THE AUTOMORPHISM GROUP OF A HOMOGENEOUS ALMOST COMPLEX MANIFOLD (1)

BY
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1. **Introduction.** Let M be a compact simply connected manifold of nonzero Euler characteristic that carries a homogeneous almost complex structure. We determine the largest connected group $A_0(M)$ of almost analytic automorphisms of M.

Our hypotheses represent M as a coset space G/K where G is a maximal compact subgroup of the Lie group $A_0(M)$ and K is a closed connected subgroup of maximal rank in G. In §2 we collect some information, decomposing $M=M_1\times\cdots\times M_t$ as a product of "irreducible" factors along the decomposition of G as a product of simple groups; then every invariant almost complex structure or riemannian metric decomposes and every invariant riemannian metric is hermitian relative to any invariant almost complex structure; furthermore the decomposition is independent of G in a certain sense. In §3 we choose an invariant riemannian metric and determine the largest connected groups $H_0(M_i)$ of almost hermitian isometries of the M_i. Then $A_0(M)$ is determined in §4. There it is shown that $A_0(M)=A_0(M_1)\times\cdots\times A_0(M_t)$, that $A_0(M_i)=H_0(M_i)$ if the almost complex structure on M_i is not integrable, and that $A_0(M_i)=H_0(M_i)^C$ if the almost complex structure on M_i is induced by a complex structure. $A_0(M)$ thus is a centerless semisimple Lie group whose simple normal analytic subgroups are just the $A_0(M_i)$.

2. **Decomposition.** Let M be an effective coset space of a compact connected Lie group G by a connected subgroup K of maximal rank. In other words $M=G/K$ is compact, simply connected and of nonzero Euler characteristic; G is a compact centerless semisimple Lie group, rank $K=$rank G, and K contains no simple factor of G. Then

$$\text{(2.1a)}\quad G = G_1\times\cdots\times G_t,\qquad K = K_1\times\cdots\times K_t \quad\text{and}\quad M = M_1\times\cdots\times M_t$$

where

$$\text{(2.1b)}\qquad G_i \text{ is simple},\qquad K_i = K\cap G_i \quad\text{and}\quad M_i = G_i/K_i.$$

G_i is a compact connected centerless simple Lie group, K_i is a connected subgroup of maximal rank, and $M_i=G_i/K_i$ is a simply connected effective coset space of nonzero Euler characteristic. The decomposition of M is unique up to order of the factors because it is determined by the decomposition of G.

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We call (2.1) the *canonical decomposition* of the coset space $M=G/K$. The factors $M_i=G_i/K_i$ are the *irreducible factors* of $M=G/K$. If there is just one irreducible factor, i.e. if G is simple, then we say that $M=G/K$ is *irreducible.*

2.2. PROPOSITION. *Let M be an effective coset space G/K where G is a compact connected Lie group and K is a connected subgroup of maximal rank. Let $M=M_1 \times \cdots \times M_t$ be the canonical decomposition into irreducible factors $M_i=G_i/K_i$.*

1. *The G-invariant almost complex structures J on M are just the $J_1\times\cdots\times J_t$ where J_i is a G_i-invariant almost complex structure on M_i.*

2. *The G-invariant riemannian metrics ds^2 on M are just the $ds_1^2\times\cdots\times ds_t^2$ where ds_i^2 is a G_i-invariant riemannian metric on M_i; there each (M_i, ds_i^2) is an irreducible riemannian manifold, so*

$$(M, ds^2) = (M_1, ds_1^2)\times\cdots\times(M_t, ds_t^2)$$

is the de Rham decomposition.

3. *Let J be a G-invariant almost complex structure on M. If ds^2 is a G-invariant riemannian metric, then it is the real part of a G-invariant almost hermitian (for J) metric h on M, and $h=h_1\times\cdots\times h_t$ where h_i is a G_i-invariant almost hermitian (for J_i) metric on M_i and ds_i^2 is the real part of h_i.*

Proof. The Lie algebras decompose uniquely as direct sums $\mathfrak{G}=\mathfrak{K}+\mathfrak{M}$ and $\mathfrak{G}_i=\mathfrak{K}_i+\mathfrak{M}_i$, $\mathfrak{K}=\sum\mathfrak{K}_i$ and $\mathfrak{M}=\sum\mathfrak{M}_i$, with $[\mathfrak{K},\mathfrak{M}]\subset\mathfrak{M}$ and $[\mathfrak{K}_i,\mathfrak{M}_i]\subset\mathfrak{M}_i$. Let Z be the center of K, so $\mathfrak{K}$ is the centralizer of Z in $\mathfrak{G}$. Then $Z=Z_1\times\cdots\times Z_t$ where Z_i is the center of K_i and $\mathfrak{K}_i$ is the centralizer of Z_i in $\mathfrak{G}_i$.

π denotes the representation of K on $\mathfrak{M}$ and π_i is the representation of K_i on $\mathfrak{M}_i$. Then $\pi=\pi_1\oplus\cdots\oplus\pi_t$. Let $X=X_1\cup\cdots\cup X_t$ be the set of nontrivial characters on Z such that

$$\mathfrak{M}^C = \sum_X \mathfrak{M}_\chi \quad\text{and}\quad \mathfrak{M}_i^C = \sum_{X_i}\mathfrak{M}_\chi \tag{2.3a}$$

where Z acts on $\mathfrak{M}_\chi$ by the character χ. Each $\mathfrak{M}_\chi$ is ad (K)-stable, so K acts on $\mathfrak{M}_\chi$ by a representation π_χ, and

$$\pi^C = \sum_X \pi_\chi \quad\text{and}\quad \pi_i^C = \sum_{X_i}\pi_\chi. \tag{2.3b}$$

The point [7, Theorem 8.13.3] is that

(2.3c) the π_χ are irreducible and mutually inequivalent.

We transform the complex decomposition (2.3) to a real decomposition. Let $X=S\cup T$, $S=S_1\cup\cdots\cup S_t$ and $T=T_1\cup\cdots\cup T_t$ where S_i consists of the nonreal characters in X_i and T_i consists of the real ones. By *real partition* of X we mean a disjoint $X=S'\cup S''\cup T$ where $S''=\bar{S}'$. If $\chi\in S_i$ then $\bar{\chi}\in S_i$; thus the real partition induces real partitions $X_i = S_i' \cup S_i'' \cup T_i$. If $|S| = 2n$ then X has 2^n real partitions. Now choose a real partition $X = S' \cup S'' \cup T$ and define

$$\chi \in S'\colon K \text{ acts on } \mathfrak{M}_\chi^R = \mathfrak{M} \cap (\mathfrak{M}_\chi + \mathfrak{M}_{\bar\chi}) \quad \text{by } \pi_\chi^R$$
$$\chi \in T\colon\ K \text{ acts on } \mathfrak{M}_\chi^R = \mathfrak{M} \cap \mathfrak{M}_\chi \qquad \text{by } \pi_\chi^R.$$

Then (2.3abc) becomes

(2.4a) $$\mathfrak{M} = \sum_{S'} \mathfrak{M}_\chi^R + \sum_T \mathfrak{M}_\chi^R \quad \text{and} \quad \mathfrak{M}_i = \sum_{S_i'} \mathfrak{M}_\chi^R + \sum_{T_i} \mathfrak{M}_\chi^R,$$

(2.4b) $$\pi = \sum_{S'} \pi_\chi^R + \sum_T \pi_\chi^R \quad \text{and} \quad \pi_i = \sum_{S_i'} \pi_\chi^R + \sum_{T_i} \pi_\chi^R,$$

(2.4c) the π_χ^R are real-irreducible and mutually inequivalent.

Let A be the commuting algebra of π on $\mathfrak{M}$. By (2.4c), $A = \sum_{S'} \boldsymbol{C} + \sum_T \boldsymbol{R}$, for π_χ^R has commuting algebra $\boldsymbol{C}$ if $\chi \in S'$, $\boldsymbol{R}$ if $\chi \in T$. Invariant almost complex structures are in obvious correspondence with elements of square $-I$ of the commuting algebra, which now are seen to exist if and only if T is empty, and (1) follows. Similarly, the decomposition of ds^2 in (2), and the existence and decomposition of h in (3), are immediate.

It remains only to show the (M_i, ds_i^2) irreducible as riemannian manifolds in (2). That fact is known [3, §5.1], but in our present context we can give a short proof for the convenience of the reader. If (M_i, ds_i^2) reduces, then it is a riemannian product $M' \times M''$ because it is complete and simply connected, so we have an ad (K_i)-stable decomposition $\mathfrak{M}_i = \mathfrak{M}' \oplus \mathfrak{M}''$ with the properties

$$[\mathfrak{M}', \mathfrak{M}''] \subset \mathfrak{K}_i, \qquad \mathfrak{M}'^C = \sum_{X'} \mathfrak{M}_\chi, \qquad \mathfrak{M}''^C = \sum_{X''} \mathfrak{M}_\chi, \qquad X_i = X' \cup X''.$$

Here X' and X'' are disjoint and self conjugate. If $\chi' \in X'$ and $\chi'' \in X''$ with $[\mathfrak{M}_{\chi'}, \mathfrak{M}_{\chi''}] \neq 0$, then $\chi'\chi'' = 1$ so $\chi' = \bar\chi'' \in X''$ which is absurd. Thus $[\mathfrak{M}', \mathfrak{M}''] = 0$, and it follows that the simple Lie algebra $\mathfrak{G}_i$ is direct sum of ideals

$$\mathfrak{G}' = \{\mathfrak{K}_i \cap [\mathfrak{M}', \mathfrak{M}']\} + \mathfrak{M}' \quad \text{and} \quad \mathfrak{G}'' = \{\mathfrak{K}_i \cap [\mathfrak{M}'', \mathfrak{M}'']\} + \mathfrak{M}''.$$

That being absurd, irreducibility is proved. Q.E.D.

2.5. Remark. In the notation of the proof of Proposition 2.2, M has a G-invariant almost complex structure if and only if $X = S$, and then those structures J correspond to the real partitions $X = S' \cup S''$ by: $\sum_{S'} \mathfrak{M}_\chi$ and $\sum_{S''} \mathfrak{M}_\chi$ are the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of J on $\mathfrak{M}^C$.

3. **Almost hermitian isometries.** Let M be a manifold with an almost hermitian metric h. Then $h = ds^2 + (-1)^{1/2}\omega$ where the riemannian metric ds^2 is the real part of h and $\omega(u, v) = ds^2(u, Jv)$ is the imaginary part; that determines the almost complex structure J. By *almost hermitian isometry* of (M, h) we mean a diffeomorphism that preserves h, i.e. that is a riemannian isometry of (M, ds^2) which preserves J.

Let $I(M)$ denote the (Lie) group of all isometries of (M, ds^2), $H(M)$ the closed subgroup consisting of those isometries that preserve J. Then $H(M)$ is the (Lie) group of all almost hermitian isometries of (M, h). In particular its identity component $H_0(M)$ is an analytic subgroup of the identity component $I_0(M)$ of $I(M)$. If $(M, h)=(M_1, h_1)\times\cdots\times(M_t, h_t)$ hermitian product, then the de Rham decomposition says that $I_0(M)$ preserves each noneuclidean factor, so those factors are stable under $H_0(M)$.

Let $M=G/K$ as in Proposition 2.2. Let h be a G-invariant almost hermitian metric on M. The canonical decomposition induces $(M, h)=(M_1, h_1)\times\cdots\times(M_t, h_t)$ hermitian product where each (M_i, ds_i^2), $ds_i^2=\operatorname{Re} h_i$, is an irreducible noneuclidean riemannian manifold. Thus $H_0(M)=H_0(M_1)\times\cdots\times H_0(M_t)$, and $H(M)$ is generated by its subgroup $H(M_1)\times\cdots\times H(M_t)$ and permutations of mutually isometric (M_i, h_i); so its determination is more or less reduced to the case where $M=G/K$ is irreducible. There the result is

3.1. Proposition. *Let M be an effective coset space G/K where G is a compact connected simple Lie group and K is a connected subgroup of maximal rank. Let h be a G-invariant almost hermitian metric on M, so $M=H_0(M)/B$ where $G\subset H_0(M)$ and $B\cap G=K$. If $G\neq H_0(M)$, then (M, h) is an irreducible hermitian symmetric space of compact type listed below.*

Case	G	K	$H_0(M)$	B	(M, h)
1	G_2	$U(2)$	$SO(7)$	$SO(5)\times SO(2)$	5-dimensional complex quadric
2	$Sp(r)/Z_2$	$Sp(r-1)\cdot U(1)$	$SU(2r)/Z_{2r}$	$U(2r-1)$	complex projective $(2r-1)$-space
3	$SO(2r+1)$	$U(r)$	$SO(2r+2)/Z_2$	$U(r+1)/Z_2$	unitary structures on R^{2r+2}
3′	$Spin(7)/Z_2$	$U(3)$	$SO(8)/Z_2$	$SO(6)\cdot SO(2)$	6-dimensional complex quadric

Remark 1. In the exceptional cases above, K is not R-irreducible on the tangent space, so M has another G-invariant almost hermitian metric for which $G=H_0(M)$.

Remark 2. The proof is easily reduced to the case where B is the centralizer of a toral subgroup of $H_0(M)$, and then the result can be extracted from [2, Table 5] and the Bott-Borel-Weil Theorem. But here it is convenient to reduce the proof to some classifications of Oniščik [4].

Proof. As M has nonzero Euler characteristic, B has maximal rank in $H_0(M)$, so $H_0(M)/B=G/K$ is one of the following entries in Oniščik's list [4, Table 7].

(i) $A_{2n-1}/A_{2n-2}\cdot T=C_n/C_{n-1}\cdot T$ (our Case 2),

(ii) $B_3/B_2\cdot T=G_2/A_1\cdot T$ (our Case 1),

(iii) $B_3/D_3=G_2/A_2$ (B_3 does not preserve J here),

(iv) $D_{n+1}/A_n \cdot T = B_n/A_{n-1} \cdot T$ (our Case 3),

(v) $D_4/D_3 \cdot T = B_3/A_2 \cdot T$ (our Case 3′).

The assertions follow with the observation that $\boldsymbol{H}_0(M)/B$ is an irreducible hermitian symmetric coset space of compact type in each of the admissible cases. Q.E.D.

4. **Almost analytic automorphisms.** Let M be a manifold with almost complex structure J. By *almost analytic automorphism* of M, we mean a diffeomorphism of M which preserves J. The set of all such diffeomorphisms forms a group $A(M)$. If M is compact, then [1] in the compact-open topology, $A(M)$ is a Lie transformation group of M. We denote its identity component by $A_0(M)$. If, further, we have an almost hermitian metric on M, then $\boldsymbol{H}(M)$ is a compact subgroup of $A(M)$. That will be our main tool in studying $A(M)$.

4.1. THEOREM. *Let $M = G/K$ be a simply connected effective coset space of nonzero Euler characteristic where G is a compact connected Lie group. Let J be a G-invariant almost complex structure on M. Let $M = M_1 \times \cdots \times M_t$ be the canonical decomposition into irreducible coset spaces, and decompose $J = J_1 \times \cdots \times J_t$ where J_i is a G_i-invariant almost complex structure on M_i. Then*

1. *$A_0(M) = A_0(M_1) \times \cdots \times A_0(M_t)$.*

2. *M has a G-invariant riemannian metric $ds^2 = ds_1^2 \times \cdots \times ds_t^2$ for which $\boldsymbol{H}_0(M)$ is a maximal compact subgroup of $A_0(M)$.*

3. *If J_i is integrable then $A_0(M_i) = \boldsymbol{H}_0(M_i)^C$. If J_i is not integrable then $A_0(M) = \boldsymbol{H}_0(M)$.*

Proof. For the second statement, enlarge G to a maximal compact subgroup H of $A_0(M)$ and choose an H-invariant riemannian metric ds^2 on M. Then $ds^2 = ds_1^2 \times \cdots \times ds_t^2$ as required, by Proposition 2.2, and $H = \boldsymbol{H}_0(M)$ by construction.

We simplify notation for the proofs of the first and third statement by enlarging G to $\boldsymbol{H}_0(M)$ and writing A for $A_0(M)$. That does not change the canonical decomposition of M, for the latter is the de Rham decomposition for ds^2 according to Proposition 2.2. Now $G/K = M = A/B$ where $G \subset A$ is a maximal compact subgroup and $K = G \cap B$.

We check that A is a centerless semisimple Lie group. If L is a closed normal analytic subgroup of A with $G \cap L$ discrete, then $G \cdot L \subset A$ is effective on

$$(G \cdot L)/(K \cdot L) = M, \text{ so } L = \{1\}.$$

Let L be the radical of A: now A is semisimple. Let $\mathfrak{L}$ be the orthocomplement of $\mathfrak{G}$ in a maximal compactly embedded subalgebra of $\mathfrak{A}$: now A has finite center, so the centerless group G contains the center of A, so A is centerless.

Let A^α, $1 \leqq \alpha \leqq r$, be the simple normal analytic subgroups of A. So $A = A^1 \times \cdots \times A^r$ with A^α centerless simple. Now $G = G^1 \times \cdots \times G^r$, $K = K^1 \times \cdots \times K^r$ and $M = M^1 \times \cdots \times M^r$ where

$$G^\alpha = G \cap A^\alpha, \qquad K^\alpha = K \cap G^\alpha, \qquad M^\alpha = G^\alpha/K^\alpha.$$

If $\alpha \neq \beta$ then A^α acts trivially on M^β. For every $a \in A^\alpha$ centralizes the transitive transformation group G^β of M^β, hence induces some transformation $\bar{a}$ of M^β that is trivial or fixed point free. As A^α is connected, $\bar{a}$ is homotopic to 1 so its Lefschetz number is the (nonzero) Euler characteristic of M^β; that shows $\bar{a}=1$. Now $M^\alpha = A^\alpha/B^\alpha$, $B^\alpha = B \cap A^\alpha$, with $B = B^1 \times \cdots \times B^r$.

According to Oniščik [5, Table 1] the only possibilities for $G^\alpha/K^\alpha = M^\alpha = A^\alpha/B^\alpha$, A^α noncompact, are given in the following table.

A^α	$M^\alpha = G^\alpha/K^\alpha$	Conditions
$\mathbf{SL}(2n, \mathbf{R})/\mathbf{Z}_2$	$\mathbf{SO}(2n)/\mathbf{SO}(2n_1) \times \cdots \times \mathbf{SO}(2n_p)$	$n = \sum n_i > 1$
$\mathbf{SL}(2n+1, \mathbf{R})$	$\mathbf{SO}(2n+1)/\mathbf{SO}(2n_1) \times \cdots \times \mathbf{SO}(2n_{p-1}) \times \mathbf{SO}(2n_p+1)$	$n = \sum n_i$
$\mathbf{GL}(n, \mathbf{Q})/\mathbf{Z}_2$	$\mathbf{Sp}(n)/\mathbf{Sp}(n_1) \times \cdots \times \mathbf{Sp}(n_p) \times \mathbf{U}(1)^q$	$n = q + \sum n_i$
$\mathbf{SO}(1, 2n-1)/\mathbf{Z}_2$	$\mathbf{SO}(2n-1)/\mathbf{SO}(2n_1) \times \cdots \times \mathbf{SO}(2n_p) \times \mathbf{U}(m_1) \times \cdots \times \mathbf{U}(m_q)$	$n-1$ $= \sum n_i + \sum m_j$
$\mathbf{E}_{6,C_4}/\mathbf{Z}_2$	$\mathbf{Sp}(4)/\mathbf{Sp}(2) \times \mathbf{Sp}(2)$ and $\mathbf{Sp}(4)/[\mathbf{Sp}(1)]^4$	none
$\mathbf{E}_{6,F_4}$	$\mathbf{F}_4/\mathbf{Spin}(9)$, $\mathbf{F}_4/\mathbf{Spin}(8)$, $\mathbf{F}_4/\mathbf{U}(4)$ and $\mathbf{F}_4/[\mathbf{SU}(2)]^4$	none
$(G^\alpha)^C$	G^α/K^α where K^α is the centralizer of a nontrivial toral subgroup of G^α	G^α compact centerless simple

Note that G^α is simple except in Case 1 with $n=2$. There M^α is the product of two Riemann spheres, so A^α is the product of two copies of $\mathbf{SL}(2, \mathbf{C})/\mathbf{Z}_2$, contradicting the table entry for A^α. Thus we always have G^α simple, so each M^α is an M_i, and the first statement of our theorem is proved with $A^\alpha = A_0(M^\alpha)$.

Now we may, and do, assume M irreducible. Thus A and G are simple.

4.2. Lemma. *The invariant almost complex structure J is integrable if and only if $A = G^C$. In that case B is a complex parabolic subgroup of A and J is induced either from the natural complex structure on A/B or from the conjugate structure.*

Proof of lemma. Let J be integrable; we check $\mathfrak{G}^C \subset \mathfrak{A}$. For if $\xi \in \mathfrak{G}$ and ξ^* denotes the holomorphic vector field induced on M, then $J(\xi^*)$ is holomorphic. Thus $\mathfrak{G}^C$ acts on M by $\xi + i\eta \to \xi^* + J(\eta^*)$, and this action integrates to G^C because M is compact; that shows $G^C \subset A$ so $\mathfrak{G}^C \subset \mathfrak{A}$.

Let $\mathfrak{A} = \mathfrak{G}^C$. As $\mathfrak{K}$ is its own normalizer in $\mathfrak{G}$ because it has maximal rank, $\mathfrak{B}$ is its own normalizer in $\mathfrak{A}$, so B is an $\mathbf{R}$-algebraic subgroup of A. Thus A has an Iwasawa decomposition GSN with $B = KSN$. As $\mathfrak{A} = \mathfrak{G}^C$, the group S^C is a complex Cartan subgroup of A, so N is a complex unipotent subgroup. Now $K^C S^C N$ is the complex group generated by B and it has intersection K with G; thus $M = A/B \to A/K^C S^C N = G/K$ is trivial so B is a complex subgroup of A. As A/B is compact now B is a complex parabolic subgroup.

Decompose $B=B^r \cdot B^u$ into reductive and unipotent parts. Let Z be the identity component of the center of B^r, complex subtorus of S^C. Let D be the set of characters $\chi \neq 1$ on Z that are restrictions of positive roots, so $\mathfrak{B}^u = \sum_D \mathfrak{A}_\chi$. Define $\mathfrak{B}^{-u} = \sum_D \mathfrak{A}_{-\chi}$ so that $\mathfrak{A}$ is the direct sum of its subspaces $\mathfrak{B}^r$, $\mathfrak{B}^u$ and $\mathfrak{B}^{-u}$. $\mathfrak{G} \cap (\mathfrak{B}^u + \mathfrak{B}^{-u})$ represents the real tangent space of M, and $\mathfrak{B}^u + \mathfrak{B}^{-u}$ represents the complexified tangent space. If $\pm\chi \in D$, then $\mathfrak{A}_\chi$ is an irreducible representation space of B^r, so J acts on $\mathfrak{A}_\chi$ either as $\sqrt{-1}$ or as $-\sqrt{-1}$. Let $\mathfrak{L}^+$ (resp. $\mathfrak{L}^-$) denote the image in $\mathfrak{A}/\mathfrak{B}$ of the $\mathfrak{A}_\chi$, $-\chi \in D$, on which J acts as $\sqrt{-1}$ (resp. $-\sqrt{-1}$). Then $\mathrm{ad}(\mathfrak{B}) \cdot \mathfrak{L}^\pm \subset \mathfrak{L}^\pm$ by invariance of J under B. If ν is the restriction to Z of the highest root, then $\mathfrak{A}/\mathfrak{B} = \sum_{n \geqq 0} \mathrm{ad}(\mathfrak{B})^n \cdot (\mathfrak{A}_{-\nu} \bmod \mathfrak{B})$, because $\mathfrak{A}$ is simple, so $\mathfrak{A}/\mathfrak{B}$ is the one of $\mathfrak{L}^+$ or $\mathfrak{L}^-$ into which $\mathfrak{A}_{-\nu}$ maps. Thus either J acts on $\mathfrak{B}^{-u}$ as $\sqrt{-1}$ and the natural complex structure of A/B induces J, or J acts on $\mathfrak{B}^{-u}$ as $-\sqrt{-1}$ and the natural structure induces $-J$. In either case J is integrable.

In general suppose $\mathfrak{G}^C \subset \mathfrak{A}$. Then $M = G^C/B \cap G^C$ is a complex flag manifold on which A is the largest connected group of analytic automorphisms. Thus A is a centerless complex semisimple group, hence the complexification of its maximal compact subgroup G.

Lemma 4.2 is proved.

4.3. Lemma. *If B^C is parabolic in A^C, then J is integrable so $A=G^C$.*

Proof of lemma. J is an element of square $-I$ in the commuting algebra of $\mathrm{ad}(\mathfrak{B})$ on $\mathfrak{A}/\mathfrak{B}$. Thus it induces an element J^C of square $-I$ in the commuting algebra of $\mathrm{ad}(\mathfrak{B}^C)$ on $\mathfrak{A}^C/\mathfrak{B}^C$. Now suppose B^C parabolic in A^C, so $M^C = A^C/B^C$ is compact and of positive Euler characteristic with invariant almost complex structure J^C.

If A is complex then $A=G^C$ and Lemma 4.2 says that J is integrable. Thus we may assume A not complex so that A^C is simple. Then Lemma 4.2 says that J^C is integrable, and in fact that either J^C or $-J^C$ is induced by the natural complex structure on A^C/B^C. Replace J by $-J$ if necessary; that does not alter integrability of J, but it replaces J^C by $-J^C$, allowing us to assume J^C induced by the natural complex structure of A^C/B^C.

Decompose $B=B^r \cdot B^u$ into reductive and unipotent parts, so $\mathfrak{B} = \mathfrak{B}^r + \mathfrak{B}^u$ and $\mathfrak{A} = \mathfrak{B} + \mathfrak{B}^{-u}$ where $\mathfrak{B}^{\pm u}$ are subalgebras normalized by $\mathfrak{B}^r$. Let $\mathfrak{B}^{-u}$ represent the real tangent space to M. Note that J^C acts on $(\mathfrak{B}^{-u})^C$ as $\sqrt{-1}$. That contradicts our arrangement that the action of J^C on $(\mathfrak{B}^{-u})^C$ is induced by the action of J on $\mathfrak{B}^{-u}$. Thus A cannot be noncomplex. Lemma 4.3 is proved.

We complete the proof of Theorem 4.1. As in the second paragraph of the proof of Lemma 4.2, B is a real algebraic subgroup of A, so there is a semidirect product decomposition $B=B^r \cdot B^u$ into reductive and unipotent parts. If rank $B^r <$ rank A, then any Cartan subalgebra of $\mathfrak{A}$ has an element ξ not contained in any isotropy subalgebra of $\mathfrak{A}$ on M so it induces a nonvanishing vector field ξ^* on M. The existence of a nonvanishing vector field ξ^* says that M has Euler characteristic zero. That contradiction proves rank $B^r = \text{rank } A$.

Let σ be the Cartan involution of $\mathfrak{A}$ with fixed point set $\mathfrak{G}$ and let $\mathfrak{A} = \mathfrak{G} + \mathfrak{P}$ be the Cartan decomposition. We may assume $\sigma(\mathfrak{B}^r) = \mathfrak{B}^r$, so $\mathfrak{B}^r = \mathfrak{K} + (\mathfrak{P} \cap \mathfrak{B}^r)$. That gives compact real forms

$$\mathfrak{A}_c = \mathfrak{G} + \sqrt{-1}\,\mathfrak{P} \quad \text{and} \quad \mathfrak{B}_c^r = \mathfrak{K} + \sqrt{-1}\,(\mathfrak{P} \cap \mathfrak{B}^r).$$

Let A_c denote the centerless group with Lie algebra $\mathfrak{A}_c$ and let B_c^r be the analytic subgroup for $\mathfrak{B}_c^r$. Then rank $B_c^r = \text{rank } B^r = \text{rank } A = \text{rank } A_c$ tells us that $X = A_c/B_c^r$ is a compact simply connected manifold of positive Euler characteristic. If $A = G$ then $B = B^r = K$, so $A_c = G$ and $B_c^r = K$, whence $X = M$.

As in the second paragraph of the proof of Lemma 4.2 we have Iwasawa decompositions $A = GSN$ and $B = KSN$. Choose a torus subgroup $T \subset K$ such that $H = T \times S \subset B^r$ is a Cartan subgroup of A. Let Δ be the root system. Now $\Delta = D \cup E \cup -E$ disjoint, and $\mathfrak{A} = \mathfrak{B}^r + \mathfrak{B}^u + \mathfrak{B}^{-u}$ direct, where

$$\mathfrak{B}^r = \mathfrak{H} + \mathfrak{A} \cap \Big\{\sum_D \mathfrak{A}_\phi\Big\}, \qquad \mathfrak{B}^u = \mathfrak{A} \cap \Big\{\sum_E \mathfrak{A}_\phi\Big\}, \qquad B^{-u} = \mathfrak{A} \cap \Big\{\sum_E \mathfrak{A}_{-\phi}\Big\}.$$

Observe that σ interchanges $\mathfrak{B}^u$ and $\mathfrak{B}^{-u}$. For $\mathfrak{B}^u \subset N$ because $N = N' \cdot B^u$ where $B^r = KSN'$, and the dual space of $\mathfrak{S}$ has an ordering such that

$$\mathfrak{N}^C = \sum_{\phi|_{\mathfrak{S}} > 0} \mathfrak{A}_\phi, \quad \text{and} \quad \phi|_{\mathfrak{S}} > 0 \quad \text{iff } \sigma\phi|_{\mathfrak{S}} < 0.$$

View the invariant almost complex structure J of M as an element of square $-I$ in the commuting algebra of ad $(\mathfrak{B})$ on $\mathfrak{A}/\mathfrak{B}$, hence in the commuting algebra of ad $(\mathfrak{B}^r)$ on $\mathfrak{B}^{-u} \cong \mathfrak{A}/\mathfrak{B}$; then extend J to an element J' of square $-I$ in the commuting algebra of ad $(\mathfrak{B}^r)$ on $\mathfrak{B}^u + \mathfrak{B}^{-u}$ by the formula

$$J'(\xi + \eta) = \sigma J(\sigma\xi) + J(\eta) \quad \text{where } \xi \in \mathfrak{B}^u,\ \eta \in \mathfrak{B}^{-u}.$$

Now J' is an A-invariant σ-invariant almost complex structure on A/B^r, so [6, Proposition 7.7] it defines an A_c-invariant σ-invariant almost complex structure on A_c/B_c^r. We have proved that $X = A_c/B_c^r$ has an invariant almost complex structure.

Suppose $A \neq G$. Note that [6, Theorem 4.10] eliminates lines 5 and 6 of the Oniščik table above, so either $A = G^C$ or A is absolutely simple and of classical type. Suppose $A \neq G^C$ so A_c is simple and of classical type. Then [6, Theorem 4.10] shows that B_c^r is the centralizer of a torus in A_c. Let $\mathfrak{Z}_c$ denote the center of $\mathfrak{B}_c^r$. Then $\sigma(\mathfrak{B}_c^r) = \mathfrak{B}_c^r$ implies $\sigma(\mathfrak{Z}_c) = \mathfrak{Z}_c$, so $\mathfrak{Z}_c = \mathfrak{U} + (-1)^{1/2}\mathfrak{V}$ with $\mathfrak{U} \subset \mathfrak{K}$ and $\mathfrak{V} \subset \mathfrak{P} \cap \mathfrak{B}^r$. Now $\mathfrak{B}^r$ has center $\mathfrak{Z} = \mathfrak{U} + \mathfrak{V} \subset \mathfrak{T} + \mathfrak{S} = \mathfrak{H}$, and $\mathfrak{B}^r$ is the centralizer of $\mathfrak{Z}$ in $\mathfrak{A}$. We order the root system Δ so that a root $\phi > 0$ whenever $\phi|_{\mathfrak{Z}} \neq 0$ and $\phi|_{\mathfrak{S}} > 0$. Then $\mathfrak{B}^C$ contains the Borel subalgebra $\mathfrak{H}^C + \sum_{\phi > 0} \mathfrak{A}_\phi$ of $\mathfrak{A}^C$ for that ordering, so $\mathfrak{B}^C$ is a parabolic subalgebra of $\mathfrak{A}^C$. Then Lemma 4.3 says $A = G^C$. We have proved that $A \neq G$ implies $A = G^C$.

If J is integrable then Lemma 4.2 says $A = G^C$. If J is not integrable then Lemma 4.2 says $A \neq G^C$, so we cannot have $A \neq G$, and that forces $A = G$. Theorem 4.1 is proved. Q.E.D.

4.3. REMARK. Theorem 4.1 extends the scope of [8, Theorem 17.4(3)], but that result remains incomplete because, as remarked at the end of [8, §17], it is not known whether

$$A_0(\boldsymbol{E}_6/\mathrm{ad}\ \boldsymbol{SU}(3)) \text{ is } \boldsymbol{E}_6 \text{ rather than } \boldsymbol{E}_6^{\boldsymbol{C}}$$

or whether

$$A_0(\boldsymbol{SO}(n^2-1)/\mathrm{ad}\ \boldsymbol{SU}(n)) \text{ is } \boldsymbol{SO}(n^2-1) \text{ rather than}$$
$$\boldsymbol{SO}(n^2-1, \boldsymbol{C}), \qquad \boldsymbol{SL}(n^2-1, \boldsymbol{R}), \quad \text{or} \quad \boldsymbol{SO}(1, n^2-1).$$

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