SMOOTH HOMOLOGY SPHERES AND THEIR FUNDAMENTAL GROUPS

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Let M^n be a smooth homology n-sphere, i.e. a smooth n-dimensional manifold such that $H_*(M^n) \cong H_*(S^n)$. The fundamental group π of M satisfies the following three conditions:

(1) π has a finite presentation,

(2) $H_1(\pi) = 0$,

(3) $H_2(\pi) = 0$,

where $H_i(\pi)$ denotes the ith homology group of π with coefficients in the trivial $Z\pi$-module Z. Properties (1) and (2) are trivial and (3) follows from the theorem of Hopf [2] which asserts that $H_2(\pi) = H_2(M)/\rho\pi_2(M)$, where ρ denotes the Hurewicz homomorphism.

For $n > 4$ we will prove the following converse

THEOREM 1. *Let π be a group satisfying the conditions* (1), (2) *and* (3) *above, and let n be an integer greater than* 4. *Then, there exists a smooth manifold M^n such that $H_*(M^n) \cong H_*(S^n)$ and $\pi_1(M) \cong \pi$.*

The proof is very similar to the proof used for the characterization of higher knot groups in [5]. Compare also the characterization by K. Varadarajan of those groups π for which Moore spaces $M(\pi, 1)$ exist [9].

Not much seems to be known for $n \leqq 4$. If M^3 is a 3-dimensional smooth manifold with $H_*(M) \cong H_*(S^3)$, then $\pi = \pi_1(M)$ possesses a presentation with an equal number of generators and relators. (Take a Morse function f on M with a single minimum and a single maximum. Then f possesses an equal number of critical points of index 1 and 2.)

Also, under restriction to *finite* groups there is the following

THEOREM 2. *Let M^3 be a* 3*-dimensional manifold such that $H_*(M) \cong H_*(S^3)$. Suppose that $\pi_1(M)$ is finite. Then, either $\pi_1(M) = \{1\}$ or else, $\pi_1(M)$ is isomorphic to the binary icosahedral group with presentation*

$$(x, y; x^2 = y^3 = (xy)^5).$$

This is implicitly well known: The hypotheses imply that $\pi = \pi_1(M)$ is a group of fix-point free transformations of a homotopy 3-sphere. Any such group belongs to a list established by Suzuki [8] and even to the shorter list of Milnor [7]. The

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binary icosahedral group is the only (nontrivial) group in Milnor's list satisfying $H_1\pi=0$.

REMARK. Curiously enough, it seems that the trivial group and the binary icosahedral group are the only available examples of finite groups π satisfying $H_1\pi=0$ and having a presentation with an equal number of generators and relators.

For $n=4$, it follows from the proof below that every finitely presented group π with an equal number of generators and relators and satisfying $H_1\pi=0$ is the fundamental group of some homology 4-sphere. It seems unlikely that these conditions should characterize the fundamental groups of homology 4-spheres, but I do not know of a counterexample.

1. **Proof of Theorem 1.** We start with a finite presentation of π:

$$(x_1, \dots, x_\alpha; R_1, \dots, R_\beta)$$

and the manifold

$$M_0 = (S^1 \times S^{n-1}) \# \cdots \# (S^1 \times S^{n-1}),$$

connected sum of α copies of $S^1 \times S^{n-1}$. Choosing as usual a contractible open set U in M_0 as "base point", the condition $n>2$ implies that $\pi_1(M, U)$ is free on α generators. After identification of free generators of $\pi_1(M, U)$ with $x_1, \dots, x_\alpha$, the elements $R_1, \dots, R_\beta$ of the free group on $x_1, \dots, x_\alpha$ can be represented by disjoint differentiable imbeddings $\phi_i: S^1 \times D^{n-1} \to M_0$, $i=1, \dots, \beta$. Moreover, these can be chosen so that the spherical modification $\chi(\phi_1, \dots, \phi_\beta)$ can be framed (see [6]). The resulting manifold $M_1=\chi(M_0; \phi_1, \dots, \phi_\beta)$ is stably parallelizable as was M_0 and $\pi_1(M_1)\cong\pi$ since $n>3$.

From the homology exact sequences of the pairs $(M_0, \bigcup_i \phi_i(S^1 \times D^{n-1}))$ and $(M_1, \bigcup_i \phi'_i(D^2 \times S^{n-2}))$, where ϕ'_i denotes the natural imbedding $D^2 \times S^{n-2} \to M_1$ with $\phi'_i|S^1 \times S^{n-2}=\phi_i|S^1 \times S^{n-2}$, one concludes that $H_i(M_1)=0$ for $i\neq 0, 2, n-2, n$, and that $H_2(M_1)$ is free abelian of rank $\gamma=\beta-\alpha$. (Observe that $\beta \geqq \alpha$ since π abelianized is trivial.) Hence, there exist bases $\xi_1, \dots, \xi_\gamma$ of $H_2(M_1)$ and $\eta_1, \dots, \eta_\gamma$ of $H_{n-2}(M_1)$ respectively such that $\xi_i \cdot \eta_j=\delta_{ij}$, where $\xi_i \cdot \eta_j$ is the homology intersection number, and δ_{ij} is the Kronecker delta.

By the theorem of Hopf mentioned above, $H_2\pi=H_2(M_1)/\rho\pi_2(M_1)$, and so by condition (3) on π, the Hurewicz homomorphism $\rho: \pi_2(M_1) \to H_2(M_1)$ is surjective.

Since $n>4$, the classes $\xi_1, \dots, \xi_\gamma$ can be represented by disjoint differentiable imbeddings $f_i: S^2 \to M_1$, $i=1, \dots, \gamma$, and M_1 being stably parallelizable, these extend to disjoint imbeddings $\psi_i: S^2 \times D^{n-2} \to M_1$. It follows from the arguments in [6, §5] that the manifold $M=\chi(M_1; \psi_1, \dots, \psi_\gamma)$ obtained by spherical modification is a homology sphere with $\pi_1(M)\cong\pi$.

2. Many of the homology spheres occurring in the literature are constructed as the boundary of a contractible manifold. We investigate this question whether, in general, homology spheres bound contractible manifolds.

The following well-known construction provides an example of a 3-dimensional homology sphere which does not bound a contractible manifold.

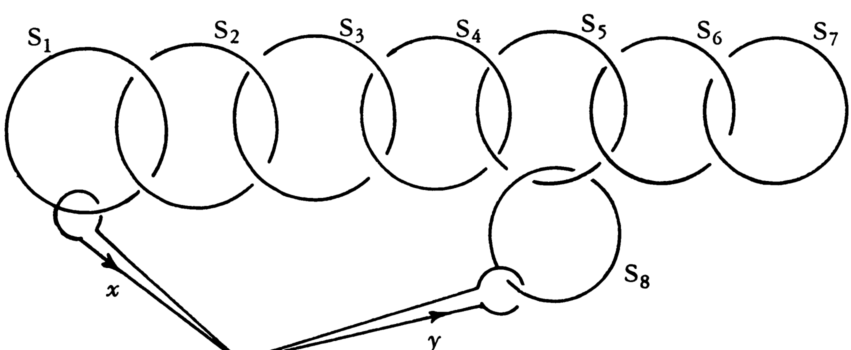
Let $W = D^4 + (\phi_1) + \cdots + (\phi_8)$ be the manifold obtained from the 4-disc by attaching eight handles of type 2 using unknotted imbeddings (with disjoint images) $\phi_i \colon S^1 \times D^2 \to S^3$, such that the matrix of linking numbers

$$L(\phi_i(S^1 \times x_0), \phi_j(S^1 \times x_1))$$

with $x_0 \neq x_1$, $x_0, x_1 \in D^2$, is:

$$L = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

This matrix is unimodular and has signature 8. The picture shows the images



$S_i = \phi_i(S^1 \times 0)$ in $R^3 = S^3 - \text{point}$. From the handle decomposition of W one sees that $\pi_1 W = \{1\}$, and $H_2 W$ is free abelian on 8 generators. An easy calculation shows that $\pi_1(bW)$ is the group with presentation:

$$(x, y; x^5 = y^2 = (x^{-1}y)^3),$$

where x and y are the classes of the loops shown on the picture. Hence, bW is a homology sphere. If bW were the boundary of a contractible manifold V, one could form the closed 4-manifold $M = W \cup V$, and the above matrix L would be the matrix of intersection numbers of $H_2 M$. From L one reads off the Stiefel-Whitney class $w_2 M = 0$ and the signature $\sigma(M) = 8$, contradicting Rohlin's theorem which says that if $w_2M=0$ for a closed orientable 4-manifold M, then $\sigma(M)\equiv 0$ mod 16. (Cf. [4].)

In contrast, we have

THEOREM 3. *Every 4-dimensional homology sphere bounds a contractible manifold. If M^n is a smooth oriented homology sphere with $n\geqq 5$, there exists a unique smooth homotopy sphere Σ^n_M such that $M^n \# \Sigma^n$ bounds a contractible smooth manifold.*

This has been obtained independently by the Hsiang brothers [3].

Let first M^4 be a smooth 4-dimensional homology sphere which we think of as imbedded in S^{k+4} with k large. It is easy to see that M has trivial normal bundle. Indeed, the only obstruction to trivializing the normal bundle is an element $\mathcal{O}\in H^4(M;\pi_3(SO_k))=Z$. It is known that $2\mathcal{O}=\pm p_1[M]$, where p_1 is the Pontryagin class. (Cf. e.g. [4].) The latter vanishes by the Thom-Hirzebruch formula $\frac{1}{3}p_1[M]=\sigma(M)=0$. Now, choosing a trivialization of the normal bundle of M^4 in S^{k+4}, the Thom construction provides an element in $\pi_{k+4}(S^k)$. Since $\pi_{k+4}(S^k)=0$, we see that M^4 is the boundary of a parallelizable manifold W^5. This manifold can be modified to get a new 5-manifold V^5 with $bV^5=bW^5=M^4$ such that $\pi_1V=\{1\}$ and $\pi_2V=0$. (Cf. [6, §5].) Clearly, V is contractible.

Assume then that $n\geqq 5$.

We associate with the given smooth homology n-sphere M^n some smooth homotopy sphere Σ^n. Observe that M^n is stably parallelizable. Since

$$H^i(M;\pi_{i-1}(SO))=0 \qquad \text{for } i<n,$$

there is only one possible obstruction $\mathcal{O}\in H^n(M;\pi_{n-1}(SO))=\pi_{n-1}(SO)$ to producing a trivialization of the stable tangent bundle of M. It is known that $\mathcal{O}$ belongs to the kernel of the homomorphism $J:\pi_{n-1}(SO)\to\Pi_{n-1}$, where $\Pi_{n-1}=\pi_{n+k-1}(S^k)$, k large. Thus $\mathcal{O}=0$ by the same arguments as in [6, §3].

We obtain Σ^n from M^n by framed surgery in dimensions 1 and 2. Let $x_1,\ldots,x_\alpha$ be a finite set of generators of $\pi_1(M,U)$, where U is a contractible open "base set" in M. Let $\phi_1,\ldots,\phi_\alpha$ be smooth imbeddings of $S^1\times D^{n-1}$ into M^n with disjoint images representing $x_1,\ldots,x_\alpha$ respectively. We use $\phi_1,\ldots,\phi_\alpha$ to attach α handles of type 2 to $I\times M$ along $(1)\times M$, where $I=[0,1]$. Let

$$V_0=I\times M+(\phi_1)+\cdots+(\phi_\alpha)$$

be the resulting $(n+1)$-manifold. We may assume that the imbeddings $\phi_1,\ldots,\phi_\alpha$ have been chosen so that V_0 is parallelizable. The manifold $N=bV_0-(0)\times M$ will be called the right-hand boundary of V_0. It is easily checked that $\pi_1N=\pi_1V_0=\{1\}$, and $H_iN=0$ for $3\leqq i\leqq n-3$ (if $n\geqq 6$). The groups H_2N and $H_{n-2}N$ are free abelian of rank α and the inclusion $N\subsetneq V_0$ induces an isomorphism $H_2N\cong H_2V_0$.

Let $\xi_1,\ldots,\xi_\alpha$ and $\eta_1,\ldots,\eta_\alpha$ be bases of H_2N and $H_{n-2}N$ respectively such that $\xi_i\cdot\eta_j=\delta_{ij}$. Then, representing the classes $\xi_1,\ldots,\xi_\alpha$ by disjoint differentiable imbeddings $\psi_1,\ldots,\psi_\alpha$ of $S^2\times D^{n-2}$ into N we construct

$$V_1=I\times M+(\phi_1)+\cdots+(\phi_\alpha)+(\psi_1)+\cdots+(\psi_\alpha)$$

by attaching α handles of type 3 to V_0 along N using $\psi_1, \ldots, \psi_\alpha$. Again this can be done so that V_1 is parallelizable. Now, the right-hand boundary $bV_1-(0)\times M$ is a smooth homotopy n-sphere Σ^n. We think of it as oriented so that the (oriented) boundary of V_1 is $-\Sigma^n-(0)\times M$, where the orientation of V_1 is given by the one of $I\times M$.

Claim: $M^n \# \Sigma^n$ bounds a contractible manifold. Indeed, let $t: (I\times D^n, bI\times D^n) \to (V_1, bV_1)$ be a smooth imbedding with $t(\text{int } I\times D^n)\subset \text{int } V_1$, then after rounding off corners we get a smooth contractible manifold $V=V_1-t(I\times \text{int } D^n)$ whose boundary is diffeomorphic to $M\#\Sigma$.

It remains to prove the uniqueness of Σ. Let Σ_1 be a homotopy n-sphere such that $M\#\Sigma_1$ is the boundary of a contractible manifold W. Using a connected sum of V_1 (as constructed above) and $I\times\Sigma_1$ along $t(I\times S^{n-1})$ and $I\times\sigma^{n-1}$, where σ^{n-1} is the boundary of a smooth n-disc δ^n in Σ_1, one gets a manifold W' whose homotopy type is S^n, and $bW'=(-\Sigma)\#\Sigma_1+(-M)\#\Sigma_1$. ($W'$ is obtained from the disjoint union

$$\{V_1-t(I\times(0))\}+\{I\times\Sigma_1-I\times(0)\}$$

under the identification of $t(x, ry)\in t(I\times D^n)$ with $(x, (1-r)y)\in I\times\delta^n$ for $x\in I$, $y\in S^{n-1}$ and $0<r<1$.)

Now, paste the manifold W along the left boundary $((0)\times M)\#\Sigma_1$ of W' by the identity diffeomorphism

$$bW = M\#\Sigma_1 \to ((0)\times M)\#\Sigma_1.$$

The resulting union $W\cup W'$ is a contractible manifold as follows easily using the van Kampen and Mayer-Vietoris theorems. The boundary of $W\cup W'$ is $(-\Sigma)\#\Sigma_1$. Therefore, Σ and Σ_1 are h-cobordant, and thus diffeomorphic since we assume $\dim\Sigma=\dim\Sigma_1\geqq 5$.

COROLLARY. *Every combinatorial homology sphere K^n of dimension $n\neq 3$ is the boundary of a contractible combinatorial manifold.*

By Hirsch's obstruction theory [1, Theorem 3.1], every combinatorial manifold K^n admits a smoothness structure in the neighborhood of its 7-skeleton. We need a smoothness structure in a neighborhood N of the 2-skeleton of K^n. Then N is parallelizable and we can apply the above surgery arguments requiring the imbeddings to have their images in the smooth subset. We obtain a manifold W^{n+1} such that $bW=\Sigma^n+M^n$, where Σ^n is a combinatorial homotopy n-sphere and the inclusion $\Sigma\subset W$ is a homotopy equivalence. Since $n\geqq 5$, Σ^n is PL-homeomorphic to $b\Delta^{n+1}$, and pasting Δ^{n+1} to W^{n+1} by such a homeomorphism provides a contractible manifold V^{n+1} with boundary M^n.

Thus, as a by-product, we have proved the probably well-known fact that every combinatorial homology sphere admits a smoothness structure.

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