Let $M^n$ be a smooth homology $n$-sphere, i.e. a smooth $n$-dimensional manifold such that $H_*(M^n) \cong H_*(S^n)$. The fundamental group $\pi$ of $M$ satisfies the following three conditions:

1. $\pi$ has a finite presentation,
2. $H_1(\pi) = 0$,
3. $H_2(\pi) = 0$,

where $H_i(\pi)$ denotes the $i$th homology group of $\pi$ with coefficients in the trivial $\mathbb{Z}_n$-module $\mathbb{Z}$. Properties (1) and (2) are trivial and (3) follows from the theorem of Hopf [2] which asserts that $H_2(\pi) = H_2(M) / \rho \pi_2(M)$, where $\rho$ denotes the Hurewicz homomorphism.

For $n > 4$ we will prove the following converse

**Theorem 1.** Let $\pi$ be a group satisfying the conditions (1), (2) and (3) above, and let $n$ be an integer greater than 4. Then, there exists a smooth manifold $M^n$ such that $H_*(M^n) \cong H_*(S^n)$ and $\pi_1(M) \cong \pi$.

The proof is very similar to the proof used for the characterization of higher knot groups in [5]. Compare also the characterization by K. Varadarajan of those groups $\pi$ for which Moore spaces $M(n, 1)$ exist [9].

Not much seems to be known for $n \leq 4$. If $M^3$ is a 3-dimensional smooth manifold with $H_*(M) \cong H_*(S^3)$, then $\pi = \pi_1(M)$ possesses a presentation with an equal number of generators and relators. (Take a Morse function $f$ on $M$ with a single minimum and a single maximum. Then $f$ possesses an equal number of critical points of index 1 and 2.) Also, under restriction to finite groups there is the following

**Theorem 2.** Let $M^3$ be a 3-dimensional manifold such that $H_*(M) \cong H_*(S^3)$. Suppose that $\pi_1(M)$ is finite. Then, either $\pi_1(M) = \{1\}$ or else, $\pi_1(M)$ is isomorphic to the binary icosahedral group with presentation

$$(x, y; x^2 = y^3 = (xy)^5).$$

This is implicitly well known: The hypotheses imply that $\pi = \pi_1(M)$ is a group of fix-point free transformations of a homotopy 3-sphere. Any such group belongs to a list established by Suzuki [8] and even to the shorter list of Milnor [7].

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binary icosahedral group is the only (nontrivial) group in Milnor's list satisfying
\( H_1\pi = 0 \).

**Remark.** Curiously enough, it seems that the trivial group and the binary
icosahedral group are the only available examples of finite groups \( \pi \) satisfying
\( H_1\pi = 0 \) and having a presentation with an equal number of generators and relators.

For \( n=4 \), it follows from the proof below that every finitely presented group \( \pi \)
with an equal number of generators and relators and satisfying \( H_1\pi = 0 \) is the
fundamental group of some homology 4-sphere. It seems unlikely that these con-
ditions should characterize the fundamental groups of homology 4-spheres, but I
do not know of a counterexample.

1. **Proof of Theorem 1.** We start with a finite presentation of \( \pi \):

\[
(x_1, \ldots, x_\alpha; R_1, \ldots, R_\beta)
\]

and the manifold

\[
M_0 = (S^1 \times S^{n-1}) \# \cdots \# (S^1 \times S^{n-1}),
\]

connected sum of \( \alpha \) copies of \( S^1 \times S^{n-1} \). Choosing as usual a contractible open
set \( U \) in \( M_0 \) as "base point," the condition \( n>2 \) implies that \( \pi_1(M, U) \) is free on \( \alpha \)
generators. After identification of free generators of \( \pi_1(M, U) \) with \( x_1, \ldots, x_\alpha \), the
elements \( R_1, \ldots, R_\beta \) of the free group on \( x_1, \ldots, x_\alpha \) can be represented by disjoint
differentiable imbeddings \( \phi_i : S^1 \times D^{n-1} \to M_0, i=1, \ldots, \beta \). Moreover, these can
be chosen so that the spherical modification \( \chi(\phi_1, \ldots, \phi_\beta) \) can be framed (see [6]).
The resulting manifold \( M_1 = \chi(M_0; \phi_1, \ldots, \phi_\beta) \) is stably parallelizable as was \( M_0 \)
and \( \pi_1(M_1) \cong \pi \) since \( n>3 \).

From the homology exact sequences of the pairs \((M_0, \bigcup_i \phi_i(D^2 \times S^{n-2}))\) and
\((M_1, \bigcup_i \phi_i(D^2 \times S^{n-2}))\), where \( \phi'_i \) denotes the natural imbedding \( D^2 \times S^{n-2} \to M_1 \)
with \( \phi'_i|S^1 \times S^{n-2} = \phi_i|S^1 \times S^{n-2} \), one concludes that \( H_i(M_1) = 0 \) for \( i \neq 0, 2, n-2, n \),
and that \( H_2(M_1) \) is free abelian of rank \( \gamma = \beta - \alpha \). (Observe that \( \beta \geq \alpha \) since \( \pi \)
abelianized is trivial.) Hence, there exist bases \( \xi_1, \ldots, \xi_\gamma \) of \( H_2(M_1) \) and \( \eta_1, \ldots, \eta_\gamma \)
of \( H_{n-3}(M_1) \) respectively such that \( \xi_i \cdot \eta_j = \delta_{ij} \), where \( \xi_i \cdot \eta_j \) is the homology inter-
section number, and \( \delta_{ij} \) is the Kronecker delta.

By the theorem of Hopf mentioned above, \( H_2\pi = H_2(M_1) / \rho \pi_2(M_1) \), and so by
condition (3) on \( \pi \), the Hurewicz homomorphism \( \rho : \pi_2(M_1) \to H_2(M_1) \) is surjective.

Since \( n>4 \), the classes \( \xi_1, \ldots, \xi_\gamma \) can be represented by disjoint differentiable
imb eddings \( f_i : S^2 \to M_1, i=1, \ldots, \gamma \), and \( M_1 \) being stably parallelizable, these
extend to disjoint imbeddings \( \psi_i : S^2 \times D^{n-2} \to M_1 \). It follows from the arguments
in [6, §5] that the manifold \( M = \chi(M_1; \psi_1, \ldots, \psi_\gamma) \) obtained by spherical modification
is a homology sphere with \( \pi_1(M) \cong \pi \).

2. Many of the homology spheres occurring in the literature are constructed
as the boundary of a contractible manifold. We investigate this question whether,
in general, homology spheres bound contractible manifolds.
The following well-known construction provides an example of a 3-dimensional homology sphere which does not bound a contractible manifold.

Let \( W = D^4 + (\phi_1) + \cdots + (\phi_8) \) be the manifold obtained from the 4-disc by attaching eight handles of type 2 using unknotted imbeddings (with disjoint images) \( \phi_i: S^1 \times D^2 \to S^3 \), such that the matrix of linking numbers

\[
L(\phi(S^1 \times x_0), \phi(S^1 \times x_1))
\]

with \( x_0 \neq x_1, x_0, x_1 \in D^2 \), is:

\[
L = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2
\end{bmatrix}
\]

This matrix is unimodular and has signature 8. The picture shows the images \( S_i = \phi_i(S^1 \times 0) \) in \( R^3 = S^3 \) - point. From the handle decomposition of \( W \) one sees that \( \pi_1 W = \{1\} \), and \( H_2 W \) is free abelian on 8 generators. An easy calculation shows that \( \pi_1(bW) \) is the group with presentation:

\[
(x, y; x^5 = y^2 = (x^{-1}y)^3),
\]

where \( x \) and \( y \) are the classes of the loops shown on the picture. Hence, \( bW \) is a homology sphere. If \( bW \) were the boundary of a contractible manifold \( V \), one could form the closed 4-manifold \( M = W \cup V \), and the above matrix \( L \) would be the matrix of intersection numbers of \( H_2 M \). From \( L \) one reads off the Stiefel-Whitney class \( w_2 M = 0 \) and the signature \( \sigma(M) = 8 \), contradicting Rohlin’s theorem.
which says that if \( \omega_2 M = 0 \) for a closed orientable 4-manifold \( M \), then \( \alpha(M) \equiv 0 \mod 16 \). (Cf. [4].)

In contrast, we have

**Theorem 3.** Every 4-dimensional homology sphere bounds a contractible manifold. If \( M^n \) is a smooth oriented homology sphere with \( n \geq 5 \), there exists a unique smooth homotopy sphere \( \Sigma^5 \) such that \( M^n \# \Sigma^n \) bounds a contractible smooth manifold.

This has been obtained independently by the Hsiang brothers [3].

Let first \( M^4 \) be a smooth 4-dimensional homology sphere which we think of as imbedded in \( S^{k+4} \) with \( k \) large. It is easy to see that \( M \) has trivial normal bundle. Indeed, the only obstruction to trivializing the normal bundle is an element \( \theta \in H^4(M; \pi_3(SO_3)) = \mathbb{Z} \). It is known that \( 2\theta = \pm p_1[M] \), where \( p_1 \) is the Pontryagin class. (Cf. e.g. [4].) The latter vanishes by the Thom-Hirzebruch formula \( \frac{1}{2}p_1[M] = \alpha(M) = 0 \). Now, choosing a trivialization of the normal bundle of \( M^4 \) in \( S^{k+4} \), the Thom construction provides an element in \( \pi_{k+4}(S^8) \). Since \( \pi_{k+4}(S^8) = 0 \), we see that \( M^4 \) is the boundary of a parallelizable manifold \( W^5 \). This manifold can be modified to get a new 5-manifold \( V^5 \) with \( bV^5 = bW^5 = M^4 \) such that \( \pi_1 V = \{1\} \) and \( \pi_2 V = 0 \). (Cf. [6, §5].) Clearly, \( V \) is contractible.

Assume then that \( n \geq 5 \).

We associate with the given smooth homology \( n \)-sphere \( M^n \) some smooth homotopy sphere \( \Sigma^n \). Observe that \( M^n \) is stably parallelizable. Since

\[
H^i(M; \pi_{i-1}(SO)) = 0 \quad \text{for } i < n,
\]

there is only one possible obstruction \( \theta \in H^n(M; \pi_{n-1}(SO)) = \pi_{n-1}(SO) \) to producing a trivialization of the stable tangent bundle of \( M \). It is known that \( \theta \) belongs to the kernel of the homomorphism \( J: \pi_{n-1}(SO) \to \Pi_{n-1} \), where \( \Pi_{n-1} = \pi_{n+k-1}(S^k) \), \( k \) large. Thus \( \theta = 0 \) by the same arguments as in [6, §3].

We obtain \( \Sigma^n \) from \( M^n \) by framed surgery in dimensions 1 and 2. Let \( x_1, \ldots, x_a \) be a finite set of generators of \( \pi_1(M, U) \), where \( U \) is a contractible open “base set” in \( M \). Let \( \phi_1, \ldots, \phi_a \) be smooth imbeddings of \( S^1 \times D^{n-1} \) into \( M^n \) with disjoint images representing \( x_1, \ldots, x_a \) respectively. We use \( \phi_1, \ldots, \phi_a \) to attach \( a \) handles of type 2 to \( I \times M \) along \( (1) \times M \), where \( I = [0, 1] \). Let

\[
V_0 = I \times M + (\phi_1) + \cdots + (\phi_a)
\]

be the resulting \((n+1)\)-manifold. We may assume that the imbeddings \( \phi_1, \ldots, \phi_a \) have been chosen so that \( V_0 \) is parallelizable. The manifold \( N = bV_0 - (0) \times M \) will be called the right-hand boundary of \( V_0 \). It is easily checked that \( \pi_1 N = \pi_1 V_0 = \{1\} \), and \( \Pi_i N = 0 \) for \( 3 \leq i \leq n-3 \) if \( n \geq 6 \). The groups \( H_2 N \) and \( H_{n-2} N \) are free abelian of rank \( a \) and the inclusion \( N \hookrightarrow V_0 \) induces an isomorphism \( H_2 N \cong H_2 V_0 \).

Let \( \xi_1, \ldots, \xi_a \) and \( \eta_1, \ldots, \eta_a \) be bases of \( H_2 N \) and \( H_{n-2} N \) respectively such that \( \xi_i \eta_j = \delta_{ij} \). Then, representing the classes \( \xi_1, \ldots, \xi_a \) by disjoint differentiable imbeddings \( \psi_1, \ldots, \psi_a \) of \( S^2 \times D^{n-2} \) into \( N \) we construct

\[
V_1 = I \times M + (\psi_1) + \cdots + (\psi_a) + (\psi_1) + \cdots + (\psi_a)
\]
by attaching \(\alpha\) handles of type 3 to \(V_0\) along \(N\) using \(\psi_1, \ldots, \psi_\alpha\). Again this can be done so that \(V_1\) is parallelizable. Now, the right-hand boundary \(bV_1 - (0) \times M\) is a smooth homotopy \(n\)-sphere \(\Sigma^n\). We think of it as oriented so that the (oriented) boundary of \(V_1\) is \(-\Sigma^n - (0) \times M\), where the orientation of \(V_1\) is given by the one of \(I \times M\).

Claim: \(M^n \# \Sigma^n\) bounds a contractible manifold. Indeed, let \(t: (I \times D^n, bI \times D^n) \to (V_1, bV_1)\) be a smooth imbedding with \(t(\text{int } I \times D^n) \subset \text{int } V_1\), then after rounding off corners we get a smooth contractible manifold \(V = V_1 - t(\text{int } I \times D^n)\) whose boundary is diffeomorphic to \(M \# \Sigma\).

It remains to prove the uniqueness of \(\Sigma\). Let \(\Sigma_1\) be a homotopy \(n\)-sphere such that \(M \# \Sigma_1\) is the boundary of a contractible manifold \(W\). Using a connected sum of \(V_1\) (as constructed above) and \(I \times \Sigma_1\) along \(t(I \times S^{n-1})\) and \(I \times \sigma^{n-1}\), where \(\sigma^{n-1}\) is the boundary of a smooth \(n\)-disc \(\delta^n\) in \(\Sigma_1\), one gets a manifold \(W'\) whose homotopy type is \(\Sigma^n\), and \(bW' = (-\Sigma) \# \Sigma_1 + (-M) \# \Sigma_1\). \((W')\) is obtained from the disjoint union

\[
\{V_1 - t(I \times (0))\} \cup \{I \times \Sigma_1 - I \times (0)\}
\]

under the identification of \(t(x, ry) \in t(I \times D^n)\) with \((x, (1-r)y) \in I \times \delta^n\) for \(x \in I, y \in S^{n-1}\) and \(0 < r < 1\).

Now, paste the manifold \(W\) along the left boundary \((0) \times M\) \# \(\Sigma_1\) of \(W'\) by the identity diffeomorphism

\[
bW = M \# \Sigma_1 \to ((0) \times M) \# \Sigma_1.
\]

The resulting union \(W \cup W'\) is a contractible manifold as follows easily using the van Kampen and Mayer-Vietoris theorems. The boundary of \(W \cup W'\) is \((-\Sigma) \# \Sigma_1\). Therefore, \(\Sigma\) and \(\Sigma_1\) are \(h\)-cobordant, and thus diffeomorphic since we assume \(\dim \Sigma = \dim \Sigma_1 \geq 5\).

**Corollary.** Every combinatorial homology sphere \(K^n\) of dimension \(n \neq 3\) is the boundary of a contractible combinatorial manifold.

By Hirsch's obstruction theory [1, Theorem 3.1], every combinatorial manifold \(K^n\) admits a smoothness structure in the neighborhood of its 7-skeleton. We need a smoothness structure in a neighborhood \(N\) of the 2-skeleton of \(K^n\). Then \(N\) is parallelizable and we can apply the above surgery arguments requiring the imbeddings to have their images in the smooth subset. We obtain a manifold \(W^{n+1}\) such that \(bW = \Sigma^n + M^n\), where \(\Sigma^n\) is a combinatorial homotopy \(n\)-sphere and the inclusion \(\Sigma \subset W\) is a homotopy equivalence. Since \(n \geq 5\), \(\Sigma^n\) is PL-homeomorphic to \(b\Delta^n + S\), and pasting \(\Delta^n + S\) to \(W^{n+1}\) by such a homeomorphism provides a contractible manifold \(V^{n+1}\) with boundary \(M^n\).

Thus, as a by-product, we have proved the probably well-known fact that every combinatorial homology sphere admits a smoothness structure.
BIBLIOGRAPHY


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