

ON THE NULLITY AND ENCLOSURE GENUS OF WILD KNOTS

BY
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The fundamental group of the complement of a wild knot in a 3-sphere can be expressed as the colimit (direct limit) of a suitable family of groups and homomorphisms (Crowell [4]). To each group in the family we assign a Jacobian module, and in §1 we prove that this assignment is functorial and preserves colimits. This is used in §2 to show that the nullity of the Alexander module of a knot with one wild point is bounded above by its enclosure genus. This can be used in some cases to calculate the enclosure genus and the penetration index of the knot in a purely algebraic way. In §3 we give examples to show that the upper bound of Theorem 2 is the best possible, along with some results concerning the penetration index of an arc relative to surfaces of genus r .

1. The Alexander module.

1. For the categorical notions used here, see [10]. The proof of Theorem 1 will depend on the following standard result ([10, Chapter II, 6.2]):

(1.1) Let \mathcal{A} be a category with coproducts and let \mathcal{B} be any category. Then the functor $T: \mathcal{A} \rightarrow \mathcal{B}$ is colimit preserving iff it preserves coproducts and coequalisers.

Let H be an arbitrary group which will remain fixed; we define a category \mathcal{H} of H -groups $[G, \alpha]$ (cf. [8]) where $G \in \mathcal{G}$ (the category of all groups and homomorphisms), and $\alpha: G \rightarrow H$ is an epimorphism. A morphism $\theta: [G_1, \alpha_1] \rightarrow [G_2, \alpha_2]$ is a group homomorphism $\theta: G_1 \rightarrow G_2$ such that the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\theta} & G_2 \\ \alpha_1 \searrow & & \swarrow \alpha_2 \\ & H & \end{array}$$

commutes, i.e. $\alpha_2\theta = \alpha_1$.

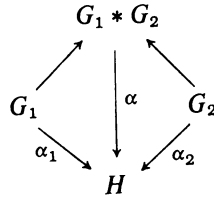
(1.2) \mathcal{H} admits coproducts.

Proof. In \mathcal{G} , the coproduct of G_1 and G_2 is just the free product $G_1 * G_2$. By the universality of the coproduct diagram in \mathcal{G} [9, p. 17], there exists a unique map

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$\alpha: G_1 * G_2 \rightarrow H$ making the diagram



commute.

Since the α_i are epimorphisms, α is an epimorphism and $[G_1 * G_2, \alpha]$ is an H -group which is the coproduct in \mathcal{H} of $[G_1, \alpha_1]$ and $[G_2, \alpha_2]$.

(1.3) \mathcal{H} admits coequalisers.

Proof. In \mathcal{G} , the coequaliser of $\theta_1, \theta_2: G \rightarrow L$ is the canonical map $\tau: L \rightarrow L/\Theta$, where Θ is the normal subgroup of L generated by all elements of the form $\theta_1(g)\theta_2(g^{-1})$ for $g \in G$.

For H -groups $[G, \alpha]$ and $[L, \beta]$, since θ_1 and θ_2 are H -morphisms, we have $\beta\theta_1 = \alpha = \beta\theta_2$. Then in \mathcal{G} , β factors uniquely through the coequaliser of θ_1 and θ_2 , i.e. $\beta = \sigma\tau$ for a unique σ ; but β is an epimorphism, so σ is also. It follows that $\tau: [L, \beta] \rightarrow [L/\Theta, \sigma]$ is an H -morphism and is the coequaliser in \mathcal{H} of the maps θ_1 and θ_2 .

2. Let $[G, \alpha] \in \mathcal{H}$ and suppose $K = \ker \alpha$, i.e. the sequence

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\alpha} H \longrightarrow 1$$

is exact. Then we have an exact sequence

$$0 \longrightarrow K/K' \longrightarrow G/K' \xrightarrow{\alpha} H \longrightarrow 1$$

with abelian kernel, where $\alpha: G/K' \rightarrow H$ is induced by α . By [5], we have a uniquely determined exact sequence of JH -modules

$$0 \longrightarrow K/K' \longrightarrow H_1(G, 1: JH) \xrightarrow{\alpha} JH \xrightarrow{\varepsilon} J \longrightarrow 0$$

where the relative homology group $H_1(G, 1: JH)$ has the Jacobian matrix of G evaluated at α (see [3]) as a relation matrix over the integer group ring JH of H , and $\varepsilon: JH \rightarrow J$ is the trivialiser.

Alternatively, we may use MacLane's statement [9, p. 120] of the same sequence

$$0 \longrightarrow K/K' \longrightarrow M_\alpha \xrightarrow{\alpha^*} JH \longrightarrow J \longrightarrow 0$$

where M_α is the JH -module with generators $[g]$ for $g \in G/K'$, and relations $[1] = 0_2[g_1g_2] = \alpha(g_1)[g_2] + [g_1]$. The map α^* is defined by $\alpha^*[g] = \alpha(g) - 1$.

It follows from Trotter's work [11] that M_α is naturally isomorphic to

$$H_1(G, 1: JH).$$

We say that $M_\alpha = M[G, \alpha]$ is the *Jacobian module of G evaluated at α* ; if $H = G/G'$ and α is the abelianiser $\alpha: G \rightarrow G/G'$, M_α is called the *Alexander module of G* . When there is no danger of confusion, we write MG for $M[G, \alpha]$. In this paragraph we will mainly be using MacLane's definition of the Jacobian module, and in §2 we will use the relative homology definition.

3. Let $\theta: [G, \alpha] \rightarrow [L, \beta]$ be an H -morphism; then we have induced maps of exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\alpha} & H \longrightarrow 1 \\ & & \downarrow & & \downarrow \theta & & \parallel \\ 1 & \longrightarrow & N & \longrightarrow & L & \xrightarrow{\beta} & H \longrightarrow 1 \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & K/K' & \longrightarrow & G/K' & \xrightarrow{\alpha} & H \longrightarrow 1 \\ & & \downarrow & & \downarrow \theta & & \parallel \\ 0 & \longrightarrow & N/N' & \longrightarrow & L/N' & \xrightarrow{\beta} & H \longrightarrow 1 \end{array}$$

We define a map $M\theta: M[G, \alpha] \rightarrow M[L, \beta]$ by $(M\theta)[gK'] = [\theta(g)N']$ for $g \in G$, and linearity. Obviously $M\theta[K'] = [\theta(1)N'] = 0$ and $M\theta[g_1g_2K'] = \alpha(g_1K')M\theta[g_2K'] + M\theta[g_1K']$ since $\beta\theta = \alpha$ and $\alpha(K') = 1$. Thus $M\theta$ is a properly defined JH -morphism.

Moreover, $M1 = 1$ and $M(\theta\phi) = M\theta M\phi$, so if \mathcal{M} is the category of JH -modules, $M: \mathcal{H} \rightarrow \mathcal{M}$ is a functor, called the *Jacobian module functor*.

THEOREM 1. *The Jacobian module functor preserves colimits.*

Proof. (i) M preserves coproducts.

For, let G have presentation $(x_i: r_i)$, $i = 1, 2$. Then $G_1 * G_2$ has presentation $(x_1 \cup x_2: r_1 \cup r_2)$ and $H_1(G_1 * G_2, 1: JH)$ has a relation matrix of the form

$$\begin{vmatrix} A_1 & 0 \\ 0 & A_2 \end{vmatrix}$$

where A_i is a relation matrix for $H_1(G_i, 1: JH)$. Thus

$$H_1(G_1 * G_2, 1: JH) \cong H_1(G_1, 1: JH) \oplus H_1(G_2, 1: JH).$$

(ii) M preserves coequalisers.

Let $\tau: [L, \beta] \rightarrow [L/\Theta, \sigma]$ be the coequaliser of $\theta_1, \theta_2: [G, \alpha] \rightarrow [L, \beta]$, as in the proof of (1.3). We want to prove that the Jacobian module of $[L/\Theta, \sigma]$ is the quotient of the Jacobian module of $[L, \beta]$ by the submodule generated by all elements of the form $M\theta_1(m) - M\theta_2(m)$ for $m \in MG$. It is sufficient to show that this submodule contains $\ker M\tau$, since $M\theta_1(m) - M\theta_2(m) \in \ker M\tau$ for all m .

So let $[lN'] \in \ker M\tau$; then $[\tau(lN')] = [\tau(l)P'] = 0$, where $P = \ker \sigma: L/\Theta \rightarrow H$. Thus $\tau(l) \in P'$, and so $l \in N$, since $\beta(l) = \sigma\tau(l) = 0$. $\tau: N' \rightarrow P'$ is onto, and $\tau(l) \in P'$, so $l = Wl_1$ where $l_1 \in N'$ and $W \in \ker \tau$. Therefore $[lN'] = [WN']$, and since $W \in \ker \tau$, W can be written

$$W = \prod_j l_j^{e_j} \theta_1(g_j) \theta_2(g_j^{-1}) l_j^{-e_j}$$

for $g_j \in G$. Using the relations

$$[l^{-1}] = -\beta(l^{-1})[l]$$

and

$$[l^m] = \sum_{j=0}^{m-1} \beta(l^j)[l]$$

for $l \in L/N'$, it follows by direct calculation that

$$[lN'] = \sum_j \beta(l_j^{e_j}) \{M\theta_1[g_j] - M\theta_2[g_j]\}$$

as required.

Thus M preserves coproducts and coequalisers, and the result follows from (1.1). Q.E.D.

4. Let R be an integral domain and let M be a finitely presented R -module $(X; P)$ where $\rho_i \in P$ is a relation of the form

$$\rho_i = \sum_{j=1}^m \alpha_{ij} x_j, \quad x_j \in X, \alpha_{ij} \in R.$$

The matrix (α_{ij}) is a *relation matrix* for the module M . Let A be a finite matrix over R , with $m(A)$ columns and rank $r(A)$; we define the *nullity* $\nu(A)$ of A to be the difference $m(A) - r(A)$. It is easy to see that this definition is equivalent to that given in [3]. If A is a relation matrix for the module M , we say that M has nullity $\nu(A)$, since the Tietze operations on matrices preserve nullity. If M is infinitely presented and R is Noetherian, we take a sequence $\{M_i\}$ of finitely presented submodules of M , such that $M_i \subset M_{i+1}$ and $\bigcup_i M_i = M$, and define

$$\nu(M) = \sup \nu(M_i).$$

(4.1) Let $A = (\alpha_{ij})$ and $B = (\beta_{kj}), j = 1, \dots, m$, be finite matrices over R . If A has l rows, the matrix $C = (\gamma_{ij})$ defined by

$$\begin{aligned} \gamma_{ij} &= \alpha_{ij}, & i \leq l, \\ &= \beta_{i-l,j}, & i > l, \end{aligned}$$

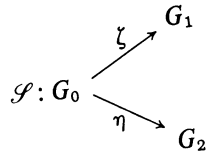
has nullity at least equal to $\nu(A) - r(B)$.

Proof. Immediate since $r(C) \leq r(A) + r(B)$.

(4.2) Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of R -modules, and suppose A is generated by $\{n_1, \dots, n_s\}$. Then $\nu(B) - s \leq \nu(C) \leq \nu(B)$.

Proof. B can be represented as a quotient of a free module F ; let $\phi: F \rightarrow B$ be the presentation map. A , as a submodule of B , is generated by s elements of the form $\rho_i = \sum_j \alpha_{ij}\phi(x_j)$ if F is free on the alphabet $\{x_j\}$. But $\sum \alpha_{ij}\phi(x_j) = \phi(\sum \alpha_{ij}x_j)$, so $C \cong B/A$ means that if we add the matrix $(\alpha_{ij}), i=1, \dots, s$, to some relation matrix for B , we obtain a relation matrix for C . The results then follow from (4.1) and [3, 2.2].

Let $[G_i, \alpha_i]$ be H -groups, $i=0, 1, 2$, and suppose G_0 is a free group on m generators. If G is the colimit of the system



where ζ and η are H -morphisms, then $[G, \alpha]$ is an H -group for a unique $\alpha: G \rightarrow H$, since H admits coproducts and coequalisers. In fact, if $w_i: G_i \rightarrow G_1 * G_2$, then G is the quotient of $G_1 * G_2$ by the normal subgroup $\langle\langle w_1\zeta(g)w_2\eta(g^{-1}): g \in G_0 \rangle\rangle$. Since the Jacobian module functor preserves colimits, $M[G, \alpha]$ is the quotient of $M[G_1, \alpha_1] \oplus M[G_2, \alpha_2]$ by the submodule K generated by all elements of the form $Mw_1\zeta[g] - Mw_2\eta[g]$ for $[g] \in M[G_0, \alpha_0]$. The elements $Mw_1\zeta[x_i] - Mw_2\eta[x_i], i=1, 2, \dots, m$, generate K if G_0 is free on x_1, \dots, x_m ; thus K is generated by $r \leq m$ elements.

The sequence

$$0 \rightarrow K \rightarrow MG_1 \oplus MG_2 \rightarrow MG \rightarrow 0$$

is exact, so

$$\nu(MG_1 \oplus MG_2) - m \leq \nu(MG_1 \oplus MG_2) - r \leq \nu(MG).$$

But $MG_1 \oplus MG_2$ has a relation matrix of the form

$$\begin{vmatrix} A_1 & 0 \\ 0 & A_2 \end{vmatrix}$$

so $\nu(MG_1 \oplus MG_2) = \nu(MG_1) + \nu(MG_2)$. Hence,

$$\nu(MG_1) \leq \nu(MG) + m - \nu(MG_2).$$

2. Nullity and enclosure genus.

5. Our “knot” is an oriented knot in a 3-sphere S , with one wild point w . The extension to knots with finitely many wild points is easily accomplished.

A “handlebody (Henkelkorper) of genus g ” is a 3-cell with g oriented handles. A sequence $H_0 \supset H_1 \supset H_2 \supset \dots$ of handlebodies is an *admissible sequence* for the knot k if

- (i) H_i is a tame closed neighbourhood of w , for each i ;
- (ii) $H_{i+1} \subset \text{Int } H_i$ for all i , and $\bigcap H_i = k$;
- (iii) k does not meet the boundary of any of the H_i .

The *enclosure genus* of k is then defined to be $\min \lim \text{genus}(H_i)$, where the minimum is taken over all admissible sequences $\{H_i\}$. We will denote the enclosure genus of k by $e(k)$.

We choose an admissible sequence for k which is of minimal genus (i.e. $\text{genus}(H_i) = e(k)$ for all i and set $G_i = \pi_1(S - H_i)$, and $G = \pi_1(S - k)$. Then the inclusions $S - H_i \subset S - H_{i+1}$ and $S - H_i \subset S - k$ induce maps $\phi_i: G_i \rightarrow G_{i+1}$ and $\psi_i: G_i \rightarrow G$, with $\psi_{i+1}\phi_i = \psi_i$.

Let $\alpha: \mathcal{G} \rightarrow \mathcal{A}$ (the category of abelian groups) be the abelianisation functor, given by $\alpha(G) = G/G'$. We note that $\alpha(G_i) \cong H_1(S - H_i)$ and $\alpha(G) \cong H_1(S - k)$, so we have a commuting diagram

$$\begin{array}{ccc} G_i & \xrightarrow{\psi_i} & G \\ \downarrow & & \downarrow \\ H_1(S - H_i) & \xrightarrow{\alpha(\psi_i)} & H_1(S - k) \cong Z. \end{array}$$

Now $\alpha(\psi_i)$ is an epimorphism, and so is the map $G_i \rightarrow H_1(S - H_i)$, so we make G_i a Z -group $[G_i, \alpha_i]$ by taking for α_i the composition of ψ_i with the canonical map $\tau: G \rightarrow G/G'$ and the Hurewicz isomorphism $G/G' \rightarrow H_1(S - k)$. The composition of these last two maps will serve for our map $\alpha: G \rightarrow Z$, making $[G, \alpha]$ a Z -group. Then the maps $\phi_i: [G_i, \alpha_i] \rightarrow [G_{i+1}, \alpha_{i+1}]$ and $\psi_i: [G_i, \alpha_i] \rightarrow [G, \alpha]$ are morphisms in the category \mathcal{Z} of Z -groups. In fact,

(5.1) $[G, \alpha]$ is the colimit in \mathcal{Z} of the sequence

$$\cdots \longrightarrow [G_i, \alpha_i] \xrightarrow{\phi_i} [G_{i+1}, \alpha_{i+1}] \longrightarrow \cdots$$

Proof. By [4], G is the colimit in \mathcal{G} of the sequence $\cdots \rightarrow G_i \xrightarrow{\phi_i} G_{i+1} \rightarrow \cdots$. Suppose $[H, \beta]$ is the colimit in \mathcal{Z} of the sequence of Z -groups. Since each Z -map is a group homomorphism, and G is the colimit, there exists a unique map $G \rightarrow H$ in \mathcal{G} . By the same token, since $[G, \alpha]$ and the maps $\psi_i: [G_i, \alpha_i] \rightarrow [G, \alpha]$ form a commuting system of Z -groups, there exists a unique Z -map $[H, \beta] \rightarrow [G, \alpha]$. There is thus a unique map in \mathcal{G} from $H \rightarrow G$ —so $G \cong H$ and $[G, \alpha] \cong [H, \beta]$.

From (5.1) and Theorem 1, we see that the Alexander module of G is the colimit of the Jacobian modules of the G_i , evaluated at α_i .

Consider the sequence

$$\cdots M\psi_i(MG_i) \subset M\psi_{i+1}(MG_{i+1}) \cdots$$

of finitely presented JZ -modules; there is a unique (inclusion) map $\bigcup M\psi_i(MG_i) \subset MG$, and a unique map $MG \rightarrow \bigcup M\psi_i(MG_i)$, since each diagram

$$\begin{array}{ccc} MG_i & \longrightarrow & M\psi_i(MG_i) \\ M\phi_i \downarrow & & \downarrow \\ MG_{i+1} & \longrightarrow & M\psi_{i+1}(MG_{i+1}) \end{array}$$

commutes, and MG is the colimit of the MG_i . Hence $MG \cong \bigcup M\psi_i(MG_i)$, and from ([3], 2.2) we have

$$\nu(MG) \leq \sup \nu(MG_i).$$

6. It is known that the Alexander module of any knot has nullity at least one, and that the nullity of a tame knot is equal to one [3, 4.5, 4.6].

THEOREM 2. *Let k be a knot in a 3-sphere S , with one wild point. If ν is the nullity of the Alexander module of $\pi_1(S-k)$, then $1 \leq \nu \leq e(k)$.*

Proof. To prove the theorem, it is sufficient to show that $\nu(MG_i) \leq e(k)$ for all i .

H_i can be written as a 3-cell U and $e(k)$ handles h_1, \dots, h_e . $S-H_i$ is of the same homotopy type as the complement of the (tame) 3-cell U and e arcs $\alpha_1, \dots, \alpha_e$ obtained by shrinking the solid handles of H to arcs from $\text{Bd } U$ to $\text{Bd } U$.

We orient the arcs $\alpha_1, \dots, \alpha_e$ and join the $2e$ points of $\text{Bd } U \cap \bigcup \alpha_i$ by e arcs β_1, \dots, β_e in $\text{Int } U$ so that $\bigcup \{\alpha_i \cup \beta_i\}$ is a tame knot k^* . Thickening U appropriately, so we can apply the van Kampen theorem, we see that $\pi_1(S-k^*)$ is the colimit of the system

$$\mathcal{S}: \pi_1((S-(U \cup k^*)) \cup (U - \bigcup \beta_i)) \begin{cases} \xrightarrow{\zeta} \pi_1(S-k^* \cup U) \\ \xrightarrow{\eta} \pi_1(U - \bigcup \beta_i) \end{cases}$$

where ζ and η are induced by inclusion. But $((S-(U \cup k^*)) \cap (U - \bigcup \beta_i))$ is of the same homotopy type as a 2-sphere with $2e$ oriented holes, and so has fundamental group free on $2e-1$ generators. Moreover, we can choose the β_i so they are unknotted and do not link each other; then $\pi_1(U - \bigcup \beta_i)$ is a free group on e generators. From subsection 4, we see that

$$\nu(MG_i) \leq 1 + (2e(k) - 1) - \nu(M\pi_1(U - \bigcup \beta_i)) = e(k). \quad \text{Q.E.D.}$$

3. Examples.

7. We will show that the upper bound of Theorem 2 is the best possible, i.e. for each n , there exists a knot of enclosure genus n which also has nullity equal to n .

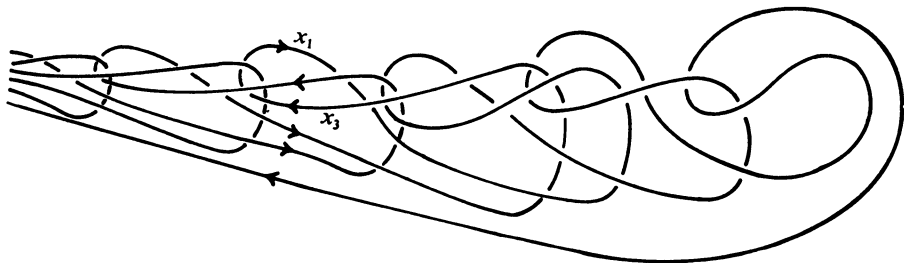


FIGURE 1

We consider a variation of the “remarkable simple closed curve” of Fox [7]; instead of passing the arc b_i of Figure 1 of [7] over just one tongue of the knot, we pass it over n tongues. A projection of the knot so obtained when $n=2$ is shown in Figure 1, and it is easy to see that k_2 has enclosure genus at most equal to three. In general, $e(k_n) \leq n+1$. We will show that the Alexander module of k_n has nullity equal to $n+1$, thus showing that $e(k_n) = n+1$. Further, because $e(k) \leq P(k, w)/2$ for all knots k , we see that $P(k_n, w) = 2n+2$.

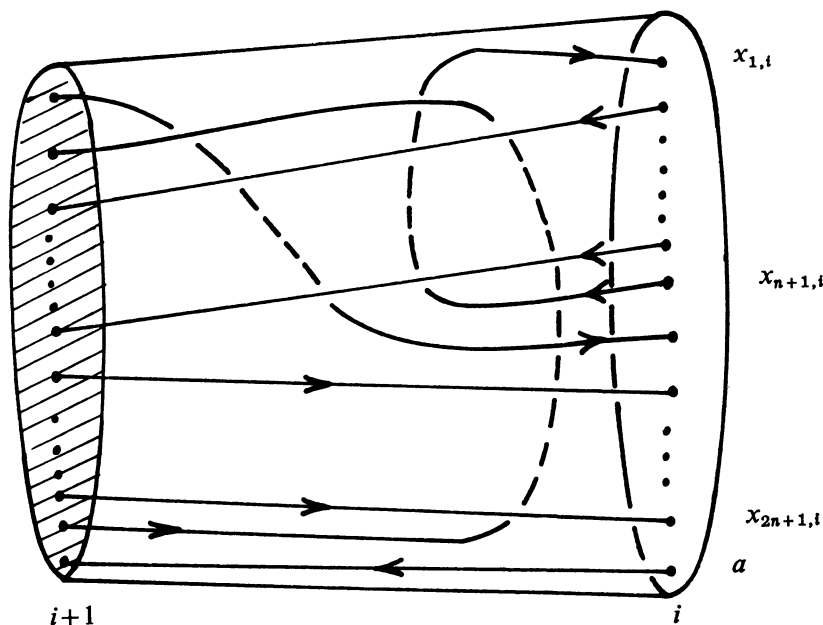


FIGURE 2

Using the techniques of [6], k_n can be constructed from an infinite family of cylinders of the type shown in Figure 2. Then $\pi_1(S-k_n)$ has a presentation of the form

$$\left(\bigcup_{i=0}^{\infty} \{x_{1,i}, x_{2,i}, \dots, x_{2n+1,i}, x_{2n+2,i} = a\}; r_0, r_1, \dots \right)$$

where r_0 is the set

$$(\alpha_0) \quad x_{l,0} = x_{2n+3-l,0}, \quad l=1, 2, \dots, n+1,$$

and r_i consists of the following relations ($i=0, 1, 2, \dots$):

- (1_i) $x_{1,i+1}x_{2,i+1} \cdots x_{n+1,i+1} = x_{2,i+1} \cdots x_{n+1,i+1}x_{n+2,i}$,
- (2_i) $x_{2,i+1} \cdots x_{n+1,i+1}x_{n+1,i} = x_{1,i}x_{2,i+1} \cdots x_{n+1,i+1}$,
- (3_i) $x_{2n+1,i+1}W_i = W_i x_{2,i+1}$, where $W_i = x_{2n+1,i} \cdots x_{n+2,i} \bar{x}_{n+1,i} \cdots \bar{x}_{2,i}$,
- (4_i) $x_{l,i+1} = x_{l-1,i}$, $l=3, \dots, n+1$,
- (5_i) $x_{n+l,i+1} = x_{n+l+1,i}$, $l=2, \dots, n$,
- (6_i) $x_{2n+2,i} = a = W_i x_{1,i}$ (relation about the vertex).

Let (α_i) be a set of relations of the form

$$x_{l,i} = x_{2n+3-l,i}, \quad l = 1, 2, \dots, n+1.$$

Then if (α_i) and r_i are true, so are (α_{i+1}) ; since (α_0) holds, (α_i) is true for all i . Using this, I calculate that $\pi_1(S-k_n)$ has a presentation of the form

$$\left(\{x_{2,0}, x_{3,0}, \dots, x_{n+1,0}\} \cup \bigcup_{i=1} \{x_{2,i}\}; (\beta_i) \right)$$

where for $i \geq n-1$, (β_i) is a relation of the form

$$\begin{aligned} x_{2,i+1} \cdots x_{2,i+2-n} x_{2,i+1-n} \bar{x}_{2,i+2-n} \cdots \bar{x}_{2,i+1} \\ = x_{2,i+2} \cdots x_{2,i+3-n} x_{2,i+2-n} \bar{x}_{2,i+3-n} \cdots \bar{x}_{2,i+2} \end{aligned}$$

with a corresponding form for $i < n-1$. The Alexander matrix computed from this presentation has the infinite form shown in Figure 3, which has nullity $n+1$, as required.

$$\begin{bmatrix} 1-t & -(t-1)^2 & -t(t-1)^2 & \cdots & t^{n-1}(2t-1) & -t^n & 0 & 0 \\ 0 & 1-t & -(t-1)^2 & \cdots & -t^{n-2}(t-1)^2 & t^{n-1}(2t-1) & -t^n & 0 \\ 0 & 0 & 1-t & \cdots & -t^{n-3}(t-1)^2 & -t^{n-2}(t-1)^2 & t^{n-1}(2t-1) & -t^n \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

FIGURE 3

8. Let k be a knot or an arc with one wild points w . Then the penetration index of k at w , relative to surfaces of genus r , (cf. [2]), is the smallest integer n such that there exist arbitrarily small neighbourhoods of w which are handlebodies of genus r , each meeting k on its boundary in n points. We will denote this penetration index by $P_r(k, w)$.

(8.1) Let $H_0 \supset H_1 \supset \dots$ be an admissible sequence for k of minimal genus, and let E be a 3-cell neighbourhood of w . Then there exists an index N such that only one handle of H_N meets $\text{Bd } E$.

Proof. $k \cap E$ consists of two arcs from $\text{Bd } E$ to w , and several arcs joining the points of $k \cap \text{Bd } E$. Let δ be the minimum distance from w to one of these arcs, and choose a 3-cell neighbourhood of w , E' say, which is centred at w and is of radius less than δ . Then only one arc α of $k \cap \text{Cl}(S-E')$ meets the boundary of E .

We may construct a handlebody H by adjoining to E' a tubular neighbourhood of this arc α , and handles about the other arcs of k lying in $\text{Int}(E)-E'$. Since $\bigcap H_i = k$, there exists an index N such that $H_N \subset \text{Int } H$. We put H_N and $\text{Bd } E$ into general position so that $H_N \cap \text{Bd } E$ is a family of discs on $\text{Bd } E$. Each of these discs contains at most one point of k , so all but one handle of H_N lies in the interior of E .

THEOREM 3. Let $k = k_1 \cup k_2$, where k_i is an arc with endpoint w . If

$$r_i = \min \{n : P_n(k_i) = 1\},$$

then $e(k_1 \cup k_2) > \max(r_1, r_2)$.

Proof. Suppose $r_1 \geq r_2$, and let E be a 3-cell neighbourhood of w . Then let k'_1 be the subarc of k_1 which joins $k_1 \cap \text{Bd } E$ to w in the interior of E . k'_1 is of the same local type as k_1 .

Let $H_0 \supset H_1 \supset \dots$ be an admissible sequence for k of genus $e(k)$. Then there exists an index N such that only one handle of H_N meets $\text{Bd } E$. We put H_N and $\text{Bd } E$ into general position so that $H_N \cap \text{Bd } E$ is a family of discs on $\text{Bd } E$, and from the proof of (8.1) we may assume that each of these discs meets k in at most one point. Then the component of $\text{Cl}(H_N - H_N \cap \text{Bd } E)$ which contains w is a handlebody H_N^* of genus $e(k) - 1$ which contains only one point of k'_1 on its boundary.

We take another 3-cell $E' \subset \text{Int } H_N$, and construct a handlebody H' as in the proof of (8.1); we may assume $H' \subset \text{Int } H_N$. There exists an index M such that $H_M \subset \text{Int } H'$, and proceeding as above we obtain a handlebody $H_M^* \subset H_N^*$ of genus $e(k) - 1$ meeting k'_1 on its boundary in only one point.

So we construct a sequence $H_N^* \supset H_M^* \supset \dots$ which contains arbitrarily small handlebodies of genus $e(k) - 1$ which meet k'_1 on their boundaries in one point. That is, $e(k) - 1 \geq r_1$. Q.E.D.

COROLLARY. *Let k be an arc which is wild at one endpoint, with $P_r = 1$ and $P_{r-1} > 1$. Then the "mixed composite" of the trivial knot k_0 with respect to k [8, p. 336] has enclosure genus equal to $r + 1$.*

Proof. The mixed composite k^* is really the union of two copies of the arc k , so $e(k^*) \geq r + 1$. But one can construct an admissible sequence for k^* from a sequence of arbitrarily small handlebodies of genus r , by sewing on an extra handle. Therefore, $e(k^*) \leq r + 1$.

To prove that the lower bound of Theorem 2 is the best possible, it is only necessary to produce a family of arcs k_n , such that $P_n(k_n) = 1$ and $P_{n-1}(k_n) > 1$. For then the mixed composite of k_0 with respect to k_n has enclosure genus $n + 1$; by V.B.6 of [8], there is an epimorphism from $\pi_1(S - k_0) = Z$ to $\pi_1(S - k_n^*)$, so the Alexander module of $\pi_1(S - k_n^*) \simeq Z$ has unit nullity.

It would appear that the arcs A_n of Alford and Ball [1] have these properties, although I can find no proof of this.

REMARK. Our examples have all been "almost unknotted" in the sense of Fox [7], so this geometric property does not determine the nullity of a knot (cf. [3, §6]).

Added in proof. (i) There is a gap in the proof of Theorem 1, in that we must prove H admits all coproducts over infinite families of H -groups, and that the functor M preserves these coproducts. For the family $\{[G_i, \alpha_i], i \in I\}$, consider the coproduct $G = \bigoplus G_i$ in G , and define the map $\alpha: G \rightarrow H$ by $\alpha(g) = \prod_i \alpha_i(g)$. There is an obvious isomorphism from $M[G, \alpha]$ to $\bigoplus M[G_i, \alpha_i]$.

(ii) M. D. Taylor (doctoral dissertation, Florida State University, 1962) has exhibited a family of arcs k_n with $P_n(k_n) = 1$ and $P_{n-1}(k_n) > 1$. So the lower bound of Theorem 2 is the best possible.

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