t, C_r^{[p]}).$$

The inner products indicated in (5.46) exist for any h in $L^1(SU(2))$ because $D_r^{[p]}(0)=D_r^{[p]}(\pi)=\sin (m+\frac{1}{2})(0)=\cos \pi/2=0$, and the inner products indicated in (5.47) exist for any $h \in L^1(SU(2))$ which is continuous at 0 because $\cos \pi/2 = \cos (m+\frac{1}{2})\pi=0$. Let h be a function in $L^1(SU(2))$ that is continuous at e. Then by (5.26)

$$(h, (1-r)Z_{mpr})^* = \sum_{j=1}^{5} (h, \csc^2 \theta F_{rmpj})^*.$$

Proposition 5.14 follows from this formula and Lemmas 5.32, 5.36, 5.45 and equations (5.46) and (5.47).

LEMMA 5.48. *Let U be a faithful representation of $SU(2)$ of type (q, p), and let f be a function in $L^1(SU(2))$ such that f is continuous at e, and the U-Fourier series for f at e converges to the value L. If p is odd then*

$$(5.49) \qquad \begin{aligned} L = 2p \lim_{r\to 1} [(&\sin \tfrac{1}{2}(p+q+3)t \cos \tfrac{1}{2}(p-q+1)t[Qf], D_r^{[p]}) \\ &-(\cos \tfrac{1}{2}(p+q+3)t \cos \tfrac{1}{2}(p-q+1)t[Qf], C_r^{[p]})]+f(e). \end{aligned}$$

If p is even then (5.49) *holds if we make the additional assumption that $[Qf] \sin t$ is in $L^1[-\pi, \pi]$.*

Proof. The result follows from Lemma 5.8, Proposition 5.14 and the trigonometric identities

$$\begin{aligned}\cos \tfrac{1}{2}t \sum_{m=q+1}^{p+1} (-1)^{m+p+1} \sin (m+\tfrac{1}{2})t &= \sin \tfrac{1}{2}(p+q+3)t \cos \tfrac{1}{2}(p-q+1)t; \\ \cos \tfrac{1}{2}t \sum_{m=q+1}^{p+1} (-1)^{m+p+1} \cos (m+\tfrac{1}{2})t &= \cos \tfrac{1}{2}(p+q+3)t \cos \tfrac{1}{2}(p-q+1)t.\end{aligned} \tag{5.50}$$

LEMMA 5.51. *Let F be an even analytic function of period 2π such that $F(\pi)=F'(\pi)=0$, let f be a function in $L^1(SU(2))$ that is continuous at the identity, and let p be an integer. Suppose that for each integer j satisfying $1 \leqq j \leqq [\tfrac{1}{2}(p-1)]$ either $[Qf]$ is bounded near $(2\pi j/p)$ or $F(2\pi j/p)=0$. Suppose also that $\lim_{r\to 1} (F[Qf], C^{[p]})$ exists. If p is odd then*

$$\lim_{r\to 1} (F[Qf], C^{[p]}) = 0. \tag{5.52}$$

If p is even then (5.52) *holds if we make the additional assumption that* $[Qf] \sin t$ *is in* $L^1[-\pi, \pi]$.

Proof. The assumptions that $F(\pi)=F'(\pi)=0$ and that f is continuous at e imply that $F[Qf]$ is in $L^1[-\pi, \pi]$. Let $\bar{J}(p)=\{j \in \mathbf{Z} : |j| \leqq [\tfrac{1}{2}(p-1)]\}$. Since $F[Qf]$ is an even function of period 2π, and $[Qf]$ is locally in L^1 at each point of the open interval $(-\pi, \pi)$, our assumptions imply that for each integer j in $\bar{J}(p)$ either $F[Qf]$ is bounded near $2\pi j/p$ or else there exists a neighborhood N_j of $2\pi j/p$ such that $[Qf]$ is in L^1 on N_j and $F(2\pi j/p)=0$. If p is odd then $\bar{J}(p)=J(p)$ where $J(p)$ is defined in (4.25), and hence (5.52) holds for odd p by Lemma 4.26. If p is even then $J(p) =\bar{J}(p) \cup \{-[\tfrac{1}{2}p]\}$. If we assume that $[Qf] \sin t$ is in $L^1[-\pi, \pi]$ then since $F(\pi) =F'(\pi)=0$ it follows that $F[Qf]$ can be written as the product of a function which is analytic at $-\pi=2\pi(-[\tfrac{1}{2}p]/p)$ and vanishes at $-\pi$, and a function in $L^1[-\pi, \pi]$. Hence in this case (5.52) again follows from Lemma 4.26.

LEMMA 5.53. *Let F and H be respectively even and odd analytic functions of period 2π, and suppose that $F(\pi)=F'(\pi)=0$. Let f be a function in $L^1(SU(2))$ that is continuous at the identity, let p be an integer and let $J^*(p)=\{n \in \mathbf{Z} : 1 \leqq n \leqq [\tfrac{1}{2}(p-1)]\}$. Suppose that $J^*(p)$ can be written as a disjoint union $J^*(p)=I^* \cup K^* \cup L^*$ where for each j in I^*, $[Qf]$ has left- and right-hand limits at $2\pi j/p$; for each j in K^*, $F(2\pi j/p) =H(2\pi j/p)=0$; and for each j in L^*, $H(2\pi j/p)=0$ and $[Qf]$ is bounded near $2\pi j/p$. Suppose that the limit*

$$\lim_{r\to 1} (2p(H[Qf], D_r^{[p]})-2p(F[Qf], C_r^{[p]})) = L \tag{5.54}$$

exists. If p is odd then

$$L = \frac{2}{\pi} \sum_{j\in I^*} H(2\pi j/p)([Qf]_+(2\pi j/p)-[Qf]_-(2\pi j/p)). \tag{5.55}$$

If p is even then (5.55) *still holds if we make the additional assumption that*

$$[Qf] \in L^1[-\pi, \pi].$$

Proof. The inner product $(H[Qf], D_r^{[p]})$ exists because $H(\pi) = D_r^{[p]}(\pi) = 0$ (we use the fact that any odd function of period 2π vanishes at π), and we saw in the previous lemma that the other inner product in (5.54) exists. Let N be the interval $[-\pi + \frac{1}{2}\pi p^{-1}, \pi - \frac{1}{2}\pi p^{-1}]$, and let M be the complement of N in $[-\pi, \pi]$. Let K_M, K_N be the characteristic functions of M, N respectively, made periodic of period 2π. Let $P = H[Qf]K_M$, $G = H[Qf]K_N$ where H is as in the statement of the lemma. Then $G \in L^1[-\pi, \pi]$ and G vanishes on a neighborhood of $-\pi$. Since $H(\pi) = 0$ it follows that $P \sin t \in L^1[-\pi, \pi]$. Apply Lemma 4.29 to the function G with $I = \{0\} \cup I^* \cup -I^*$ and $K = K^* \cup -K^* \cup L^* \cup -L^* \cup (\{-[\frac{1}{2}p]\} \cap J(p))$, and we obtain

$$\lim_{r \to 1} 2p(G, D_r^{[p]}) = \frac{1}{\pi} \sum_{j \in I} (G_+(2\pi j/p) - G_-(2\pi j/p)) \tag{5.56}$$
$$= \frac{2}{\pi} \sum_{j \in I^*} H(2\pi j/p)([Qf]_+(2\pi j/p) - [Qf]_-(2\pi j/p)).$$

Now

$$2p(P, D_r^{[p]}) = \frac{1}{\pi} \int_{-\pi}^{\pi} P\left(\frac{t}{p} - \pi\right) D_r(t)\, dt \qquad \text{for } p \text{ even,} \tag{5.57}$$

$$2p(P, D_r^{[p]}) = \frac{\varepsilon(p)}{\pi} \int_{-\pi}^{\pi} P\left(\frac{t+\pi}{p} - \pi\right) D_r(t)\, dt \quad \text{for } p \text{ odd,} \tag{5.58}$$

where $\varepsilon(p) = 1$ if $p = 1$ and $\varepsilon(p) = 2$ if $p \geqq 3$. (To derive (5.58) we have used the fact that P and D_r are both odd functions.) It follows from the facts that P vanishes on $[-\pi + \frac{1}{2}\pi p^{-1}, \pi - \frac{1}{2}\pi p^{-1}]$ and $P \sin t \in L^1[-\pi, \pi]$ that $P((t+\pi)/p - \pi)$ vanishes on a neighborhood of $t = 0$, and $P((t+\pi)/p - \pi) \sin t \in L^1[-\pi, \pi]$. Hence we can apply Lemma 4.21 to the function $P((t+\pi)/p - \pi) \csc t$ to show that

$$\lim_{r \to 1} 2p(P, D_r^{[p]}) = 0 \qquad \text{for } p \text{ odd.} \tag{5.59}$$

Suppose now that p is even and $[Qf]$ is in $L^1[-\pi, \pi]$. Then we can write $P(t/p - \pi)$ as the product of a function $[Qf](t/p - \pi)K_M(t/p - \pi)$ in $L^1[-\pi, \pi]$ and an analytic function $H(t/p - \pi)$ that vanishes at 0. Hence by Lemma 4.13

$$\lim_{r \to 1} 2p(P, D_r^{[p]}) = 0 \quad \text{for } p \text{ even if } [Qf] \in L^1. \tag{5.60}$$

Since $P + G = H[Qf]$, it follows from (5.56)–(5.60) that

$$\lim_{r \to 1} 2p(H[Qf], D_r^{[p]}) = \frac{2}{\pi} \sum_{j \in I^*} H(2\pi j/p)([Qf]_+(2\pi j/p) - [Qf]_-(2\pi j/p)). \tag{5.61}$$

This result combined with our assumption (5.54) shows that $\lim_{r\to 1}(F[Qf], C_r^{[p]})$ exists. By Lemma 5.51 we thus have

$$\lim_{r\to 1}(F[Qf], C_r^{[p]}) = 0, \tag{5.62}$$

and Lemma 5.53 follows from (5.61) and (5.62).

THEOREM 5.63. *Let U be a faithful representation of $SU(2)$ of type (q,p), let $x\in SU(2)$, and let f be a function in $L^1(SU(2))$ such that f is continuous at x and the U-Fourier series for f at x converges to the limit L. Let ${}_xf$ be the function on $SU(2)$ defined by ${}_xf(y)=f(xy)$ and let*

$$J(q,p) = \left\{ j\in \boldsymbol{Z} : 1 \leqq j \leqq [\tfrac{1}{2}(p-1)],\ \sin\left(\frac{j\pi(q+3)}{p}\right)\cos\left(\frac{j\pi(q-1)}{p}\right) \neq 0\right\}. \tag{5.64}$$

Suppose that $[Q({}_xf)]$ has left- and right-hand limits at $2\pi j/p$ for each $j\in J(q,p)$, and $[Q({}_xf)]$ is bounded at $2\pi j/p$ for all j such that $1\leqq j\leqq [\frac{1}{2}(p-1)]$ and $\sin(j\pi(q+3)/p)=0$ but $\cos(j\pi(q-1)/p)\neq 0$. Then if p is odd

$$\begin{aligned} L &= f(x) \\ &\quad + \frac{2}{\pi}\sum_{j\in J(q,p)} \sin\left(\frac{(q+3)j\pi}{p}\right)\cos\left(\frac{(q-1)j\pi}{p}\right)\left([Q({}_xf)]_+\left(\frac{2\pi j}{p}\right) - [Q({}_xf)]_-\left(\frac{2\pi j}{p}\right)\right), \end{aligned}$$

and if p is even the same formula holds provided we make the additional assumption that $[Q({}_xf)]$ is in $L^1[-\pi,\pi]$.

Proof. Since f is continuous at x, we see that ${}_xf$ is continuous at e, and since $U_nf(x)=U_n({}_xf)(e)$ it follows that the U-Fourier series for ${}_xf$ at e converges to L. The theorem follows from Lemma 5.48 and Lemma 5.53 with $I^*=J(p,q)$ and $K^*=\{j\in J^*(p) : \cos j\pi(q-1)/p=0\}$.

COROLLARY 5.65. *Let U be any faithful representation of $SU(2)$, and let $C(SU(2))$ be the space of continuous functions on $SU(2)$. Then U-Fourier series are honest for functions in $C(SU(2))$.*

Proof. This follows immediately from Theorem 5.63 since $Q({}_xf)$ is continuous for any $x\in SU(2)$ and any $f\in C(SU(2))$.

COROLLARY 5.66. *Let U be a faithful representation of $SU(2)$. Then U-Fourier series are honest for functions in $L^\infty(SU(2))$ if and only if U is of type* (1, 1), (2, 2) *or* (3, 3).

Proof. We verify immediately from (5.64) that $J(1,1)=J(2,2)=J(3,3)=\varnothing$, so it follows from Theorem 5.63 that U-Fourier series are honest for functions in $L^\infty(SU(2))$ if U is of type (1, 1), (2, 2) or (3, 3). On the other hand let U be of type

(q, p) different from $(1, 1)$, $(2, 2)$ or $(3, 3)$ and for $1 \leqq j \leqq \frac{1}{2}[p-1]$ let f_{jp} be the bounded class function on $SU(2)$ defined by

$$(5.67)\qquad \begin{aligned} f_{jp}(x) &= \theta(x) \csc \theta(x) && \text{if } 0 < \theta(x) \leqq 2\pi j/p \\ &= (\theta(x)-\pi) \csc \theta(x) && \text{if } 2\pi j/p < \theta(x) < \pi \\ &= 1 && \text{if } x = e \\ &= -1 && \text{if } x = -e. \end{aligned}$$

Then f_{jp} is continuous at e. Using (3.6) and (5.4) we see that

$$(5.68)\qquad \begin{aligned} \left(f_{jp} * \sum_{k=1}^{n} k\chi_k\right)(e) &= \sum_{k=1}^{n} k(f_{jp}, \chi_k)^* = -2 \sum_{k=1}^{n} \cos(2\pi jk/p) \\ &= 1 - \csc(\pi j/p) \sin((n+\tfrac{1}{2})(2\pi j/p)). \end{aligned}$$

Using this result in (3.8), and noting the trigonometric identity (5.50) we get

$$(5.69)\qquad \begin{aligned} U_n(f_{jp})(e) &= \sum_{m=q+1}^{p+1} (-1)^{m+p+1}(f_{jp} * D_{(n-1)p+m})(e) \\ &= 1 - \csc(\pi j/p) \sum_{m=q+1}^{p+1} (-1)^{m+p+1} \sin((m+\tfrac{1}{2})(2\pi j/p)) \\ &= 1 - 2\csc(2\pi j/p) \sin((q+3)\pi j/p) \cos((q-1)\pi j/p) \end{aligned}$$

for all $n \geqq 3$. Note that the right-hand side of (5.69) does not depend on n. It follows from (5.69) and our restrictions on (q, p) that the U-Fourier series for f_{1p} converges deceptively at e unless $p = 2(q-1)$. If $p = 2(q-1)$, then our restrictions on (q, p) imply that $p \geqq 6$ so f_{2p} is defined, and (5.69) shows that the U-Fourier series for f_{2p} converges deceptively at e. Corollary 5.66 follows from these remarks.

PROPOSITION 5.70. *Let a, b be integers satisfying $a > b \geqq 0$. Let f be a bounded function in $L^1(SU(2))$ which is continuous at $x \in SU(2)$, and suppose that the limit*

$$L = \lim_{n\to\infty} \sum_{k=1}^{an+b} (f * k\chi_k)(x)$$

exists. If $a \leqq 2$ or if $a = 2b+1$ we can conclude that $L = f(x)$. For any other pair (a, b) with $a > b \geqq 0$ there is a bounded function f such that $L \neq f(x)$.

Proof. $L = \lim_{n\to\infty} ({}_xf, D_{an+b})^* = \lim_{r\to 1} ({}_xf, (1-r)Z_{bar})^*$ where Z_{bar} is defined by (5.9). Suppose $a = 2b+1$. Then $\sin(b+\frac{1}{2})(2\pi j/a) = 0$ for all integers j. By Proposition 5.14 and Lemma 5.53 (with $p = a$, $H = \cos\frac{1}{2}t \sin(b+\frac{1}{2})t$, $F = \cos\frac{1}{2}t \cos(b+\frac{1}{2})t$, and $K^* = I^* = \varnothing$) we get $L = {}_xf(e) = f(x)$. If $a \leqq 2$ we again use Proposition 5.14 and Lemma 5.53 (but now $J^*(p) = \varnothing$) to get $L = f(x)$. Now suppose $a > 2$ and $a \neq 2b+1$.

Then $\sin (b+\frac{1}{2})(2\pi/a)\neq 0$. Let f_{1a} be defined by (5.67). Then for any integer n we have by (5.68)

$$\sum_{k=1}^{an+b} (f_{1a} * k\chi_k)(e) = 1-\csc (\pi/a) \sin (b+\tfrac{1}{2})(2\pi/a)$$

and hence $L\neq f(e)$.

6. **Honesty of U-Fourier series for a certain class of functions.** Let A, B be the functions on $SU(2)$ defined by

$$x = \begin{bmatrix} A(x) & B(x) \\ -\bar{B}(x) & \bar{A}(x) \end{bmatrix}, \qquad x \in SU(2). \tag{6.1}$$

If x and y are elements of $SU(2)$ define

$$d(x, y) = [\mathrm{Tr}\,(x-y)(x-y)^*]^{1/2}. \tag{6.2}$$

Then d is a metric on $SU(2)$ such that the map of $SU(2)$ into C^4 defined by $x \to (A(x), B(x), \bar{A}(x), \bar{B}(x))$ is an isometry for the Euclidean metric on C^4. This metric is easily seen to be left and right translation invariant. We will denote the open ball of radius r about $x \in SU(2)$ for this metric by $B(x, r)$. In the following discussion the Hausdorff dimension of subsets of $SU(2)$ will be with respect to the metric d. (See Chapter VII of [3] for definition and properties of Hausdorff dimension and Hausdorff measure.)

PROPOSITION 6.3. *Let S be the set of all functions in $L^\infty(SU(2))$ whose set of discontinuities has Hausdorff dimension $\leqq 2$, and let T be the set consisting of those functions in S whose set of discontinuities has Hausdorff dimension <2. Let U be a faithful representation of $SU(2)$. Then U-Fourier series are honest for functions in T, but U-Fourier series are honest for functions in S if and only if U is of type* (1, 1), (2, 2) *or* (3, 3).

Proof. Let U be of type (q, p), let $f \in T$, and let x be a point of continuity of f. We will show that $[Q({}_xf)]$ is continuous on $[0, \pi)$, and in particular $[Q({}_xf)]$ is continuous on $(2\pi/p)J(q, p)$ (see (5.64)). It will then follow from Theorem 5.63 that if the U-Fourier series for f at x converges, it must converge to $f(x)$. Note that $[Q({}_xf)]$ is continuous at 0 because ${}_xf$ is continuous at e. Let E be the set of points at which ${}_xf$ is discontinuous. Since T is translation invariant the Hausdorff dimension of E is less than 2, and hence the two-dimensional measure of E is zero. This means that for each positive integer n there is a sequence $\{x_{nk} : 1 \leqq k < \infty\}$ of points in E and a sequence $\{r_{nk} : 1 \leqq k < \infty\}$ of positive numbers such that

$$E \subseteq \bigcup_{k=1}^{\infty} B(x_{nk}, r_{nk}) \quad \text{and} \quad \sum_{k=1}^{\infty} r_{nk}^2 < \frac{1}{n}. \tag{6.4}$$

Let $B_n = \bigcup_{k=1}^{\infty} B(x_{nk}, r_{nk})$. Then the complement B_n' of B_n is a closed set on which ${}_xf$ is continuous, and by the Tietze extension theorem we can find a continuous function f_n on $SU(2)$ such that $\|f_n\|_\infty \leqq \|{}_xf\|_\infty = \|f\|_\infty$ and ${}_xf = f_n$ on B_n'. Since f_n is continuous, $[Q(f_n)]$ is also continuous. For any $t \in [0, \pi]$

$$[Qf_n](t) - [Q({}_xf)](t) = \int_{SU(2)} f_n(yx(t)y^{-1}) - {}_xf(yx(t)y^{-1})\, d\mu(y) \tag{6.5}$$

(see (5.2) and (5.6)). Since $f_n - {}_xf$ vanishes off of B_n we get

$$|[Qf_n](t) - [Q_xf](t)| \leqq 2\|f\|_\infty \mu(E_n) \tag{6.6}$$

where

$$E_n = \{y : yx(t)y^{-1} \in B_n\} \subseteq \bigcup_{k=1}^{\infty} \{y : yx(t)y^{-1} \in B(x_{nk}, r_{nk})\}. \tag{6.7}$$

It follows from (6.7) that

$$\mu(E_n) \leqq \sum_{k=1}^{\infty} \mu\{y : yx(t)y^{-1} \in B(x_{nk}, r_{nk})\}. \tag{6.8}$$

In order to estimate the sum (6.8) we will need to prove a few lemmas.

Lemma 6.9. *Let A and B be the functions on $SU(2)$ defined in* (6.1). *Then there exists a constant K such that $\mu\{x \in SU(2) : |B(x)| < \varepsilon\} < K\varepsilon^2$ for every $\varepsilon > 0$.*

Proof. If $x = (x_1, x_2, x_3, x_4)$ is any point in $\boldsymbol{R}^4$, let $\bar{x}$ be the matrix defined by

$$\bar{x} = \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix}. \tag{6.10}$$

Then $SU(2)$ acts as a group of orthogonal transformations on $\boldsymbol{R}^4$ by left translations, $\bar{x} \to g\bar{x}$ for $g \in SU(2)$. Also we may identify the unit sphere S^3 in $\boldsymbol{R}^4$ with $SU(2)$ by (6.10). Now S^3 can be parametrized by

$$\begin{aligned} &x_1 = \sin\theta_1, && x_2 = \cos\theta_1 \sin\theta_2, && -\frac{\pi}{2} \leqq \theta_1, \theta_2 \leqq \frac{\pi}{2}, \\ &x_3 = \cos\theta_1 \cos\theta_2 \sin\theta_3, && x_4 = \cos\theta_1\cos\theta_2\cos\theta_3, && 0 \leqq \theta_3 < 2\pi. \end{aligned} \tag{6.11}$$

In [1, p. 116] it is shown that there is a unique measure $d\omega$ on S^3 which is invariant under all orthogonal maps of $\boldsymbol{R}^4$ and has total mass $=1$, and this measure is

$$d\omega = \tfrac{1}{2}\pi^{-2} \cos^2\theta_1 \cos\theta_2\, d\theta_1\, d\theta_2\, d\theta_3. \tag{6.12}$$

Thus $d\omega$ is Haar measure on $SU(2)$. Using the identifications (6.10) and (6.11) we have $|B|^2 = x_3^2 + x_4^2 = \cos^2\theta_1 \cos^2\theta_2$, and hence

$$\begin{aligned} \mu\{x : |B(x)| < \varepsilon\} &= \tfrac{1}{2}\pi^{-2} \int_R \cos^2\theta_1 \cos\theta_2\, d\theta_1\, d\theta_2\, d\theta_3 \\ &\leqq 4\varepsilon\pi^{-1} \int_V \cos\theta_1\, d\theta_1\, d\theta_2 \end{aligned} \tag{6.13}$$

where

$$R = \{(\theta_1, \theta_2, \theta_3) : \cos\theta_1 \cos\theta_2 < \varepsilon, -\pi/2 \leqq \theta_1, \theta_2 \leqq \pi/2, 0 \leqq \theta_3 < 2\pi\},$$
$$V = \{(\theta_1, \theta_2) : \cos\theta_1 \cos\theta_2 < \varepsilon, 0 \leqq \theta_1, \theta_2 \leqq \pi/2\}.$$

Now

$$\int_V \cos\theta_1 \, d\theta_1 \, d\theta_2 = \int_0^{\operatorname{arc}\cos\varepsilon} (\operatorname{arc}\sin(\varepsilon\sec\theta_1))\cos\theta_1 \, d\theta_1 + \int_{\operatorname{arc}\cos\varepsilon}^{\pi/2} \cos\theta_1 \, d\theta_1,$$

and since $\operatorname{arc}\sin t \leqq \frac{1}{2}\pi t$ for $0 \leqq t \leqq 1$ we see that $\int_V \cos\theta_1 \, d\theta_1 \, d\theta_2 < 3\varepsilon$. This result combined with (6.13) proves Lemma 6.9.

LEMMA 6.14. *Let $z \in SU(2)$, let $t \in (0, \pi)$ and let ε be a positive number. Let*

$$F(t, z, \varepsilon) = \{y \in SU(2) : yx(t)y^{-1} \in B(z, \varepsilon)\}.$$

Then there exists an absolute constant K such that

$$\mu(F(t, z, \varepsilon)) \leqq K\varepsilon^2 \csc^2 t.$$

Proof. Let $y_0 \in F(t, z, \varepsilon)$ (if $F(t, z, \varepsilon) = \varnothing$ the lemma is clearly true). If y is any point in $F(t, z, \varepsilon)$ then

$$\begin{aligned} d(y_0^{-1}yx(t), x(t)y_0^{-1}y) &= d(yx(t)y^{-1}, y_0x(t)y_0^{-1}) \\ &\leqq d(yx(t)y^{-1}, z) + d(z, y_0x(t)y_0^{-1}) < 2\varepsilon. \end{aligned} \tag{6.15}$$

A straightforward calculation shows that for any $a \in SU(2)$

$$d(ax(t), x(t)a) = 8^{1/2}|B(a)| \sin t. \tag{6.16}$$

If we take $a = y_0^{-1}y$ in (6.16) and use (6.15) we obtain

$$|B(y_0^{-1}y)| < \varepsilon \csc t \cdot 2^{-1/2} < \varepsilon \csc t, \quad y \in F(t, z, \varepsilon). \tag{6.17}$$

Let $H(t, \varepsilon) = \{x \in SU(2) : |B(x)| < \varepsilon \csc t\}$. It follows from (6.17) that $F(t, z, \varepsilon) \subseteq y_0 H(t, \varepsilon)$, and hence by Lemma 6.9

$$\mu(F(t, z, \varepsilon)) \leqq \mu(H(t, \varepsilon)) \leqq K\varepsilon^2 \csc^2 t$$

which proves Lemma 6.14.

Applying Lemma 6.14 to (6.8) we get

$$\mu(E_n) \leqq K \csc^2 t \sum_{k=1}^{\infty} r_{nk}^2 \leqq Kn^{-1} \csc^2 t \tag{6.18}$$

so by (6.6) we have

$$|[Qf_n](t) - [Q_x f](t)| \leqq 2K\|f\|_\infty n^{-1} \csc^2 t.$$

It follows that $[Qf_n]$ converges uniformly to $[Q_x f]$ on any compact subinterval of $(0, \pi)$, and hence the limit function $Q_x f$ is continuous on $(0, \pi)$. This completes the proof that U-Fourier series are honest for functions in T. It follows from Corollary

5.66 that U-Fourier series are honest for functions in S if U is of type (1, 1), (2, 2) or (3, 3). If U is of type (q, p) different from (1, 1), (2, 2) or (3, 3), then we saw in the proof of Corollary 5.66 that some function f_{jp} defined as in (5.67) has a U-Fourier series that converges deceptively at e, so to complete the proof of Proposition 6.3 it will suffice to show that $f_{jp} \in S$. The set of discontinuities of f_{jp} is

$$\Delta(j, p) = \{x \in SU(2) : \theta(x) = 2\pi j/p\}.$$

Use (6.10) to identify $SU(2)$ with the unit sphere in $\boldsymbol{R}^4$. Then by (3.5) we see that $\Delta(j, p)$ is the intersection of the sphere $x_1^2+x_2^2+x_3^2+x_4^2=1$ and the plane $x_1=\cos(2\pi j/p)$. This intersection is a 2-sphere which has Hausdorff dimension 2 for the Euclidean metric on $\boldsymbol{R}^4$. Since $d(\bar{x}, \bar{y})=2^{1/2}|x-y|$ for all $x, y \in S^3 \subseteq \boldsymbol{R}^4$ we conclude that the Hausdorff dimension of $\Delta(j, p)$ is also 2, so $f_{jp} \in S$. This completes the proof of Proposition 6.3.

7. **Deception sets.** Let U be a faithful representation of $SU(2)$ of type (q, p). The *deception set* D_U or $D(q, p)$ of U is the subset of $SU(2)$ given by

$$(7.1)\quad \begin{aligned} D(q, p) &= \{x : e^{ip\theta(x)} = 1\} && \text{if } p \text{ is odd} \\ &= \{-e\} \cup \{x : e^{ip\theta(x)} = 1 \text{ and } e^{i(q-1)\theta(x)} \neq -1\} && \text{if } p \text{ is even.} \end{aligned}$$

Since two elements x, y of $SU(2)$ are conjugate if and only if $\theta(x)=\theta(y)$ we see that $xD_U=D_Ux$ for all $x \in SU(2)$.

THEOREM 7.2. *Let U be a faithful representation of $SU(2)$ with deception set D_U, let $x \in SU(2)$ and let $f \in L^1(SU(2))$. If f is continuous at each point of xD_U then the U-Fourier series for f does not converge deceptively at x. For each point $y \in xD_U$ such that $y \neq x$ there is a function in $L^1(SU(2))$ which is analytic except at y, and whose U-Fourier series converges deceptively at x.*

Proof. Let j be an integer such that $1 \leqq j \leqq [\frac{1}{2}(p-1)]$ and $\cos((q-1)j\pi/p) \neq 0$, and let $K_j=\{x \in SU(2) : \theta(x)=2\pi j/p\}$. It is easy to verify that $K_j \subseteq D_U$. Suppose now that f is continuous at each point of xD_U. Then ${}_xf$ is continuous at each point of D_U so for any $\varepsilon>0$ and any $g \in K_j$ there exists a $\delta(g)>0$ such that

$$(7.3)\qquad d(y, g) < \delta(g) \Rightarrow |{}_xf(y)-{}_xf(g)| < \tfrac{1}{2}\varepsilon.$$

The family of sets $\{B(g, \delta(g)) : g \in K_j\}$ form an open cover for K_j. Let δ be the Lebesgue number for this cover. Let $g \in K_j$ and let h be any element of $SU(2)$ such that $d(g, h)<\delta$. There exists a $g_0 \in K_j$ such that g and h are both contained in $B(g_0, \delta(g_0))$, by the definition of Lebesgue number. Thus

$$(7.4)\qquad |{}_xf(g)-{}_xf(h)| \leqq |{}_xf(g)-{}_xf(g_0)|+|{}_xf(g_0)-{}_xf(h)| < \varepsilon$$

by (7.3). Now let $s, t \in [0, \pi]$. If $|s-t|<\frac{1}{2}\delta$ then for all $y \in SU(2)$

$$(7.5)\qquad d(yx(s)y^{-1}, yx(t)y^{-1}) = d(x(s), x(t)) = 8^{1/2}|\sin\tfrac{1}{2}(s-t)| < \delta.$$

Now $yx(2\pi j/p)y^{-1} \in K_j$ for all $y \in SU(2)$, and hence if $|s-(2\pi j/p)| < \frac{1}{2}\delta$ it follows from (7.5) and (7.4) that

$$|{}_xf(yx(s)y^{-1}) - {}_xf(yx(2\pi j/p)y^{-1})| < \varepsilon. \tag{7.6}$$

Using the definition (5.6) of Q together with (7.6) we get

$$|[Q_xf](s) - [Q_xf](2\pi j/p)| < \varepsilon \quad \text{whenever } |s-(2\pi j/p)| < \delta/2,$$

and we have shown that $[Q_xf]$ is continuous at $2\pi j/p$ for all j such that $1 \leqq j \leqq [\frac{1}{2}(p-1)]$ and $\cos((q-1)\pi j/p) \neq 0$. Thus it follows from Theorem 5.63 that if the U-Fourier series for f converges at x it must converge to $f(x)$. (Note that if p is even, then $-e \in D_U$ so ${}_xf$ is continuous at $-e$, and hence $[Q_xf] \in L^1[-\pi, \pi]$.) Now let

$$f = (1-A)^{-1} \tag{7.7}$$

where A is defined by (6.1). It is shown in [5, p. 670] that $f \in L^1(SU(2))$, and an argument almost identical with an argument given there shows that the sequence $\{f_n\}$ defined by

$$f_n = \sum_{j=0}^{n} A^j \tag{7.8}$$

converges to f in $L^1(SU(2))$. In [9, p. 164] it is shown that A^{j-1} is a coordinate function of the j-dimensional irreducible representation of $SU(2)$. Hence $n\chi_n * A^{j-1} = \delta_{nj}A^{j-1}$, and it follows that $n\chi_n * f_j = A^{n-1}$ for all $j \geqq n-1$. Thus $n\chi_n * f = \lim_{j\to\infty} n\chi_n * f_j = A^{n-1}$ and

$$D_n * f = \sum_{j=1}^{n} j\chi_j * f = f_{n-1}. \tag{7.9}$$

If g is any element of $SU(2)$ we can write

$$g = u^{-1}x(\theta(g))u \tag{7.10}$$

for some $u \in SU(2)$. Equation (7.10) does not determine u uniquely, but we will choose some u and keep it fixed in this discussion. If $g \in SU(2)$ we define a function g^* in $L^1(SU(2))$ by

$$g^*(y) = f(uyu^{-1}x(-\theta(g))) \tag{7.11}$$

where f is defined by (7.7). g^* is analytic except when $uyu^{-1}x(-\theta(g)) = e$, i.e. except when $y = g$. Also

$$g^*(e) = f(x(-\theta(g))) = (1-\exp(-i\theta(g)))^{-1}, \tag{7.12}$$

so that $g^*(e)$ is finite for any $g \neq e$, and $g^*(e) \neq 0$ for all $g \in SU(2)$. From now on we assume that $g \neq e$. For any $y \in SU(2)$, $D_n * g^*(y) = f_{n-1}(uyu^{-1}x(-\theta(g)))$, and in particular, from (7.8) and (7.12) we have

$$D_n * g^*(e) = f_{n-1}(x(-\theta(g))) = g^*(e)[1-\exp(-in\theta(g))].$$

Using this result together with (2.1) and (3.8) we obtain

$$U_n g^*(e) = g^*(e) - E_{qp}(g) \exp(-i(n-1)p\theta(g)) \tag{7.13}$$

where

$$\begin{aligned} E_{qp}(g) &= g^*(e)(-1)^{q+1}(p-q+1) && \text{if } \theta(g) = \pi \\ &= g^*(e)\frac{e^{-i(q+1)\theta(g)}(1+e^{-i(p-q+1)\theta(g)})}{1+e^{-i\theta(g)}} && \text{if } \theta(g) \neq \pi. \end{aligned} \tag{7.14}$$

Suppose now that the eigenvalues of g are pth roots of unity, so that $\theta(g)=2\pi j/p$ for some integer j. Then $\exp(-i(n-1)p\theta(g))=1$ for all n, and it follows from (7.13) that

$$\lim_{n\to\infty} U_n g^*(e) = g^*(e) - E_{qp}(g). \tag{7.15}$$

If $\theta(g)=\pi$ (i.e. if $g=-e$) then it follows from (7.14) that $E_{qp}(g)\neq 0$, and (7.15) shows that the U-Fourier series for g^* converges deceptively at e. If $\theta(g)\neq\pi$ then (7.14) shows that $E_{qp}(g)\neq 0$ if and only if $\exp(-i(p-q+1)\theta(g))\neq -1$. It follows that the U-Fourier series for g^* converges deceptively at e whenever g is in the deception set D_U (and $g\neq e$). If f is any function on $SU(2)$, and $x\in SU(2)$ let f_x be the function on $SU(2)$ defined by $f_x(g)=f(x^{-1}g)$ for all $g\in SU(2)$. Let $x\in SU(2)$ and let $y=xz$ be a point in xD_U such that $y\neq x$. Then $(z^*)_x$ is a function in $L^1(SU(2))$ that is analytic except at y, and $(z^*)_x(x)=z^*(e)$, but the U-Fourier series for $(z^*)_x$ at x converges to $z^*(e)-E_{qp}(z)\neq z^*(e)$. This completes the proof of Theorem 7.2.

Corollary 7.16. *Let U and V be two faithful representations of $SU(2)$ which are series equivalent. Then U and V have the same deception sets.*

Proof. Suppose $D_U\neq D_V$, and let g be an element of $SU(2)$ which is in exactly one of the sets D_U, D_V. Say $g\in D_U$, $g\notin D_V$. If g^* is constructed as in (7.11) then the U-Fourier series for g^* converges deceptively at e, but since g^* is analytic on D_V the V-Fourier series for g^* does not converge deceptively at e. Thus U and V are not series equivalent.

Corollary 7.17. *Let U be a faithful representation of $SU(2)$. Then U-Fourier series are honest for functions in $L^1(SU(2))$ if and only if U is of type* (1, 1).

Proof. Theorem 7.2 shows that U-Fourier series are honest for functions in $L^1(SU(2))$ if and only if $D_U=\{e\}$. It follows from the definition of D_U that $-e\in D_U$ if p is even, D_U contains all elements of $SU(2)$ whose eigenvalues are pth roots of unity if p is odd, and $D_U=\{e\}$ if and only if U is of type (1, 1).

Proposition 7.18. *Two faithful representations of $SU(2)$ are of the same type if and only if they are series equivalent.*

Proof. By Corollary 3.19 we know that representations of the same type are series equivalent. Let U be a faithful representation of type (q, p) and let V be a faithful representation of type (r, s) and suppose that $(q, p)\neq(r, s)$. First consider the case where $p=s$. Then $q\neq r$, so $p>2$. Let $g=x(2\pi/p)$, so $\theta(g)=2\pi/p\neq\pi$. It follows from (7.14) and (7.15) that $\lim_{n\to\infty} U_n g^*(e)$ and $\lim_{n\to\infty} V_n g^*(e)$ both exist, and

$$
\begin{aligned}
(7.19)\quad & \lim_{n\to\infty} U_n g^*(e) - \lim_{n\to\infty} V_n g^*(e) \\
&\quad = g^*(e)(\exp(2\pi i/p)+1)^{-1}(\exp(-2\pi ir/p)-\exp(-2\pi iq/p)).
\end{aligned}
$$

Since $1\leqq r, q\leqq p$ and $q\neq r$ the right side of (7.19) is not zero, and hence U and V are not series equivalent if $p=s$. Now suppose that $p\neq s$. Say, for example $s<p$ so $p\geqq 2$. Let $g=x(2\pi/p)$. Then from definition (7.1) we see that $g\notin D(r,s)$, and $g\in D(q,p)$ unless p is even and $q-1=\frac{1}{2}p$. Thus by Corollary 7.16, U and V are not series equivalent if $q-1\neq\frac{1}{2}p$. Suppose now that $q-1=\frac{1}{2}p$. If s is odd we have $-e\in D(q,p)$ and $-e\notin D(r,s)$ and by Corollary 7.16 U and V are not series equivalent. Suppose therefore, that s is even. Since $p>s$, we have $p\geqq 4$, and since $p=2(q-1)$ where q is even we have actually $p\geqq 6$. Let $h=x(4\pi/p)\neq -e$. Then $h\in D(q,p)$ but $h\notin D(r,s)$ unless $s=\frac{1}{2}p$. It follows that U and V are not series equivalent unless possibly $(q-1)=\frac{1}{2}p=s$. If $q-1=\frac{1}{2}p=s$ then it follows from (7.14) and (7.15) that $\lim_{n\to\infty} U_n h^*(e)$ and $\lim_{n\to\infty} V_n h^*(e)$ both exist and

$$
\begin{aligned}
(7.20)\quad & \lim_{n\to\infty} U_n h^*(e) - \lim_{n\to\infty} V_n h^*(e) \\
&\quad = h^*(e)\exp(-8\pi i/p)(1+\exp(-4\pi i/p))^{-1}(\exp(-4(r-1)\pi i/p)-1).
\end{aligned}
$$

Since our hypotheses imply that $0<2(r-1)/p<1$ the right side of equation (7.20) is not zero, and U and V are again not series equivalent. Proposition 7.18 now follows.

8. **Deceptive convergence for L^2 functions.** If f is a function in $L^1(SU(2))$ then the *Riemann Lebesgue set of f* is defined to be

$$
r(f) = \left\{x\in SU(2) : \lim_{n\to\infty} n\chi_n * f(x) = 0\right\}.
$$

It follows from [6, Lemma 3] that $\mu(r(f))=1$ for any $f\in L^2(SU(2))$.

THEOREM 8.1. *Let U be a faithful representation of $SU(2)$, and let f be a function in $L^1(SU(2))$. Then the U-Fourier series for f does not converge deceptively at any point of the Riemann Lebesgue set of f. In particular, if $f\in L^2(SU(2))$ then the set of points where the U-Fourier series for f converges deceptively has measure zero.*

Proof. Let x be a point of continuity of f which is contained in $r(f)$, and suppose that the U-Fourier series for f at x converges to L. Let U be of type (q,p) and let V be a representation of $SU(2)$ of type $(1,1)$. For any positive integer n let m be the largest integer such that $(m-1)p+q+1\leqq n$. Then $m\to\infty$ as $n\to\infty$. By (3.18) and the relationship $D_r-D_{r-1}=r\chi_r$ we have

$$
\begin{aligned}
(8.2)\quad V_n f(x) - U_m f(x) &= \sum_{j=(m-1)p+q+2}^{n+1} (j\chi_j * f)(x) \\
&\quad - \sum_{j=1}^{(p-q)/2} ((m-1)p+q+2j+1)\chi_{(m-1)p+q+2j+1} * f(x).
\end{aligned}
$$

By the definition of m we have $0 \leqq (n+1)-((m-1)p+q+2) < p$. Since

$$\lim_{j\to\infty} (j\chi_j * f)(x) = 0$$

it follows from (8.2) that $\lim_{n\to\infty} V_n f(x) - U_m f(x) = 0$, and hence

$$\lim_{n\to\infty} V_n f(x) = \lim_{m\to\infty} U_m f(x) = L.$$

By Corollary 7.17 we have $L = f(x)$, and the U-Fourier series for f does not converge deceptively at x.

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