THE POISSON KERNELS AND THE CAUCHY PROBLEM FOR ELLIPTIC EQUATIONS WITH ANALYTIC COEFFICIENTS

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1. **Introduction.** It is well known that the Cauchy problem for elliptic equations with C^∞-data cannot always be solved, even if their coefficients are analytic and the problem is local. In this paper we shall discuss relations between Cauchy data with which the Cauchy problem has at least a local solution. Obviously one of the sufficient conditions is that Cauchy data are all analytic (the Cauchy-Kowalewski theorem). In [2] J. Hadamard emphasizes that in order that the Cauchy problem for the laplacian Δ have a solution, it is necessary for the Cauchy data to satisfy some relation, and, in fact, states such a relation for Δ. Our result is an extension of that for Δ to general linear elliptic operators.

Let Ω be a bounded domain containing in the half space R_+^{n+1} of the $(n+1)$-dimensional euclidian space R^{n+1} ($n \geqq 1$) with coordinates $x_1, \ldots, x_n$, t defined by the relation $t>0$. We assume that the boundary of Ω, $\partial\Omega$, is of class C^∞ (for the definition see §2) and contains a domain ω of the hyperplane $t=0$. By α we denote multi-indices $(\alpha_1, \ldots, \alpha_n)$ of nonnegative integers. Their sum is denoted by $|\alpha|$. With $D_j = \partial/\partial x_j$ and real numbers ξ_j, we set $D_x = (D_1, \ldots, D_n)$, $D_x^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

Consider a linear partial differential operator of even order

$$L(x, t; D_x, D_t) = \sum_{|\alpha|+k \leqq 2m} a_{\alpha k}(x, t) D_x^\alpha D_t^k,$$

where $D_t = \partial/\partial t$ and $a_{\alpha k}(x, t)$ are complex-valued analytic functions in a domain containing the closure of Ω, $\bar{\Omega}$. We suppose further that L is *properly elliptic* in $\bar{\Omega}$, i.e., if $l(x, t; D_x, D_t)$ is the part of order $2m$ of $L(x, t; D_x, D_t)$, then for every real $(n+1)$-vector $(\xi_1, \ldots, \xi_n, \tau) \neq 0$ and for every point (x, t) in $\bar{\Omega}$, we have $l(x, t; \xi, \tau) \neq 0$, and for every $\xi \neq 0$ and for every $(x, t) \in \bar{\Omega}$ the polynomial of τ, $l(x, t; \xi, \tau)$, has exactly m roots $\tau_1(x, t; \xi), \ldots, \tau_m(x, t; \xi)$ with positive imaginary parts. It is clear that every elliptic operator is properly elliptic if $n \geqq 2$.

Consider next the set of linear partial differential operators with coefficients in $C^\infty(\partial\Omega)$ $(\partial\Omega = \bar{\Omega} - \Omega)$

$$B_j(x, t; D_x, D_t) = \sum_{|\alpha|+k \leqq \mu_j} b_{j,\alpha k}(x, t) D_x^\alpha D_t^k, \qquad j = 1, \ldots, 2m,$$

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where μ_j are exact order of B_j and $\mu_j < 2m$. We assume that $\{B_j\}_{j=1}^{2m}$ is *a Dirichlet set*, i.e., (i) $\mu_j \neq \mu_k$ for $j \neq k$ and (ii) if $b_j(x, t; \xi, \tau)$ is the part of order μ_j of B_j, then we have $b_j(x, t; \boldsymbol{n}) \neq 0$ when $\boldsymbol{n} \neq 0$ is normal to $\partial\Omega$ at (x, t), and that the set $B_1, \ldots, B_m$ *covers* L, i.e., for every fixed real $\xi \neq 0$ and $(x, t) \in \partial\Omega$ the polynomials $b_j(x, t; \xi, \tau)$ in $\tau (j=1, \ldots, m)$ are linearly independent mod $\prod_{j=1}^{m} (\tau - \tau_j(x, t; \xi))$.

In §2 we shall prove the existence of the Poisson kernels for the system $\{L, B_1, \ldots, B_m, \Omega, \omega\}$ (see Theorem 2), using Theorem 1 which is nothing but a simple application of the works of Schechter [7] and [8]. For homogeneous L, B_j with constant coefficients and for $\Omega = R_+^{n+1}$, the Poisson kernels have been already constructed explicitly by Agmon-Douglis-Nirenberg in [1]. In the last section we consider the Cauchy problem

$$(1) \qquad \begin{aligned} Lu &= 0 \qquad \text{in } t > 0, \\ \left(\frac{\partial}{\partial t}\right)^k u &= \phi_k(x) \quad \text{on } \omega, \qquad k = 1, \ldots, 2m, \end{aligned}$$

and characterize relations between the Cauchy data $\phi_0, \ldots, \phi_{2m-1}$ with which the problem (1) is locally solvable in the direction $t > 0$, making use of the Poisson kernels and of the regularity theory stated in the Appendix, where we give a theorem concerning the analytic regularity at the boundary which is a slight improvement of the results of Morrey-Nirenberg [6] (cf. Matsuzawa [4], [5]).

2. **Poisson kernels.** Throughout this section we suppose that Ω, ω, L and $\{B_j\}_{j=1}^{2m}$ have the same meaning as in the introduction. Let L^* be the formal adjoint of L:

$$L^* v = \sum_{|\alpha| + k \leqq 2m} (-1)^{|\alpha| + k} D_x^\alpha D_t^k (\bar{a}_{\alpha k} v),$$

where $\bar{a}_{\alpha k}$ is the complex conjugate of $a_{\alpha k}$. Then we can find another Dirichlet set $\{B'_j\}_{j=1}^{2m}$ such that the equality

$$(2) \qquad (Lu, v) - (u, L^* v) = \sum_{j=1}^{2m} \int_{\partial\Omega} B_j u (B'_{2m-j+1} v)^- \, dS, \qquad u, v \in C^\infty(\bar{\Omega})$$

holds and $\{B'_j\}_{j=1}^{m}$ covers L^*, where dS is the element of surface and (u, v) is the usual inner product in $L^2(\Omega)$ (for the existence of such $\{B'_j\}_{j=1}^{2m}$ see [7]).

For nonnegative integer s, we define the norm $\|\cdot\|_s$ by

$$\|u\|_s = \left(\sum_{|\alpha| + k \leqq s} \iint_{\Omega} |D_x^\alpha D_t^k u|^2 \, dx \, dt \right)^{1/2}.$$

By $H^s(\Omega)$ we mean the Hilbert space resulting from the completion of the set $C^\infty(\bar{\Omega})$ by the norm $\|\cdot\|_s$. By the fact that $\partial\Omega$ is of class C^∞ we mean the existence of a finite open covering in R^{n+1} of $\partial\Omega$, $\{U_k\}$, such that each $\bar{\Omega} \cap U_k$ can be mapped in a one-to-one way onto a hemisphere $V_R: y_1^2 + \cdots + y_n^2 + \tau^2 < R^2$, $\tau \geqq 0$, by a mapping T_k which together with its inverse is infinitely differentiable and transforms

$\partial\Omega \cap U_k$ onto $\sigma_R : y_1^2 + \cdots + y_n^2 < R^2$. Let $\{\alpha_k(x, t)\}$ be a set of functions in $C_0^\infty(R^{n+1})$ such that supp $[\alpha_k] \subset U_k$ and $\sum_k \alpha_k = 1$ on $\partial\Omega$. For $g \in C^\infty(\partial\Omega)$, we set

$$\begin{aligned} g_k(y) &= \alpha_k(T_k^{-1}(y, 0))g(T_k^{-1}(y, 0)) \quad &&\text{on } \sigma_R, \\ &= 0 &&\text{on } R^n - \sigma_R. \end{aligned}$$

For a nonnegative integer s, we define

$$\langle g, h\rangle_s = \sum_k \int |\xi|^{2s-1} \hat{g}_k(\xi)(\hat{h}_k(\xi))^- \, d\xi, \qquad g, h \in C^\infty(\partial\Omega), \tag{3}$$

$$\langle g\rangle_s^2 = \langle g, g\rangle_s, \qquad g \in C^\infty(\partial\Omega), \tag{4}$$

where $\hat{g}_k$ is the Fourier transform of g_k:

$$\hat{g}_k(\xi) = \int e^{-iy\xi} g_k(y)\, dy \qquad (y\xi = y_1\xi_1 + \cdots + y_n\xi_n).$$

It follows that (3) and (4) satisfy all the properties of an inner product and norm, respectively, and that there exists a constant C_s such that, for all $v \in C^\infty(\partial\Omega)$ satisfying $v = g$ on $\partial\Omega$,

$$\langle g\rangle_s \leqq C_s \|v\|_s. \tag{5}$$

Now we can state

THEOREM 1. *Let l be an integer satisfying $l > (n+1)/2$. There then exists a linear mapping M on $H^l(\Omega)$ into $H^{l+2m}(\Omega)$ such that for every $f \in H^l(\Omega)$*

$$\begin{aligned} L(Mf) &= f \quad \text{in } \Omega, \\ B_k(Mf) &= 0 \quad \text{on } \omega, \qquad k = 1, \ldots, m, \end{aligned} \tag{6}$$

$$\|Mf\|_{l+2m} \leqq C_l' \|f\|_l, \tag{7}$$

where C_l' is a constant not depending on f and it should be noted that $H^{l+2m}(\Omega) \subset C^{2m}(\bar{\Omega})$ (cf. Sobolev's lemma).

Proof. We employ the same reasoning as in the proof (of sufficiency) of Theorem 5.2 in [8]. Let ζ be in $C^\infty(\partial\Omega)$ such that $\zeta = 0$ on ω and $\zeta \neq 0$ on $\partial\Omega - \omega$. It then follows from (5) and the coercive inequality for L^*, $\{B_j'\}_{j=1}^m$ that the estimate

$$c^{-1}\|v\|_{2m}^2 \leqq \|L^*v\|_0^2 + \sum_{j=1}^m \langle B_j'v\rangle_{2m-\mu_j'}^2 + \sum_{j=m+1}^{2m} \langle \zeta B_j'v\rangle_{2m-\mu_j'}^2 + \|v\|_0^2 \leqq c\|v\|_{2m}^2 \tag{8}$$

holds for all $v \in C^\infty(\bar{\Omega})$ and so for all $v \in H^{2m}(\Omega)$, c being a positive constant not depending on v and μ_j' the order of B_j'. The ellipticity of L^* and the analyticity of its coefficients guarantee that v vanishes identically in Ω if it satisfies

$$\begin{aligned} L^*v &= 0 \quad \text{in } \Omega, \\ B_j'v &= 0 \quad \text{on } \omega, \text{ for } j = 1, \ldots, m, \\ &= 0 \quad \text{on } \partial\Omega - \omega, \text{ for } j = 1, \ldots, 2m, \end{aligned}$$

noting that the set $\{B_j'\}_{j=1}^{2m}$ is a Dirichlet set. Therefore (8) holds without the term $\|v\|_0^2$.

Application of Lax-Milgram's theorem gives us that for every $f \in H^l(\Omega)$ we can find a $w \in H^{2m}(\Omega)$ such that

$$
(9) \qquad \begin{aligned} [w, v] &= (L^*w, L^*v) + \sum_{j=1}^{m} \langle B_j'w, B_j'v\rangle_{2m-\mu_j'} + \sum_{j=m+1}^{2m} \langle \zeta B_j'w, \zeta B_j'v\rangle_{2m-\mu_j'} \\ &= (f, v), \qquad v \in H^{2m}(\Omega). \end{aligned}
$$

Moreover the regularity theory shows us $w \in H^{l+4m}(\Omega)$ (see [7]). It will then be clear that such a w is uniquely and linearly determined for every $f \in H^l(\Omega)$. Define M by $Mf = L^*w$. Then we see easily with the aid of (2) and (9) that M is a linear mapping on $H^l(\Omega)$ into $H^{l+2m}(\Omega)$ and satisfies (6) if $l > (n+1)/2$.

Next we must prove the continuity of M. Taking $v = w$ in (9) we can deduce, from (8) without the term $\|v\|_0^2$, the inequality

$$
c^{-1}\|w\|_{2m}^2 \leqq (f, w) \leqq c\|w\|_{2m}^2.
$$

Thus we have, with some constant $c' > 0$,

$$
\|Mf\|_0 \leqq c'\|f\|_0, \qquad f \in H^l(\Omega).
$$

From this we obtain immediately that M is closed, i.e., if a sequence f_j converges to f in $H^l(\Omega)$ and Mf_j does to u in $H^{l+2m}(\Omega)$, then $u = Mf$. By the closed graph theorem we can assert that M is a continuous mapping on $H^l(\Omega)$ into $H^{l+2m}(\Omega)$. Thus (7) is obtained. Q.E.D.

Finally we state a theorem on existence of the Poisson kernels:

THEOREM 2. *Let q be an integer with the same parity as n satisfying $q > 4m + (n+1)/2$. There then exist functions $W_{j,q}(x, t; y)$ in $C^{2m}(\bar{\Omega}_{x,t})$ $(j = 1, \ldots, m)$ for every fixed y in R^n such that*

(i) *$D_x^\alpha D_t^k W_{j,q}(x, t; y)$ are continuous in $(x, t; y) \in \bar{\Omega}_{x,t} \times R_y^n$ for every α, k satisfying $|\alpha| + k \leqq 2m$,*

(ii) *if we regard $W_{j,q}(x, t; \cdot)$ as a distribution on R^n for fixed (x, t), then kernels*

$$
(10) \qquad K_j(x, t; y) = \Delta_y^{(n+q)/2}\, W_{j,q}(x, t; y) \qquad (j = 1, \ldots, m)
$$

$(\Delta_y = \sum_{j=1}^{m} (\partial/\partial y_j)^2)$ are the Poisson kernels for the system $\{L, B_1, \ldots, B_m, \Omega, \omega\}$, i.e., for any $\phi_j \in C_0^\infty(R^n)$ the function defined by

$$
u(x, t) = \sum_{j=1}^{m} \int K_j(x, t; y)\phi_j(y)\, dy
$$

satisfies

$$
\begin{aligned} Lu &= 0 \quad \textit{in } \Omega, \\ B_k u &= \phi_k \quad \textit{on } \omega, \qquad k = 1, \ldots, m. \end{aligned}
$$

Proof. From (7) and Sobolev's lemma we can derive that, if $l>(n+1)/2$, $H^{2m+l}(\Omega)\subset C^{2m}(\bar{\Omega})$ and

$$|Mf|_{2m,\Omega} \leqq C_l''\|f\|_l, \qquad f\in H^l(\Omega), \tag{11}$$

where C_l'' is a constant and

$$|u|_{s,\Omega} = \sum_{|\alpha|+k\leqq s} \sup_{(x,t)\in\Omega} |D_x^\alpha D_t^k u(x,t)|.$$

Let q be as in Theorem 2 and $E_q(x)$ be a fundamental solution of $\Delta_x^{(n+q)/2}$ belonging to $C^{q-1}(R^n)$ (as such one we can take, following Agmon-Douglis-Nirenberg [1],

$$-\frac{1}{(2\pi i)^n q!}\int_{|\xi|=1} (x\xi)^q \log\frac{x\xi}{i}\, d\omega_\xi,$$

where the principal branch of the logarithm in complex plane slit along the negative real axis is taken, $d\omega_\xi$ is the area element on the unit sphere $|\xi|=1$ and $x\xi=x_1\xi_1+\cdots+x_n\xi_n$). Since the set $\{B_j\}_{j=1}^{2m}$ is a Dirichlet set, we can then find m functions $V_{j,q}(x,t;y)$ $(j=1,\ldots,m)$ such that

(a) $D_x^\alpha D_t^k V_{j,q}(x,t;y)$ are continuous in $(x,t;y)\in\bar{\Omega}_{x,t}\times R_y^n$ for all α, k satisfying $|\alpha|+k\leqq q-2m$,

(b) if we denote by δ_{jk} the Kronecker delta, then

$$B_k V_{j,q}(x,t;y) = \delta_{jk}E_q(x-y) \quad \text{on } \omega, \qquad k=1,\ldots,m.$$

Put

$$f_{j,q}^{(y)}(x,t) = -L(x,t;D_x,D_t)V_{j,q}(x,t;y).$$

This and all its derivatives up to order $N=[(n+1)/2]+1$ are continuous in $\bar{\Omega}_{x,t}\times R_y^n$. Applying Theorem 1 with $l=N$ to $f(x,t)=f_{j,q}^{(y)}(x,t)$ and putting

$$H_{j,q}(x,t;y) = M(f_{j,q}^{(y)}),$$

we obtain from (11)

$$|H_{j,q}(\cdot,\cdot;y)|_{2m,\Omega} \leqq C_l''\|f_{j,q}^{(y)}\|_N.$$

From this and (a) we can immediately conclude (i) of Theorem 2 if we put

$$W_{j,q}(x,t;y) = V_{j,q}(x,t;y)+H_{j,q}(x,t;y).$$

Moreover we then have, with (b),

$$\begin{aligned} L_{x,t}W_{j,q}(x,t;y) &= 0 && \text{in } \Omega,\\ B_k W_{j,q}(x,0;y) &= \delta_{jk}E_q(x-y) && \text{on } \omega. \end{aligned} \tag{12}$$

Let $\phi_j\in C_0^\infty(R^n)$ $(j=1,\ldots,m)$. It then follows from (i) of Theorem 2 and (12) that, for each j,

$$\int K_j(x,t;y)\phi_j(y)\,dy = \int W_{j,q}(x,t;y)\Delta^{(n+q)/2}\phi_j(y)\,dy$$

satisfies

$$Lu = 0 \quad \text{in } \Omega$$
$$B_k u = \delta_{jk}\phi_j \quad \text{on } \omega, \qquad k = 1, \ldots, m.$$

This gives the proof of (ii). Q.E.D.

3. **Cauchy problem.** Let L, Ω and ω be the same as those in §1 and let us take $(\partial/\partial n)^{j-1}$ $(j=1,\ldots,2m)$ as boundary operators B_j in §1, where $\partial/\partial n$ is the inner normal derivative to $\partial\Omega$. It is clear that the set $\{(\partial/\partial n)^{j-1}\}_{j=1}^{2m}$ is a Dirichlet set. A sequence of integers $\nu=(\nu_1,\ldots,\nu_m)$ is said to be *admissible* if $0\leqq\nu_1<\cdots<\nu_m\leqq 2m-1$ and if the set $(\partial/\partial n)^{\nu_1},\ldots,(\partial/\partial n)^{\nu_m}$ covers all properly elliptic operators (cf. the notion "completely elliptic" in Hörmander [3]). If ν is admissible, we can see from §2 that the Poisson kernels of the form (10):

$$K_j^{(\nu)}(x, t; y) = \Delta_y^{(n+q)/2} W_{j,q}^{(\nu)}(x, t; y)$$

for the system $\{L, (\partial/\partial n)^{\nu_1},\ldots,(\partial/\partial n)^{\nu_m}, \Omega, \omega\}$ always exist.

THEOREM 3. *Let $\nu=(\nu_1,\ldots,\nu_m)$ be an admissible sequence. With the Cauchy data $\phi_0,\ldots,\phi_{2m-1}$ in $C_0^\infty(R^n)$ the Cauchy problem* (1) *is locally solvable in the direction $t>0$, if and only if m functions*

$$\phi_{\nu_k'}(x)-\sum_{j=1}^m \int \left(\frac{\partial}{\partial t}\right)^{\nu_k'} W_{j,q}^{(\nu)}(x, 0; y)\Delta^{(n+q)/2}\phi_{\nu_j}(y)\,dy \qquad (k = 1,\ldots,m)$$

are all analytic in ω. Here $\nu_1',\ldots,\nu_m'$ is a sequence such that $0\leqq\nu_1'<\cdots<\nu_m'\leqq 2m-1$ and $\nu_k'\neq\nu_j$ for all k, j.

Proof. Let u be a solution of (1) with Cauchy data $\phi_0,\ldots,\phi_{2m-1}$ in $C_0^\infty(R^n)$. Then

$$v = u-\sum_{j=1}^m \int K_j^{(\nu)}(x, t; y)\phi_{\nu_j}(y)\,dy$$

satisfies

$$Lv = 0 \quad \text{in } t > 0,$$
$$\left(\frac{\partial}{\partial t}\right)^{\nu_k} v = 0 \quad \text{on } \omega, \qquad k = 1,\ldots,m.$$

Thus we can conclude with the aid of the Appendix that $\psi_k(x)=(\partial/\partial t)^{\nu_k'}v(x,0)$, that is,

$$\text{(13)} \quad \psi_k(x) = \phi_{\nu_k'}(x)-\sum_{j=1}^m \int \left(\frac{\partial}{\partial t}\right)^{\nu_k'} W_{j,q}^{(\nu)}(x, 0; y)\Delta^{(n+q)/2}\phi_{\nu_j}(y)\,dy \qquad (k = 1,\ldots,m)$$

are all analytic in ω. Conversely, let $\phi_0,\ldots,\phi_{2m-1}$ be arbitrarily given in $C_0^\infty(R^n)$. If m functions $\psi_k(x)$ $(k=1,\ldots,m)$ defined by (13) are all analytic in ω, we can then find a solution of (1). Indeed, let v be a solution of the Cauchy problem

$$Lv = 0 \quad \text{in } t > 0,$$
$$(\partial/\partial t)^{\nu_j} v = 0 \quad \text{on } \omega, \qquad j = 1,\ldots,m,$$
$$(\partial/\partial t)^{\nu_k'} v = \psi_k \quad \text{on } \omega, \qquad k = 1,\ldots,m$$

(we can solve this by the Cauchy-Kowalewski theorem); then we can conclude from (13) that

$$u = v+\sum_{j=1}^{m} \int W_{j,q}^{(\nu)}(x, t; y)\Delta^{(n+q)/2}\phi_{\nu_j}(y)\, dy$$

is a solution of (1). The proof is thus complete.

APPENDIX. Let $L(x, t; D_x, D_t)$ be a properly elliptic partial differential operator of order $2m$ with analytic coefficients defined in a neighborhood of a hemisphere $\Sigma_\rho : x_1^2+\cdots+x_n^2+t^2<\rho^2$, $t>0$, and $B_1(x; D_x, D_t), \ldots, B_m(x; D_x, D_t)$ be linear partial differential operators of order $\mu_j<2m$ with analytic coefficients defined in the planary boundary of Σ_ρ, $\sigma_\rho : x_1^2+\cdots+x_n^2<\rho^2$ such that the following are fulfilled:

(i) $\mu_j\neq\mu_k$ for $j\neq k$,

(ii) if $b_j(x, D_x, D_t)$ is the part of order μ_j of B_j, then we have $b_j(x; 0, 1)\neq 0$ for all $x\in\sigma_R$,

(iii) the set $B_1, \ldots, B_m$ covers L.

Before stating our theorem we have to introduce some notations and to prepare a few lemmas according to Morrey-Nirenberg [6]. For $u\in C^\infty(\Sigma_R)$ $(R<\rho)$ and $\phi\in C_0^\infty(R^n)$ we define two norms, respectively, for $j=0, 1, \ldots,$

$$|u; \Sigma_R|_j = \left(\iint_{\Sigma_R} |u^{(j)}|^2\, dx\, dt\right)^{1/2},$$

$$[\phi]_j = \inf |v; R_+^{n+1}|_j(^1),$$

where $u^{(j)}$ is a vector function $\{D_x^\alpha D_t^k u;\ |\alpha|+k=j\}$ and "inf" is taken over all functions v in $C_0^\infty(\bar{R}_+^{n+1})$ which equal ϕ on $t=0$. Finally we set

$$L^0(D_x, D_t) = l(0, 0; D_x, D_t),$$

$$B_j^0(D_x, D_t) = b_j(0; D_x, D_t),$$

where $l(x, t; D_x, D_t)$ is the leading part of L.

The following Lemma is given in [1].

LEMMA 1. *Assume u is in* $C_0^\infty(\bar{R}_+^{n+1})$ *such that* supp $[u]\subset\Sigma_R\cup\sigma_R$ $(R<\rho)$. *Then there exists a constant* $K_1(^2)$ *such that*

$$|u; \Sigma_R|_{2m} \leqq K_1\left(|L^0u; \Sigma_R|_0+\sum_{j=1}^{m} [B_j^0 u]_{2m-\mu_j}\right).$$

(1) We can easily see that this norm is equivalent to one employed in §2 (cf. (3), (4) and (5)): $\langle\phi\rangle_j=\int|\xi|^{2j-1}|\hat{\phi}(\xi)|^2\, d\xi, \phi\in C_0^\infty(R^n)$, i.e., there exists a constant c_j such that $c_j^{-1}\langle\phi\rangle_j\leqq[\phi]_j\leqq c_j\langle\phi\rangle_j$, $\phi\in C_0^\infty(R^n)$.

(2) From now on we shall use the symbols K_i, K_i' to denote constants depending only on n, L^0 and B_j^0 $(j=1, \ldots, m)$.

LEMMA 2. *Assume u is in* $C^\infty(\Sigma_R \cup \sigma_R)$. *There then exists a constant* K_2 *such that, for* $0<r<r+\delta<R$, $\delta<r$,

$$|u;\Sigma_r|_{2m} \leqq K_2\left(|L^0u;\Sigma_{r+\delta}|_0+\sum_{q=1}^{2m}\delta^{-q}|u;\Sigma_{r+\delta}|_{2m-q}+\sum_{j=1}^{m}[\zeta_{r,\delta}B_j^0u]_{2m-\mu_j}\right),$$

where $\zeta_{r,\delta}(x,t)=h(((|x|^2+t^2)^{1/2}-r)/\delta)$ *with a fixed function* $h\in C^\infty(R^1)$ *satisfying* $h(\tau)=1$ *for* $\tau\leqq 0$ *and* $h(\tau)=0$ *for* $\tau\geqq 1$.

Lemma 2 is easily derived from Lemma 1 and the definition of the norm $[\cdot]_j$, noting that there exists a constant C_k^m such that

$$\sup_{(x,t)\in\Sigma_R}|\zeta_r^{(k)}(x,t)| \leqq C_k^m\delta^{-k}. \tag{1'}$$

We need some further notations: for f, u in $C^\infty(\Sigma_R)$

$$\begin{aligned} M_{R,p}(f) &= (p!)^{-1}\sup_{\varepsilon R\leqq r<R}(R-r)^{2m+p}|f;\Sigma_r|_p, && p=0,1,\ldots,\\ N_{R,p}(u) &= [p!]^{-1}\sup_{\varepsilon R\leqq r<R}(R-r)^{2m+p}|u;\Sigma_r|_{2m+p}, && p=-2m,-2m+1,\ldots, \end{aligned} \tag{2'}$$

and for $v\in C^\infty(\Sigma_R\cup\sigma_R)$

$$F_{R,p}^{(j)}(v) = (p!)^{-1}\sup_{\varepsilon R\leqq r<R}(R-r)^{2m+p}[\zeta_r v^{(p)}]_{2m-\mu_j}, \qquad p=0,1,\ldots, \tag{3'}$$

where $\varepsilon=1-1/2m$, $\zeta_r=\zeta_{r,\delta}$ with $\delta=(R-r)/(p+1)$, and $[p!]=p!$ for $p\geqq 0$ and $[p!]=1$ for $p<0$.

LEMMA 3. *Suppose u is in* $C^\infty(\Sigma_R\cup\sigma_R)$. *Then there exists a constant* K_3 *such that, for* $p>0$,

$$N_{R,p}(u) \leqq K_3\left(M_{R,p}(L^0u)+\sum_{q=1}^{2m}N_{R,p-q}(u)+\sum_{j=1}^{m}F_{R,p}^{(j)}(B_j^0u)\right).$$

This lemma is an immediate consequence of the preceding one.

Now we can prove

THEOREM 4. *Let u be a solution in* $C^{2m}(\Sigma_\rho\cup\sigma_\rho)$ *of the system*

$$\begin{aligned} Lu &= f \quad \text{in } \Sigma_\rho,\\ B_ju &= 0 \quad \text{on } \sigma_\rho, \qquad j=1,\ldots,m. \end{aligned} \tag{4'}$$

If f is analytic in a neighborhood of Σ_ρ, *then u is analytic on* $\Sigma_\rho\cup\sigma_\rho$.

Proof. To prove this theorem we have only to assert the analyticity of the u in some neighborhood of the origin. It is, first of all, clear that u is in $C^\infty(\Sigma_\rho\cup\sigma_\rho)$ (see Theorem 15.3 in [1]). As a consequence, if we can establish the following inequalities

$$N_{R,p}(u) \leqq M\lambda^p, \quad R\leqq R_1\leqq 1, \qquad p=-2m,-2m+1,\ldots, \tag{5'}$$

for some fixed constant M, $\lambda\geqq 1$ and $R_1<\rho$, we can deduce Theorem 4 from (5′) and a slight modification of the proof of Lemma 2.3 in [6].

We write the system (4′) in the form

$$(6') \qquad \begin{aligned} L^0u &= f+(L^0-L)u \quad \text{in } \Sigma_\rho, \\ B_j^0u &= (B_j^0-B_j)u \qquad \text{on } \sigma_\rho, \qquad j = 1,\dots,m, \end{aligned}$$

and L^0-L, $B_j^0-B_j$ in the forms

$$(7') \qquad \begin{aligned} (L^0-L)u &= \sum_{q=0}^{2m} a_q(x,t)u^{(2m-q)}, \\ (B_j^0-B_j)u &= \sum_{q=0}^{\mu_j} b_{j,q}(x)u^{(\mu_j-q)}, \end{aligned}$$

with $a_0(0,0)=0$ and $b_{j,0}(0)=0$. The analyticity of f, a_q and $b_{j,q}$ guarantee that we can find numbers $A\geqq 2$, H and $R_0\leqq 1$, $R_0<\rho$, such that

$$(8') \qquad \begin{aligned} &|f|_{p,\Sigma_{R_0}} \leqq p!HA^p, \\ &|a_q|_{p,\Sigma_{R_0}} \leqq p!HA^p, \qquad |b_{j,q}|_{p,\Sigma_{R_0}} \leqq p!HA^p, \\ &|a_0(x,t)| \leqq HA(|x|+|t|), \qquad |b_{j,0}(x)| \leqq HA|x| \quad \text{in } \Sigma_{R_0}. \end{aligned}$$

Application of Lemma 3 gives us, with the aid of (6′) and (7′), that if $R\leqq R_0$, for all $p>0$

$$(9') \qquad \begin{aligned} N_{R,p}(u) \leqq K_3\Bigg(M_{R,p}(f)+\sum_{q=0}^{2m} M_{R,p}(a_qu^{(2m-q)})+\sum_{q=1}^{2m} N_{R,p-q}(u) \\ +\sum_{j=1}^{m}\sum_{q=0}^{\mu_j} F_{R,p}^{(j)}(b_{j,q}u^{(\mu_j-q)})\Bigg). \end{aligned}$$

First we find, with the aid of (2′), (8′) and Leibniz' formula that for $R\leqq R_0$ and $p>0$

$$(10') \qquad \begin{aligned} M_{R,p}(f) &\leqq HK_1'R^{2m+(n+1)/2}(AR)^p, \\ \sum_{q=0}^{2m} M_{R,p}(a_qu^{(2m-q)}) &\leqq HARN_{R,p}(u)+H\sum_{t=1}^{p}\left(\frac{AR}{2m}\right)^t N_{R,p-t}(u) \\ &\quad +H\sum_{q=1}^{2m}\sum_{t=0}^{p} A^{-q}\left(\frac{AR}{2m}\right)^{q+t} N_{R,p-q-t}(u). \end{aligned}$$

From (3′) we have, recalling the definition of the norm $[\cdot]_j$,

$$F_{R,p}^{(j)}(b_{j,q}u^{(\mu_j-q)}) \leqq (p!)^{-1}K_2' \sup_{\varepsilon R\leqq r<R} (R-r)^{2m+p}|\zeta_r(b_{j,q}u^{(\mu_j-q)})^{(p)};\Sigma_{r+\delta}|_{2m-\mu_j}.$$

Using (1′), (8′) and Leibniz' formula we obtain, for j, q and r such that $1\leqq j\leqq m$, $0\leqq q\leqq \mu_j$ and $\varepsilon R\leqq r<R$,

$$\begin{aligned} I_r^{(j,q)} &= |\zeta_r(b_{j,q}u^{(\mu_j-q)})^{(p)};\Sigma_{r+\delta}|_{2m-\mu_j} \\ &\leqq K_3'\sum_{k=0}^{2m-\mu_j}\delta^{-k}\sum_{t=0}^{p+2m-\mu_j-k}\binom{p+2m-\mu_j-k}{t}|b_{j,q}^{(t)}u^{(p+2m-k-t-q)};\Sigma_{r+\delta}|_0. \end{aligned}$$

From this we obtain, for $p>0$,

$$(p!)^{-1} \sup_{\varepsilon R \leqq r < R} (R-r)^{2m+p} I_r^{(j,0)}$$
$$\leqq e\left(1+\frac{1}{p}\right)^{2m} H\left\{\sum_{k=0}^{2m-\mu_j} (k+1)^k \left(ARN_{R,p-k}(u) + \sum_{t=1}^{2m+p-\mu_j-k} (AR)^t N_{R,p-k-t}(u)\right)\right\}$$

and when $q \geqq 1$

$$(p!)^{-1} \sup_{\varepsilon R \leqq r < R} (R-r)^{2m+p} I_r^{(j,q)}$$
$$\leqq e\left(1+\frac{1}{p}\right)^{2m} H \sum_{k=0}^{2m-\mu_j} (k+1)^k \sum_{t=0}^{2m+p-\mu_j-k} A^{-q}(AR)^{t+q} N_{R,p-k-t-q}(u).$$

Consequently

$$\text{(11}')\qquad \sum_{j=1}^{m} \sum_{q=0}^{\mu_j} F_{R,p}^{(j)}(b_{j,q}u^{(\mu_j-q)}) \leqq K_4'\left(HAR \sum_{k=0}^{2m} N_{R,p-k}(u) + H \sum_{k=0}^{2m} \sum_{\tau=1}^{2m+p-k} (AR)^\tau N_{R,p-k-\tau}(u)\right).$$

It follows from (9′), (10′) and (11′) that

$$N_{R,p}(u) \leqq K_4\left(HR^{2m+(n+1)/2}(AR)^p + HARN_{R,p}(u) + HAR \sum_{k=1}^{2m} N_{R,p-k}(u)\right.$$
$$\left. + H \sum_{k=0}^{2m} \sum_{\tau=1}^{2m+p-k} (AR)^\tau N_{R,p-k-\tau}(u) + \sum_{k=1}^{2m} N_{R,p-k}(u)\right).$$

Hence, if we set

$$R_1 = \min\left(\frac{1}{2K_4HA}, R_0\right),$$

the inequality

$$\text{(12}')\qquad \begin{aligned} N_{R,p}(u) \leqq 2K_4\left(HR^{2m+(n+1)/2}(AR)^p + H \sum_{k=0}^{2m} \sum_{\tau=1}^{2m+p-k} (AR)^\tau N_{R,p-\tau-k}(u)\right) \\ +(1+2K_4) \sum_{k=1}^{2m} N_{R,p-k}(u) \end{aligned}$$

holds for $R \leqq R_1$ and $p>0$.

Using (12′) we can finally establish the inequality (5′) with

$$\lambda = 6+6K_4+3AR_1,$$
$$M = 2K_4HR_1^{2m+(n+1)/2} + \lambda^{2m}|u; \Sigma_{R_1}|_{2m}.$$

Thus the proof is completed.

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