

SYSTEMS OF SINGULAR INTEGRAL OPERATORS ON SPHERES

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I. This paper has two purposes: to develop a theory of special functions for $SO(n)/SO(n-2)$; and to apply it to the study of systems of singular integral operators on S^{n-1} having specified transformation properties under the action of rotations. The results on special functions for $SO(n)/SO(n-2)$ enable us to classify all "irreducible" systems and to decompose an arbitrary singular operator as a sum of operators equivalent (modulo the smoothing operators) to an element of one of the irreducible systems.

To motivate the study of systems of operators, and to illustrate the point of view adopted, we ask the following question about operators on $L^p(E_n)$. Which bounded operators on each $L^p(E_n)$, $1 < p < \infty$, or merely on $L^2(E_n)$, enjoy the invariance properties given below⁽¹⁾?

- (1) $\tau_h T = T \tau_h$ for each $h \in E_n$, where $(\tau_h f)(x) = f(x - h)$;
- (2) $\delta_\lambda T = T \delta_\lambda$ for each real number $\lambda > 0$, where $(\delta_\lambda f)(x) = f(\lambda^{-1}x)$;
- (3) $L_a T = T L_a$ for each rotation a , where $(L_a f)(x) = f(a^{-1}x)$.

Conditions (1) and (2) show that $(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$, where $m(\xi)$ is a homogeneous function of degree 0, and \hat{f} is the Fourier transform of f . If condition (3) is satisfied then m must be constant, and T a scalar multiple of the identity. Therefore, to obtain a nontrivial answer we must relax at least one of the three conditions. To discuss translation-invariant singular operators we keep (1) and (2). (Keeping (1) and (3) leads to the theory of "variable-kernel" operators, to which we shall return shortly.) We relax (3) by asking instead if there is a (complex) vector space V of operators such that the action $T \xrightarrow{a} L_a T L_a^{-1}$ yields an (irreducible) representation $a \rightarrow R_a$ of the rotation group $SO(n)$ on V . In other words, as an equality in V ,

$$(3') \quad L_a T L_a^{-1} = R_a T$$

for each $T \in V$ and $a \in SO(n)$. If $a \rightarrow R_a$ is the trivial representation given by $R_a = I$ for each a , we recover condition (3). If $a \rightarrow R_a^1 = a$ is the *standard* representation of $SO(n)$, the system spanned by the Riesz operators R_1, \dots, R_n defined by $(R_j f)^\wedge(\xi) = \xi_j |\xi|^{-1} \hat{f}(\xi)$ satisfies (1), (2) and (3'). Moreover, as a consideration of $(Tf)^\wedge(\xi)$ shows, any system satisfying (1), (2) and (3') for the standard representation must coincide with the span of the Riesz operators. As a generalization of the

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⁽¹⁾ The question and some of the discussion that follows is taken from [23, Chapter III] and is given here to help develop the setting for considering systems of singular operators.

two examples just given, we consider the irreducible representation $a \rightarrow R_a^s$ of $SO(n)$ on the spherical harmonics of degree s . The system spanned by all s -fold iterates of the Riesz operators has the required transformation properties, and is the only one that does. In a certain sense these are enough examples for E_n . In [5] it is shown that any translation-invariant singular operator T can be written as $T = \sum_{s=0}^{\infty} T_s$, with T_s lying in the system described above for $a \rightarrow R_a^s$. Moreover, any "variable-kernel" operator T can be written as $T = \sum_{s=0}^{\infty} A_s T_s$, where T_s is as before, and A_s is the operator given by multiplication by the bounded smooth function $a_s(x)$. In both cases the series converges in L^p -operator norm.

We now pose a similar question for singular operators on S^{n-1} . We ask which irreducible representations R of $SO(n)$ will support a system V of operators T satisfying

$$(1.1) \quad L_a T L_a^{-1} = R_a T$$

for each $a \in SO(n)$. Also, for the representations that do, we ask how many "different" ones are possible. Here the answer is different than the one sketched above for E_n . To understand why, we must consider the action of $SO(n)$ on the symbol functions. For E_n these are functions on the sphere S^{n-1} (or its Cartesian product with E_n), while for S^{n-1} these are functions on $CS(S^{n-1})$, a compact manifold that may be identified with $SO(n)/SO(n-2)$. (See (1.2) and the remarks on identification following (1.5).)

In §II we investigate in detail the Fourier analysis (Peter-Weyl expansion) of a function in $L^2(SO(n)/SO(n-2))$. The results obtained are in close analogy to the classical decomposition

$$(1.2) \quad L^2(S^{n-1}) = \sum_{s=0}^{\infty} H_s,$$

where H_s denotes the space of restrictions to S^{n-1} of homogeneous harmonic polynomials of degree s in x_1, \dots, x_n . We show that in the Peter-Weyl decomposition

$$(1.3) \quad L^2(SO(n)/SO(n-2)) = \sum_m \oplus \mathcal{H}_m$$

the space \mathcal{H}_m can be realized as the restriction to the manifold

$$(1.4) \quad CS(S^{n-1}) = \left\{ (x, \xi) \in E_n \times E_n \mid \begin{matrix} x_1^2 + \dots + x_n^2 = 1 \\ \xi_1^2 + \dots + \xi_n^2 = 1 \end{matrix} \quad x_1 \xi_1 + \dots + x_n \xi_n = 0 \right\}$$

of a space Q_m of polynomials in $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ that have the desired transformation behavior (Theorem 2). It is these spaces that are used in §IV to construct operators having specified transformation laws analogous to (1.1). We show that the polynomials in Q_m are *harmonic* in a sense related to the action of $SO(n)$ on $E_n \times E_n$ given by (2.3), and that $\sum_m Q_m$ is in fact the space of all harmonic polynomials (Theorem 3). The chief difference between (1.2) and (1.3) is that while H_s

is irreducible under the action of $SO(n)$, \mathcal{H}_m is in general a sum of more than one irreducible subspace.

Estimates for the operator Λ^{-1} , which plays a role analogous to the Λ^{-1} introduced by Calderón and Zygmund, are proved in §III. The description of Λ^{-1} in local coordinates is carried out there as well. In §IV we give definitions and pose the problem on the existence of systems in a more precise form. We construct *canonical systems* that are analogous to the examples presented above for E_n , and prove a decomposition theorem of an arbitrary singular integral operator on $L^p(S^{n-1})$ (Theorem 5). This decomposition theorem allows us to classify all possible systems of singular operators transforming according to an irreducible representation of $SO(n)$ (Theorem 6).

The contents of this paper form most of the author's doctoral dissertation, written for Princeton University under the direction of Professor E. M. Stein. It is with great pleasure that the debt owed him is here recorded.

NOTATION. Most notation will be explained as it is introduced. In general, E_k will denote k -dimensional Euclidean space and $x=(x_1, \dots, x_k)$ a point in this space. The norm of x will be denoted by $|x|$. The multi-index convention for differentiation in E_k will often be used. Thus if $\alpha=(\alpha_1, \dots, \alpha_k)$

$$\frac{\partial^{|\alpha|} f}{\partial x^\alpha} = \left(\frac{\partial}{\partial x} \right)^\alpha f = \frac{\partial^{\alpha_1 + \dots + \alpha_k}}{\partial x_1^{\alpha_1} \dots \partial x_k^{\alpha_k}} f.$$

The Fourier transform of a function f in $L^1(E_k)$ or $L^2(E_k)$ will be denoted \hat{f} and defined by

$$(1.5) \quad \hat{f}(\xi) = \int_{E_k} f(x) e^{ix\xi} dx.$$

By $SO(n)$ we denote group of rotations in E_n that preserve orientation.

IDENTIFICATIONS. The group $SO(n)$ acts transitively on $S^{n-1} = \{x \in E_n \mid |x| = 1\}$ by the action $x \rightarrow ax$, where $a=(a_{ij})$ is regarded as an $n \times n$ matrix and x as a column vector. The fixing group of the "north pole" $N=(0, \dots, 0, 1)$ is a copy of $SO(n-1)$, and so we have the identification $S^{n-1} \cong SO(n)/SO(n-1)$. The symbol $\sigma(T)$ of a singular integral operator T on $L^p(S^{n-1})$ is a function on the cotangent bundle $T^*(S^{n-1})$, which may be described as

$$T^*(S^{n-1}) = \{(x, \xi) \in E_n \times E_n \mid |x| = 1, x\xi = 0\},$$

where $x\xi = x_1\xi_1 + \dots + x_n\xi_n$. For each fixed x we may introduce a metric in the fibre over x ,

$$T_x^*(S^{n-1}) = \{(x, \xi) \in \{x\} \times E_n \mid x\xi = 0\}$$

by use of the norm

$$(1.6) \quad |\xi| = (\xi_1^2 + \dots + \xi_n^2)^{1/2},$$

where ξ stands for $(x, \xi) \in T_x^*$. This metric depends differentiably on x . Since it is homogeneous and $\sigma(T)$ is homogeneous of degree 0 on each fibre of $T^*(S^{n-1})$, $\sigma(T)$ is actually determined by its value on the cosphere bundle of S^{n-1} (see [21]), which may thus be written as

$$(1.7) \quad CS(S^{n-1}) = \{(x, \xi) \in E_n \times E_n \mid \frac{|x|}{|\xi|} = 1, x\xi = 0\}.$$

The group $SO(n)$ acts transitively on $CS(S^{n-1})$, and the fixing group of the point (N, v_0) is $\{(a_{ij}) \in SO(n) \mid a_{nn} = a_{n-1, n-1} = 1\}$, where $v_0 = (0, \dots, 0, 1, 0)$. Thus $CS(S^{n-1})$ can be diffeomorphically identified with $SO(n)/SO(n-2)$. When we use the symbol $CS(S^{n-1})$ we shall always mean the manifold given by (1.7), and when we write $CS(S^{n-1}) \cong SO(n)/SO(n-2)$ we shall regard $SO(n-2)$ as the fixing group of the point (N, v_0) defined above.

The action of $SO(n)$ on $SO(n)/SO(n-2)$ induced by $g \cdot SO(n-2) \xrightarrow{a} (ag) \cdot SO(n-2)$, when transferred to $CS(S^{n-1})$ by the identification described above, becomes

$$(1.8) \quad (x, \xi) \xrightarrow{a} (ax, a\xi).$$

If we compute the action on $T^*(S^{n-1})$ induced by the action $x \xrightarrow{a} ax$ of $SO(n)$ on S^{n-1} , the result is also (1.8). We shall rely heavily on this last remark in several places. It may be restated as follows: if $l_a(x) = ax$, then

$$l_a^*(x, \xi) = (ax, a\xi),$$

where l_a^* is the map: $T_{ax}^*(S^{n-1}) \rightarrow T_x^*(S^{n-1})$ induced by l_a .

Action of $SO(n)$ on operators and symbols. If T is a singular integral operator in a suitable class, then so is $L_a T L_a^{-1}$. To be specific, whenever we say T is a singular operator we shall mean that T lies in the class $C^\infty(S^{n-1})$ defined by Seeley in [21]. The requirements thus imposed on T may be summarized by saying that the transfer of T to any coordinate neighborhood U yields an operator of the form $(T_U f)(x) = a(x)f(x) + \int_{E_{n-1}} k(x, x-y)f(y) dy + (Rf)(x)$, $x \in U$. Here $k(x, z)$ is to be a Calderón-Zygmund kernel that is C^∞ for $z \neq 0$, $a(x)$ a bounded smooth function, and R a "smoothing" operator.

The symbols $\sigma(T)$ and $\sigma(L_a T L_a^{-1})$ are related by a transformation law:

$$(1.9) \quad \sigma(T)(x, \xi) = \sigma(L_a T L_a^{-1})(ax, a\xi).$$

This transformation law will be derived in §IV in a slightly different context (Proposition 10).

Representations of $SO(n)$ and integration against characters. The irreducible representations of $SO(n)$ are in one-to-one correspondence with *dominant weight vectors* $m = (m_1, \dots, m_k)$, where $k = [n/2]$. We require that the entries be nonnegative integers (we shall not consider double-valued representations) satisfying $m_1 \geq m_2 \geq \dots \geq m_k \geq 0$ for n odd and $m_1 \geq m_2 \geq \dots \geq |m_k|$ for n even. We shall denote the

representation corresponding to the dominant weight vector m by R^m . See [24] for a brief discussion of this correspondence, or [1] or [25] for a complete discussion.

We shall have occasion to use the following easily verified fact about the characters of irreducible representations of a compact group. Let R be an irreducible representation of the compact group G that is a subrepresentation of a representation L of G on a Hilbert space \mathcal{H} . Then the projection π_R from \mathcal{H} onto the closed subspace \mathcal{H}_R of \mathcal{H} consisting of all the copies of irreducible spaces on which the action of L is equivalent to that of R is given by

$$(1.10) \quad \pi_R v = d_R \int_G (L_a v) \bar{\chi}_R(a) da,$$

where d_R is the degree of R and χ_R is its character. Here as throughout, the Haar measure da on $SO(n)$ will be assumed to be normalized so that $\int_{SO(n)} da = 1$.

II. We begin by recapitulating in Theorem 1 the results of the Peter-Weyl theorem as they apply to $SO(n)/SO(n-2) \cong CS(S^{n-1})$. Before defining the spaces $P_{r,s}^m$ and Q_m that will be used to set up a correspondence between symbol functions and operators (in §IV), we define the notion of an $SO(n)$ -harmonic polynomial. After showing that the restriction map to $CS(S^{n-1})$ has kernel 0 on Q_m (Theorem 2), we show that the polynomials in Q_m are $SO(n)$ -harmonic (Theorem 3). The Corollary of Theorem 3 may be interpreted as explaining the fact that the restriction map has kernel 0 on Q_m .

Let L denote the left regular representation of $SO(n)$, $n \geq 3$, on

$$L^2(SO(n)/SO(n-2))$$

defined by $(L_a f)(x) = f(a^{-1}x)$ for $f \in L^2(SO(n))$.

THEOREM 1.

$$(2.1) \quad L^2(SO(n)/SO(n-2)) = \sum_m \bigoplus \mathcal{H}_m,$$

where each \mathcal{H}_m is an invariant subspace such that the restriction $L|_{\mathcal{H}_m} = \sum_{i=1}^{s_m} \bigoplus R^{m,i}$. In the first sum the summation is over all dominant weights m of the form $(m_1, m_2, 0, \dots, 0)$. In the second sum $s_m = m_1 - |m_2| + 1$ for $n \geq 4$, $s_m = 2m + 1$ for $n = 3$, and $R^{m,i}$ are copies of the irreducible representation R^m corresponding to the dominant weight m .

Proof. For $n = 3$ the statement is part of the Peter-Weyl theorem applied to the left regular representation of $SO(3)$ on $L^2(SO(3))$. For $n \geq 4$ we apply the Peter-Weyl theorem to conclude that $L = \sum_m \bigoplus c_m R^m$. The coefficients c_m can be computed as follows. Let V^m be a representation space for R^m . By applying the *branching theorem* twice, (see [1, pp. 251–253]), first on restricting R^m to $SO(n-1)$, and then on restricting each irreducible subrepresentation of $R^m|_{SO(n-1)}$ to $SO(n-2)$, we see that an $m_1 - |m_2| + 1$ -dimensional subspace of V^m is left invariant by $R^m|_{SO(n-2)}$. The conclusion then follows from the Frobenius reciprocity theorem.

We pass to the notion of a G -harmonic polynomial. The terminology and notation is from [8]. Let V be a finite dimensional real vector space and let $S(V^*) =$ the polynomial functions on V . Suppose G acts on V by the action $v \xrightarrow{g} g \cdot v$. Then it acts on V^* by $(g \cdot v^*)(v) = v^*(g^{-1} \cdot v)$, and this action can be extended to $S(V^*)$.

Now each element $X \in V$ gives rise (by parallel translation) to a vector field on V , which we regard as a differential operator $\partial(X)$ on V . This mapping $X \rightarrow \partial(X)$ extends to an isomorphism of the symmetric algebra $S(V)$ over V onto the algebra of all differential operators on V with constant complex coefficients. We let $I(V)$ denote the image of the elements of $S(V)$ invariant under the action of $SO(n)$, and $I_+(V)$ denote the image of those elements of $I(V)$ with zero constant term.

DEFINITION. An element $p \in S(V^*)$ is a G -harmonic polynomial if it is annihilated by all invariant differential operators (of positive degree):

$$(2.2) \quad \partial(J)p = 0 \quad \text{for } J \in I_+(V).$$

If we take V to be a real n -dimensional vector space with basis $\{\partial/\partial x_1, \dots, \partial/\partial x_n\}$, choose the dual basis $\{x_1, \dots, x_n\}$ for V^* and let $SO(n)$ act on V^* by $x \xrightarrow{a} ax$, where x is regarded as a column vector, then we recover the classical definition of a harmonic polynomial in x_1, \dots, x_n . For our purposes we shall take $G = SO(n)$ and let V be a real vector space of dimension $2n$ with basis $\{\partial/\partial x_1, \dots, \partial/\partial x_n, \partial/\partial \xi_1, \dots, \partial/\partial \xi_n\}$. We choose the dual basis $\{x_1, \dots, x_n, \xi_1, \dots, \xi_n\}$ for V^* and let $SO(n)$ act on V^* by the action

$$(2.3) \quad (x, \xi) \xrightarrow{a} (ax, a\xi).$$

Here x and ξ are to be regarded as column vectors. When we speak of a ‘‘harmonic’’ polynomial in $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ it will always refer to the action (2.3) of $SO(n)$.

Eventually (Theorem 3) we shall show that the spaces $P_{r,s}^m$ and Q_m we are about to define are spaces of $SO(n)$ -harmonic polynomials in the sense of the above definition. Before beginning the proofs of Theorems 2 and 3, we write down all the necessary definitions and give the statement of these theorems.

Let $n \geq 3$, and let $m = (m_1, m_2, 0, \dots, 0)$ be a dominant weight of the form discussed in Theorem 1. Let r and s be any two nonnegative integers such that

$$(2.4) \quad r + s = m_1 + |m_2|, \quad |m_2| \leq r \leq m_1.$$

Let us realize $R^{(r,0,\dots,0)}$ on $H_r(x) = \{(\text{solid}) \text{ homogeneous harmonic polynomials in variables } x_1, \dots, x_n \text{ of degree } r, \text{ complex coefficients}\}$, and similarly, let us realize $R^{(s,0,\dots,0)}$ on $H_s(\xi) = \{(\text{solid}) \text{ homogeneous harmonic polynomials in variables } \xi_1, \dots, \xi_n \text{ of degree } s, \text{ complex coefficients}\}$. Then $H_r(x) \otimes H_s(\xi)$ can be regarded as a vector space of polynomials $p(x, \xi)$ such that $p(tx, u\xi) = t^n u^s p(x, \xi)$. The tensor product $R^{(r,0,\dots,0)} \otimes R^{(s,0,\dots,0)}$ acts on $H_r(x) \otimes H_s(\xi)$, but not irreducibly.

LEMMA 1. Let r and s be nonnegative integers with $s \leq r$. Let R^λ denote the irreducible representation of $SO(n)$ corresponding to the dominant weight $(\lambda, 0, \dots, 0)$ and let

R^{λ_1, λ_2} denote the one corresponding to the dominant weight $(\lambda_1, \lambda_2, 0, \dots, 0)$. Then for $n \geq 4$ the tensor product breaks down as follows into irreducible representations, each with multiplicity 1:

$$(2.5) \quad \begin{aligned} R^r \otimes R^s = & R^{r+s} \oplus R^{r+s-2} \oplus \dots \oplus R^{r-s} \\ & \oplus R^{r+s-1,1} \oplus R^{r+s-3,1} \oplus \dots \\ & \oplus \dots \\ & \oplus R^{r,s}. \end{aligned}$$

This decomposition is given in [15, p. 276], for the group $O(n)$. All the representations on the right side of (2.5) remain irreducible on restriction to $SO(n)$, provided $n \neq 4$. If $n=4$, the irreducible representation R^{λ_1, λ_2} ($\lambda_2 > 0$) of $O(4)$ decomposes as $R^{\lambda_1, \lambda_2} \oplus R^{\lambda_1, -\lambda_2}$ on restriction to $SO(4)$. See [1, pp. 96 ff. and p. 262].

We notice that under the restrictions (2.4) on r and s the representation R^m occurs exactly once on the right side of (2.5). For such r and s we are in a position to make the following

DEFINITION. The space $P_{r,s}^m$ is the irreducible subspace of $H_r(x) \otimes H_s(\xi)$ (regarded as polynomials in $x_1, \dots, x_n, \xi_1, \dots, \xi_n$) consisting of vectors transforming according to R^m under the action (2.3) of $SO(n)$.

Summing over r and s satisfying (2.4) we have the

DEFINITION. For $n \geq 4$,

$$(2.6) \quad Q_m = \sum_{r+s=m_1+|m_2|, |m_2| \leq r \leq m_1} \oplus P_{r,s}^m.$$

We note that there are $s_m = m_1 - |m_2| + 1$ irreducible summands in (2.6). For $n=3$ we shall adopt a modification that will have $s_m = 2m + 1$ terms. See (2.14).

Let us recall the expression (1.7) for $CS(S^{n-1})$ as well as the remarks made afterwards about the identification of $CS(S^{n-1})$ and $SO(n)/SO(n-2)$.

THEOREM 2. Let $n \geq 4$ and $m = (m_1, m_2, 0, \dots, 0)$ be a dominant weight. The space Q_m defined by (2.6) is mapped by the restriction map to $CS(S^{n-1})$ into

$$\mathcal{H}_m \subseteq L^2(SO(n)/SO(n-2)).$$

The restriction map commutes with the action of $SO(n)$ induced on each of these spaces by (2.3). Furthermore, the restriction map has kernel 0 on Q_m and carries Q_m onto \mathcal{H}_m .

THEOREM 3. Let $n \geq 3$. The space $H(V^*)$ of $SO(n)$ -harmonic polynomials in $C[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ under the action (2.4) coincides with the (algebraic) direct sum

$$\sum_{(m_1, m_2, 0, \dots, 0)} Q_m.$$

Here $C[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ denotes the ring of complex coefficient polynomials in $x_1, \dots, x_n, \xi_1, \dots, \xi_n$.

COROLLARY. *If h is an $SO(n)$ -harmonic polynomial that vanishes on $CS(S^{n-1})$, then h is the zero polynomial.*

REMARK. For the Corollary in a much more general setting see [12].

The proofs of these statements will occupy most of the remainder of this chapter. After the proof of Theorem 2 we shall introduce the modifications necessary in the definition of Q_m for the case $n=3$. Theorem 4 will give the results for $n=3$ corresponding to Theorem 2, whose proof we now begin.

If $q \in C[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$, we define the polynomial q_a by

$$q_a(x, \xi) = q(a^{-1}x, a^{-1}\xi).$$

Let CS stand for $CS(S^{n-1})$, and let Res_m denote the restriction map to CS for polynomials in Q_m . Then if L is the left regular representation discussed in Theorem 1,

$$q_a|_{CS} = L_a(q|_{CS}),$$

so that restriction to CS commutes with the action of $SO(n)$, and hence Res_m carries Q_m into \mathcal{H}_m . Since Q_m and \mathcal{H}_m have the common dimension $s_m = m_1 - |m_2| + 1$ as complex vector spaces, Theorem 2 will be proved once we show that the kernel of Res_m is zero.

To do this we shall use the following two propositions, which we prove at the end of this chapter, using the methods of algebraic geometry. Let S denote the ring $C[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$.

PROPOSITION 1. *Let $n \geq 2$ and let $q \in S$ be any polynomial that vanishes on $CS(S^{n-1})$. Then there are polynomials A, B, C in S such that*

$$(2.7) \quad q = A(|x|^2 - 1) + B(|\xi|^2 - 1) + Cx\xi.$$

PROPOSITION 2. *The vector space homomorphism $\phi: S \oplus S \oplus S \rightarrow S$ defined by $\phi(A, B, C) = A|x|^2 + B|\xi|^2 + Cx\xi$ has as its kernel the ideal in S generated by the three elements*

$$(|\xi|^2, -|x|^2, 0), \quad (x\xi, 0, -|x|^2), \quad (0, x\xi, -|\xi|^2).$$

If q lies in the kernel of Res_m , we apply Proposition 1 to it. Writing A, B and C as a sum of homogeneous polynomials:

$$(2.8) \quad A = \sum_{j=0}^s A_j, \quad B = \sum_{j=0}^s B_j, \quad C = \sum_{j=0}^s C_j$$

and substituting the resulting expression (2.8) into (2.7) yields, on equating terms of equal homogeneity:

$$(2.9) \quad \begin{aligned} A_s|x|^2 + B_s|\xi|^2 + C_sx\xi &= 0 \\ A_{s-2}|x|^2 + B_{s-2}|\xi|^2 + C_{s-2}x\xi - A_s - B_s &= 0 \\ \cdot &\cdot \cdot \\ A_{p-2}|x|^2 + B_{p-2}|\xi|^2 + C_{p-2}x\xi - A_p - B_p &= q \\ \cdot &\cdot \cdot \\ A_0 + B_0 &= 0, \end{aligned}$$

where p denotes the total homogeneity $m_1 + |m_2|$ of q . Applying Proposition 2 to the top line of (2.9) yields $A_s = \alpha_{s-2}|\xi|^2 + \beta_{s-2}x\xi$, $B_s = -\alpha_{s-2}|x|^2 + \gamma_{s-2}x\xi$. (Here we may assume α_{s-2} , β_{s-2} , γ_{s-2} are homogeneous of degree $s-2$.)

Substituting into the second equation of (2.9) yields

$$(2.10) \quad (A_{s-2} + \alpha_{s-2})|x|^2 + (B_{s-2} - \alpha_{s-2})|\xi|^2 + (C_{s-2} - \beta_{s-2} - \gamma_{s-2})x\xi = 0.$$

Applying Proposition 2 to (2.10) yields

$$\begin{aligned} A_{s-2} + \alpha_{s-2} &= \alpha_{s-4}|\xi|^2 + \beta_{s-4}x\xi, \\ B_{s-2} - \alpha_{s-2} &= -\alpha_{s-4}|x|^2 + \gamma_{s-4}x\xi. \end{aligned}$$

We can now substitute these expressions into the third equation of (2.9), and so on, until we arrive at the equation

$$A_p + B_p = \alpha_{p-2}|\xi|^2 - \alpha_{p-2}|x|^2 + (\beta_{p-2} + \gamma_{p-2})x\xi,$$

where α_{p-2} , β_{p-2} and γ_{p-2} are homogeneous of degree $p-2$. But then, from the equation in (2.9) involving terms of homogeneity p ,

$$(2.11) \quad q = a|x|^2 + b|\xi|^2 + cx\xi,$$

where a , b and c are homogeneous of degree $p-2$.

DEFINITION. Let R^m be the irreducible representation of $SO(n)$, $n \geq 3$, whose highest weight is (m_1, m_2, \dots, m_k) . The richness of R^m and of m is

$$|m| = |m_1| + \dots + |m_k|.$$

It is easy to show that if $|m| = r$, then R^m occurs as a subrepresentation of $\otimes^r st_n$ acting on $\otimes^r E_n$, where st_n denotes the standard representation $a \rightarrow a$ of $SO(n)$ acting as rotations on E_n . See [24]. We shall use a converse of this fact:

LEMMA 2. Any subrepresentation of $\otimes^r st_n$ has richness at most r .

The proof is a simple argument by induction, using the fact that if R is an irreducible representation of richness t , then $R \otimes st_n$ breaks up into irreducible representations whose richness is at most $t+1$. See the discussion of Cartan composition in [24].

It follows from Lemma 2 and the use of the identification mentioned just before Lemma 1 that in the decomposition into irreducible subspaces of

$$S_t = \{p \in S \mid \text{degree } p \leq t\}$$

under the action $q \xrightarrow{a} q_a$ of $SO(n)$ none of the irreducible representations that occur has richness exceeding t .

Now it follows from the projection formula (1.10) that

$$(2.12) \quad q(x, \xi) = d_m \int_{SO(n)} q_g(x, \xi) \bar{\chi}_m(g) dg.$$

Substituting the expression (2.11) on the right side of (2.12) we obtain

$$(2.13) \quad \begin{aligned} q(x, \xi) = & d_m |x|^2 \int_{SO(n)} a_g(x, \xi) \bar{\chi}_m(g) dg + d_m |\xi|^2 \int_{SO(n)} b_g(x, \xi) \bar{\chi}_m(g) dg \\ & + d_m |x|^2 \int_{SO(n)} c_g(x, \xi) \bar{\chi}_m(g) dg. \end{aligned}$$

But each of the integrals on the right must be zero, since the richness of R^m is $p = \text{degree } q$, while a, b and c lie in S_{p-2} . This completes the proof of Theorem 2.

The case $n=3$ is atypical in that the multiplicity of H_r in \mathcal{H}_r is $2r+1$. Let us consider first the example of R^1 , which occurs three times in $L^2(SO(3))$. We may realize it in two "different" ways on $H_1(x)$ and $H_1(\xi)$ respectively. The proof of Theorem 5.2 applies so far. To find the "other" realization we use the fact that for $n=3$ the adjoint representation "degenerates" into the standard representation R^1 . Guided by the construction performed for the adjoint representation in the case $n \geq 5$ (for $n=4$ the adjoint representation is not irreducible), we look at $H_1(x) \otimes H_1(\xi)$ and select the three dimensional space spanned by the polynomials $x_1 \xi_2 - x_2 \xi_1, x_1 \xi_3 - x_3 \xi_1, x_2 \xi_3 - x_3 \xi_2$. Here we resort to using polynomials of degree 2 to get all the representations we need.

The Clebsch-Gordan formula for $SO(3)$ is

$$R^r \otimes R^s = R^{r+s} \oplus R^{r+s-1} \oplus \dots \oplus R^{r-s}$$

for $0 \leq s \leq r$ (see (2.5)). To get all the spaces we need for R^m we allow r and s to be nonnegative integers such that $0 \leq s \leq r \leq m$ and $r+s$ is either m or $m+1$. This determines $2m+1$ products that contain R^m acting on a space $P_{r,s}^m \subseteq H_r(x) \otimes H_s(\xi)$.

DEFINITION. For $n=3$ and $m \geq 0$, set

$$(2.14) \quad Q_m = \sum_{0 \leq s \leq r \leq m; r+s=m} \oplus P_{r,s}^m \oplus \sum_{0 \leq s \leq r \leq m; r+s=m+1} \oplus P_{r,s}^m.$$

Note that the total homogeneity of any polynomial in the first sum of (2.14) is exactly one less than that of any polynomial in the second.

THEOREM 4. For $n=3$ the space Q_m is mapped by the restriction map to $CS(S^2)$ into the space \mathcal{H}_m . The restriction map commutes with the action of $SO(3)$ on both spaces. Furthermore, the restriction map to $CS(S^2)$ has kernel 0 on Q_m and carries Q_m onto \mathcal{H}_m .

Proof. The demonstration repeats that of Theorem 2 until one considers the analogue of the equations (2.7) and (2.9). Now q is no longer homogeneous, but as we have just remarked, it is the sum of two homogeneous pieces q_1 and q_2 that differ by 1 in their total homogeneity. Considering separately the terms of odd and even homogeneity in (2.7), we obtain two sets of equations analogous to (2.9). By the same argument as in the proof of Theorem 2 we show that both q_1 and q_2 are identically zero.

Proof of Theorem 3. Suppose first that $n > 3$, and let $q \in Q_m$. It is a homogeneous polynomial, of degree p say. Since $S(V^*) = I(V^*)H(V^*)$ (see [8]),

$$(2.15) \quad q = h + \sum r_i h_i,$$

where h and each h_i are harmonic polynomials and the r_i are invariant polynomials. (We may assume $p > 0$, since the case $p = 0$ is trivial.)

Since $SO(n)$ acts by linear substitutions, it is easy to see that if r is an invariant polynomial so is each of its homogeneous terms. Dually, if D is an invariant differential operator, so is each term of fixed order (that is, each homogeneous part). Hence if h is harmonic each term in the expression of h as a sum of homogeneous polynomials must be annihilated by each invariant differential operator D .

Hence we may assume that in (2.15) h is homogeneous of degree $p = \text{degree } q$, while each of the h_i appearing has homogeneity strictly less than p . Now we may apply the projection formula (1.10) to q . The resulting expression (2.12) becomes

$$(2.16) \quad q(x, \xi) = d_m \int_{SO(n)} h_a(x, \xi) \bar{\chi}_m(a) da + 0.$$

The integrals of the h_i are 0 by the same argument as in Theorem 2, since their degree is lower than that of q . Finally, by applying any invariant differential operator under the integral sign, the right hand side of (2.16) is seen to be harmonic.

If $n = 3$, we write $q = q_1 + q_2$, where q_1 is homogeneous of odd degree and q_2 is homogeneous of even degree. The argument above may be carried out for each separately. (We observe that there are no invariant polynomials of odd degree. See [25, p. 32].)

Now let $h(x, \xi)$ be any $SO(n)$ -harmonic polynomial. As we have shown above, each homogeneous term of h is harmonic. Let us therefore assume that h is homogeneous of degree k , and show that h is a finite sum $h = \sum q_m$, where $q_m \in Q_m$. We begin by writing $h = \sum h_m + h_1$, where the sum is finite and

$$(2.17) \quad h_m = d_m \int_{SO(n)} h_a \bar{\chi}_m(a) da = d_m \int_{SO(n)} (h_m)_a \bar{\chi}_m(a) da,$$

while for each m the similar integral taken for h_1 vanishes.

Let us temporarily fix $m = (m_1, m_2, 0, \dots, 0)$ and consider h_m . Substituting tx for x and $t\xi$ for ξ in the last equation shows that h_m is homogeneous of degree k . Its restriction to $CS(S^{n-1})$ is in \mathcal{H}_m , and so there is a polynomial q_m in Q_m that agrees with h_m on $CS(S^{n-1})$. Let us denote the homogeneity of q_m (or in the case $n = 3$, that of the piece whose homogeneity has the same parity as k) by p . By Lemma 2 we must have $p \leq k$. Since $h_m - q_m$ vanishes on $CS(S^{n-1})$, we can, as in the proof of Theorem 2, obtain the equations

$$(2.18) \quad \begin{aligned} & A_s |x|^2 + B_s |\xi|^2 + C_s x\xi \\ h_m - q_m = & + A_{s-2} |x|^2 + B_{s-2} |\xi|^2 + C_{s-2} x\xi - A_s - B_s + \dots \\ & + A_{k-2} |x|^2 + B_{k-2} |\xi|^2 + C_{k-2} x\xi - A_k - B_k + \dots \\ & + A_{p-2} |x|^2 + B_{p-2} |\xi|^2 + C_{p-2} x\xi - A_p - B_p + \dots \end{aligned}$$

If $p=k$, the argument following equations (2.9) may once again be applied to show that $h_m - q_m$ is the zero polynomial. If $p < k$, we use a similar recursion argument to obtain

$$(2.19) \quad h_m = (A_{k-2} + \alpha_{k-2})|x|^2 + (B_{k-2} - \alpha_{k-2})|\xi|^2 + (C_{k-2} - \beta_{k-2} - \gamma_{k-2})x\xi.$$

Since $S(V^*)$ is the direct sum of $H(V^*)$ and the ideal in $S(V^*)$ generated by $I_+(V^*)$, the invariant polynomials of positive degree, h_m is the zero polynomial.

Finally, let us deal with the harmonic polynomial $h_1 = h - \sum h_m$. It is homogeneous of degree k . Since for each m and each $(x, \xi) \in CS(S^{n-1})$

$$d_m \int_{SO(n)} (h_1)(a^{-1}x, a^{-1}\xi) \bar{\chi}_m(a) da = 0,$$

the polynomial h_1 vanishes on $CS(S^{n-1})$. We may now write equations analogous to (2.18) with $h_m - q_m$ replaced by h_1 . A recursion argument like the one following equations (2.9) leads to an equation similar to (2.19), and so as above, h_1 is the zero polynomial. Thus any harmonic polynomial h is a finite sum $h = \sum q_m$, with $q_m \in Q_m$.

The Corollary to Theorem 3 is proved by writing any harmonic polynomial h as $h = \sum q_m$ with $q_m \in Q_m$. If h vanishes on $CS(S^{n-1})$, so does each q_m , since $\{\mathcal{H}_m\}$ is an orthogonal family. But by Theorem 2, this means that each q_m is identically 0.

The proof of Proposition 1 proceeds with the use of some elementary notions and theorems in algebraic geometry (over the field of complex numbers), and of the celebrated Hilbert Nullstellensatz. We begin with a brief summary of the facts that will be used. If K is a field, let $K[X_1, \dots, X_n]$ denote the ring of polynomials in indeterminates X_1, \dots, X_n .

Let A be a ring with unit (usually a polynomial ring over a field, some quotient of one, or a localization of such). An ideal I is *prime* in A if and only if A/I is an integral domain, and *maximal* if and only if A/I is a field.

If A is a quotient ring of $C[X_1, \dots, X_n]$ and m is a maximal ideal in A , then A/m is isomorphic to C (see [19, p. 5]).

If I is an ideal in $K[X_1, \dots, X_n]$, the *locus* (or *variety*) of I in K^n is the set of all points $(x_1, \dots, x_n) \in K^n$ such that $f(x_1, \dots, x_n) = 0$ for each f in I . A *variety* in K^n is the locus in K^n of some ideal of $K[X_1, \dots, X_n]$. A variety is *irreducible* if it is not the union of proper varieties. Since C is algebraically closed, the ideal J of all polynomials that vanish on V , the locus of an ideal I in $C[X_1, \dots, X_n]$ is prime if and only if V is irreducible.

The dimension d of the locus of a prime ideal I in $C[X_1, \dots, X_n]$ is the transcendence degree of $C[X_1, \dots, X_n]/I$ over C . This is equal to the dimension of the variety of I as a manifold (at its nonsingular points).

If I is an ideal in $C[X_1, \dots, X_n]$ its radical \sqrt{I} is the set of all polynomials f such that $f^e \in I$ for some positive integer e . By the Hilbert Nullstellensatz, if I is any ideal in $C[X_1, \dots, X_n]$, then the set of all polynomials f in $C[X_1, \dots, X_n]$

that vanish on the variety of I is exactly the ideal \sqrt{I} . (This uses the fact that C is algebraically closed.)

For proofs and more discussion see [9, pp. 1-7], [13], or [27, pp. 160-168 of Vol. II].

We shall be interested in the polynomial ring $S = C[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$. By (p, q, \dots) we shall mean the ideal in S generated by (p, q, \dots) . The next two lemmas will allow us to apply the Hilbert Nullstellensatz to any polynomial q that vanishes on $CS(S^{n-1})$.

LEMMA 3. *The locus in C^{2n} of each of the ideals $I_n = (|x|^2 - 1, |\xi|^2 - 1, x\xi)$ and $J_n = (|x|^2 - 1, |\xi|^2 - 1)$ is connected.*

Proof sketch. The complex locus of I_n consists of points $(z_1, \dots, z_n, w_1, \dots, w_n)$ in C^{2n} satisfying the equations

$$(2.20) \quad z_1^2 + \dots + z_n^2 = 1, \quad w_1^2 + \dots + w_n^2 = 1, \quad z_1 w_1 + \dots + z_n w_n = 0.$$

By setting $z_j = x_j + iy_j$, $w_j = \xi_j + i\eta_j$ we see that $CS(S^{n-1})$ can be identified with the "real locus" of the three generating polynomials, i.e., those points $(z_1, \dots, z_n, w_1, \dots, w_n)$ such that for each $j = 1, \dots, n$ we have $y_j = 0$ and $\eta_j = 0$. Let us solve equations (2.20) for (say) z_1, w_1 and w_2 in terms of $z_2, \dots, z_n, w_3, \dots, w_n$. We can then define a curve in the complex locus starting from an arbitrary point $(z_1^0, \dots, z_n^0, w_1^0, \dots, w_n^0)$ to some point on the real locus as follows: we let the imaginary parts of $z_2, \dots, z_n, w_3, \dots, w_n$ tend to zero one at a time. Then we can let the numbers $x_1, \dots, x_n, \xi_3, \dots, \xi_n$ vary so that the resulting values of z_1, w_1 and w_2 are real. We omit details of the (double-valued) solutions and the precautions that must be taken (including relabeling if necessary) to avoid vanishing denominators. See [14] for these details. Since the real locus is connected, and any point in the complex locus can be connected by a path to a point on the real locus, the complex locus is connected.

The proof for J_n is simpler.

LEMMA 4⁽²⁾. *Let $n \geq 3$. If a polynomial in S vanishes on the real locus of*

$$(|x|^2 - 1, |\xi|^2 - 1, x\xi)$$

in R^{2n} , then it vanishes on the complex locus (2.20) of this ideal.

Proof. As in the proof of Lemma 3, we set $z_j = x_j + iy_j$, $w_j = \xi_j + i\eta_j$ and regard the real locus as those points such that $y_j = 0, \eta_j = 0$. Let $x^0 = (\sqrt{2}/2, \sqrt{2}/2, 0, \dots, 0)$, $\xi^0 = (\sqrt{2}/2, -\sqrt{2}/2, 0, \dots, 0)$, and solve the equations (2.20) near (x^0, ξ^0) to obtain

$$(2.21) \quad \begin{aligned} z_1 &= \phi(z_2, \dots, z_n), \\ w_k &= \psi_k(z_2, \dots, z_n, w_3, \dots, w_n), \quad k = 1, 2, \end{aligned}$$

⁽²⁾ The author would like to thank Dr. Pierre Samuel for suggesting the use of this lemma as well as for sending him the proof, due to A. Weil, given for Lemma 6.

where ϕ, ψ_1 and ψ_2 are analytic single-valued functions of their arguments for $|z_2 - \sqrt{2}/2| < \delta, |z_j| < \delta, |w_j| < \delta, j \geq 3$. It can be easily verified that if z_3 is close to $\sqrt{2}/2$ and real, and $z_3, \dots, z_n, w_3, \dots, w_n$ are close to zero and real, then the corresponding values of z_1, w_1, w_2 obtained from (2.21) will also be real.

If f is any polynomial in S , then for (z, w) close to (x^0, ξ^0) we may write

$$(2.22) \quad f(z, w) = f(\phi_1, z_2, \dots, z_n, \psi_1, \psi_2, w_3, \dots, w_n) = \sum_{\alpha, \beta} c_{\alpha, \beta}^{\alpha} z_2^{\alpha_2} \dots z_n^{\alpha_n} w_3^{\beta_3} \dots w_n^{\beta_n},$$

where $\alpha = (\alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_3, \dots, \beta_n)$ are multi-indices. But each c_{β}^{α} in (2.22) can be computed by taking derivatives on the right side along real axes, i.e., along the $x_2, \dots, x_n, \xi_3, \dots, \xi_n$ axes. But if f vanishes on the real locus, all such derivatives must be zero, by the remark at the end of the last paragraph. Hence if f vanishes on the real locus it vanishes for all points on the complex locus distant (in C^{2n}) from (x^0, ξ^0) less than some positive number ϵ .

To complete the proof we note that the Jacobian

$$(2.23) \quad J = \frac{\partial(|x|^2 - 1, |\xi|^2 - 1, x\xi)}{\partial(x_1, \dots, x_n, \xi_1, \dots, \xi_n)}$$

has rank 3 at every point of the complex locus, so that the locus is an analytic manifold. Since this analytic manifold is connected (Lemma 3) any analytic function f that vanishes on an open set of it vanishes identically on it.

At this point we can apply the Hilbert Nullstellensatz to obtain (2.7) with q replaced by a positive integral power q^e . The following lemma allows us to assert that e can be taken to be 1, thus completing the proof of Proposition 1.

LEMMA 5. *The ideals I_n and J_n of Lemma 3 are each prime for $n \geq 2$.*

Proof. Since V_n , the locus in C^{2n} of I_n , is a connected analytic manifold (see proof of Lemma 4), it must be an irreducible variety, for if V_n were reducible there would be two polynomials f and g such that fg vanishes identically on V_n , but neither f nor g does. In this case, some nonempty open set in V_n would be the union of the zero sets of f and g , each of which, however, is $2n - 4$ dimensional. Since V_n is irreducible, its ideal, which by the Hilbert Nullstellensatz is $\sqrt{I_n}$, is prime. Lemma 6 together with the remarks after (2.23) shows that $I = \sqrt{I}$.

The proof for J_n is similar.

LEMMA 6. *Let k be an algebraically closed field and I an ideal in $k[X_1, \dots, X_n]$ whose radical \sqrt{I} is a prime ideal of dimension d . Suppose the Jacobian matrix $(\partial F_j / \partial X_i)$ of the generators F_1, \dots, F_{n-d} of I is of rank $n - d$ at each zero of I in k^n . Then $I = \sqrt{I}$.*

Proof. It must be shown that the affine coordinate ring $A = k[X_1, \dots, X_n] / I = k[x, \dots, x_n]$ has no nilpotent elements.

The A -module of differential forms $D = \Omega_{A/k}$ is the A -module defined by the generators dx_i and the relations

$$\sum_i \frac{\partial F}{\partial X_i}(x) dx_i = 0$$

for each $F \in I$. The hypothesis on the rank of the Jacobian implies that if m is a maximal ideal of A , then $A/m \otimes_A D$ is a vector space of dimension d over the field A/m . (Since k is algebraically closed A/m is actually isomorphic to k .)

For a fixed maximal ideal m of A , let A' denote the local ring

$$A_m = \{f/g \mid f, g \in A, g \notin m\},$$

and let m' denote its maximal ideal mA' . Since m is maximal $A/m = A'/m'$ ($=k$), and hence (by the localization of modules of differentials)

$$A/m \otimes_A D = (A'/m') \otimes_{A'} \Omega_{A'/k}.$$

But there is a canonical isomorphism: $m/m'^2 \rightarrow (A'/m') \otimes_{A'} \Omega_{A'/k}$. (See [6, exposé 17, No. 3, Theorem 5].) Hence the dimension of m'/m'^2 over A'/m' is d , so that A' is (a regular local ring and in particular is) an integral domain [6, exposé 17, No. 1, Theorem 1 and Theorem 2].

Since each A_m is a reduced ring, and the canonical homomorphism $A \rightarrow \prod_{m \text{ max}} A_m$ is injective [2, Chapter II, Section 3, No. 3, Corollary 2 of Theorem 1] it follows that A is a reduced ring.

REMARK. The proof of Lemma 6 given above involves the notion of a regular local ring. For a discussion see either [6] or [27, Vol. II, pp. 301–302].

LEMMA 7. For $n \geq 3$ the ideal $(|x|^2, |\xi|^2)$ is prime in S .

Proof. This follows from the fact that $(|x|^2)$ is prime in $C[x_1, \dots, x_n]$ and $(|\xi|^2)$ is prime in $C[\xi_1, \dots, \xi_n]$. See [13, p. 86].

Proof of Proposition 2. The ideal the three elements generate is easily seen to be contained in the kernel. Let (A, B, C) lie in the kernel. If $C \neq 0$ then $Cx\xi \in (|x|^2, |\xi|^2) \subseteq S$, while $x\xi$ does not lie in this ideal (since it is linear in x_1, \dots, x_n and in ξ_1, \dots, ξ_n). But by Lemma 7 the ideal $(|x|^2, |\xi|^2)$ is prime in S , and so $C = -\beta|x|^2 - \gamma|\xi|^2$ for $\beta, \gamma \in S$.

Hence $(A - \beta x\xi)|x|^2 + (B - \gamma x\xi)|\xi|^2 = 0$, and since $|x|^2$ is irreducible in S , and $|\xi|^2$ does not divide it,

$$B - \gamma x\xi = -\alpha|x|^2 \quad \text{for } \alpha \in S.$$

Thus $(A - \beta x\xi)|x|^2 = \alpha|x|^2|\xi|^2$ and so $A = \alpha|\xi|^2 + \beta x\xi$.

III. To construct singular operators we shall start with differential operators, and compose them with suitable powers of Λ^{-1} , where $-\Lambda^2$ approximates the Laplacian on S^{n-1} . These powers of Λ^{-1} will be considered in connection with the function spaces $L_k^p(S^{n-1})$ for $1 < p < \infty$ and k any integer. See [21] for a definition

of these spaces in terms of a partition of unity $\{\phi_i\}$. For each k , $L_k^p = L_k^p(S^{n-1})$ is a reflexive Banach space that is isomorphic to $L^p(S^{n-1})$. If $k \geq 0$, then the space of C^∞ functions is dense in L_k^p . It is not hard to show that any norm for L_k^p arising from a partition of unity as in [21] is equivalent to the norm

$$(3.1) \quad \|f\|_{p,k} = \left\{ \sum_{0 \leq h \leq k} \|D_{i_1, \dots, D_{i_h}} f\|_p^p \right\}^{1/p},$$

where the summation is over all h -fold iterates of standard basis elements $D_{ij} = x_i(\partial/\partial x_j) - x_j(\partial/\partial x_i)$ of S_n , the Lie algebra of $SO(n)$. The measure used in defining $\|f\|_{p,0} = \|f\|_p$ is $d\sigma$, the (unique) rotation invariant measure normalized so that $\int_{S^{n-1}} d\sigma = \omega_{n-1}$, the surface area of S^{n-1} .

DEFINITION. Let r be a nonnegative integer. We say that T is smoothing of order r on $L^p(S^{n-1})$ if for all integers k , T is continuous: $L_k^p(S^{n-1}) \rightarrow L_{k+r}^p(S^{n-1})$. If no particular p is mentioned it is assumed that this holds for each p with $1 < p < \infty$.

PROPOSITION 3. Let $T: C^\infty(S^{n-1}) \rightarrow C^\infty(S^{n-1})$ be an operator that commutes with the action L_a of $SO(n)$ on $C^\infty(S^{n-1})$. Suppose that T is bounded: $L^p \rightarrow L^p$. Then

- (a) For any $D \in S_n$, $TD = DT$ on $C^\infty(S^{n-1})$;
- (b) T is smoothing of order 1 if and only if for any $D \in S_n$ DT is bounded: $L^p \rightarrow L^p$;
- (c) T is smoothing of order $r \geq 1$ if and only if for any r -fold iterate D of elements of S_n , DT is bounded: $L^p \rightarrow L^p$.

Proof sketch. Conclusion (a) follows immediately from

$$(3.2) \quad (Df)(x) = \lim_{t \rightarrow 0} t^{-1} [f(\exp tDx) - f(x)],$$

since T commutes with rotations.

If T is bounded: $L_k^p \rightarrow L_{k+1}^p$ for each k , then taking $k=0$ and using (3.1) in the definition of a smoothing operator shows DT is bounded $L^p \rightarrow L^p$. For the converse we use the fact that if $h \leq k+1$,

$$(3.3) \quad D_{i_1} D_{i_2} \dots D_{i_h} T f = D_{i_1} T (D_{i_2} \dots D_{i_h}) f.$$

Thus $\|D_{i_1} T (D_{i_2} \dots D_{i_h}) f\|_p \leq C_p \|D_{i_2} \dots D_{i_h} f\|_p \leq C_p \|f\|_{p,k}$, and summing over h -fold iterates with $0 \leq h \leq k+1$ yields the desired estimate. Part (c) is proved analogously.

Let $P_r(t)$ denote the Poisson kernel

$$\frac{1}{\omega_{n-1}} (1-r^2)(1-2rt+r^2)^{-n/2}$$

for S^{n-1} . We define the operator Λ^{-1} on $C^\infty(S^{n-1})$ by

$$(3.4) \quad (\Lambda^{-1}f)(x) = \int_{S^{n-1}} K(x \cdot y) f(y) dy,$$

where

$$(3.5) \quad K(t) = - \int_0^1 P_r(t) dr.$$

PROPOSITION 4. *The operator Λ^{-1} is bounded: $L^p(S^{n-1}) \rightarrow L^p(S^{n-1})$ for $1 \leq p \leq \infty$. If $Y_s \in H_s$, the spherical harmonics of degree s , then*

$$(3.6) \quad \Lambda^{-1} Y_s = - Y_s / (s+1).$$

Proof. The first statement follows from Hölder's inequality and the fact that for each x , $K(x \cdot y)$ is an $L^1(S^{n-1})$ function of y such that $K(ax \cdot ay) = K(x \cdot y)$. (In fact, Λ^{-1} is given as a convolution on the sphere, regarded as a homogeneous space of $SO(n)$, with an L^1 function.)

If Y_s is in H_s , then

$$(3.7) \quad \int_{S^{n-1}} P_r(x \cdot y) Y_s(y) dy = r^s Y_s(x).$$

(See [3] or [22] for a discussion of spherical harmonics in n dimensions.) Interchanging the order of integration and letting $\rho \rightarrow 1$ in the formula

$$\int_0^\rho \left\{ \int_{S^{n-1}} P_r(x \cdot y) Y_s(y) dy \right\} d\rho = \frac{\rho^s}{s+1} Y_s$$

that follows from (3.7) establishes (3.6).

PROPOSITION 5. *For $1 < p < \infty$ and k an integer, the operator Λ^{-1} is bounded: $L^p_k(S^{n-1}) \rightarrow L^p_{k+1}(S^{n-1})$.*

Proof. By Proposition 3, it is sufficient to show that if D_{ij} is a standard basis element of S_n , then $D_{ij} \Lambda^{-1}$ is bounded: $L^p \rightarrow L^p$. Since $L_a \Lambda^{-1} = \Lambda^{-1} L_a$ and

$$L_a D_{ij} L_a^{-1} = \text{adj}_a(D_{ij}) = \sum_{k < l} a_{ki} a_{lj} D_{kl},$$

we have

$$(3.8) \quad L_a D_{ij} \Lambda^{-1} L_a^{-1} = \sum_{k < l} a_{ki} a_{lj} D_{kl} \Lambda^{-1}.$$

Let U be any open set containing $N = (0, \dots, 0, 1)$, and let f be any $C^\infty(S^{n-1})$ function supported in $aU = \{ax \mid x \in U\}$. Then for any basis element D_{ij} of S_n and $f_a = L_a f$ (which is supported in U),

$$(3.9) \quad \begin{aligned} \|D_{ij} \Lambda^{-1} f\|_p &= \|L_a D_{ij} \Lambda^{-1} L_a^{-1} f_a\|_p = \left\| \sum_{k < l} a_{ki} a_{lj} D_{kl} \Lambda^{-1} f_a \right\|_p \\ &\leq \frac{n(n-1)}{2} \text{Max}_{k < l} \|D_{kl} \Lambda^{-1} f_a\|_p. \end{aligned}$$

Hence it is sufficient to show that $\|\Lambda^{-1} f\|_{p,1} \leq C \|f\|_p$ for C^∞ functions f supported in some small open set of S^{n-1}

$$U_\epsilon = \left\{ (x_1, \dots, x_n) \mid \begin{array}{l} x_1^2 + \dots + x_n^2 = 1 \\ 1 - \epsilon < x_n \leq 1 \end{array} \right\},$$

where $\varepsilon > 0$ can be any fixed positive number. In fact, we shall prove the corresponding inequality for local coordinates $\bar{x} = (x_1, \dots, x_{n-1}) \in E_{n-1}$ for U_ε gotten by projecting a point $x \in U_\varepsilon$ onto the equatorial plane $x_n = 0$. If $t = x \cdot y = x_1 y_1 + \dots + x_n y_n$, we may write $t = t(\bar{x}, \bar{y})$ for $|\bar{x}|^2 = x_1^2 + \dots + x_{n-1}^2$ and $|\bar{y}|^2 = y_1^2 + \dots + y_{n-1}^2$ small (this makes t close to 1).

LEMMA 8. *The kernel $K(t)$ given by (3.5) may be written as*

$$(3.10) \quad K(t) = a_n(1-t)^{-(n-2)/2} - \omega_n^{-1} \int_0^1 (1-r^2)(1-r^{n-4})(1-2rt+r^2)^{-n/2} dr + \left(\frac{n-4}{n-2}\right) \omega_n^{-1} \int_0^1 r^{n-4}(1-2rt+r^2)^{(2-n)/2} dr.$$

Proof. Differentiation of both sides establishes the following identity given in [16], valid for $\gamma > -1$:

$$(3.11) \quad \int_0^r t^{2\lambda}(1-t^2)(1-2at+t^2)^{-\lambda-2} dt = \frac{r^{2\lambda+1}}{\lambda+1} (1-2ar+r^2)^{-\lambda-1} - \frac{\lambda}{\lambda+1} \int_0^r t^{2\lambda}(1-2at+t^2)^{-\lambda-1} dt.$$

Writing

$$P_r(t) = \omega_n^{-1} r^{n-4}(1-r^2)(1-2rt+r^2)^{-n/2} + \omega_n^{-1}(1-r^{n-4})(1-r^2)(1-2rt+r^2)^{-n/2},$$

integrating from 0 to 1 with respect to r and applying (3.11) yields (3.10), with $a_n = -2^{(2-n)/2} \omega_n^{-1} (2/n-2)$.

Let us rewrite (3.10) as

$$(3.12) \quad K(t) = a_n(1-t)^{-(n-2)/2} + A(t) + B(t).$$

For $t \neq 1$ both kernels $A(t)$ and $B(t)$ can be differentiated under the integral sign with any operator $D_{ij} = x_i(\partial/\partial x_j) - x_j(\partial/\partial x_i)$, or a similar one in the y variables. Since such operators span the tangent plane at any point of U_ε , we may estimate the gradient $\nabla_{\bar{x}, \bar{y}}$ of A or B (with respect to differentiation in local coordinates) in terms of derivatives given by the Lie algebra. A simple estimate of such derivatives shows that if $r(\bar{x}, \bar{y})$ denotes one of the kernels $A(t(\bar{x}, \bar{y}))$ or $B(t(\bar{x}, \bar{y}))$, then $r(\bar{x}, \bar{y})$ is C^∞ for $\bar{x} \neq \bar{y}$ and for $|\bar{x}|$ and $|\bar{y}|$ small

$$(3.13) \quad r(\bar{x}, \bar{y}) = O(|\bar{x} - \bar{y}|^{-n+3}), \quad \nabla_{\bar{x}, \bar{y}} r(\bar{x}, \bar{y}) = O(|\bar{x} - \bar{y}|^{-n+2}).$$

Now if $(Rf)(\bar{x}) = \int r(\bar{x}, \bar{y}) f(\bar{y}) d\bar{y}$, it follows from the mean-value theorem together with (3.13) and Lebesgue's dominated convergence theorem that $(\partial/\partial x_i) Rf$ ($i = 1, \dots, n-1$) is given by a kernel $k(\bar{x}, \bar{y})$ satisfying

$$(3.14) \quad \int |k(\bar{x}, \bar{y})| d\bar{x} = O(1) \quad \text{uniformly in } \bar{y}, \\ \int |k(\bar{x}, \bar{y})| d\bar{y} = O(1) \quad \text{uniformly in } \bar{x}.$$

Hence each operator $(\partial/\partial x_i)R$ is bounded: $L^p(U_\epsilon) \rightarrow L^p(U_\epsilon)$, and hence, Rf is bounded: $L^p(U_\epsilon) \rightarrow L^p_1(U_\epsilon)$. See [15, p. 59].

REMARK. In the proof of Proposition 8 we shall show that $A(t)$ and $B(t)$ in (3.12) give operators that are actually smoothing of order 2. (The proof will use the result that Λ^{-1} is smoothing of order 1.)

To complete the proof of Proposition 6 we now show that the first term in (3.12), which we denote by $K_1(t)$, gives a bounded operator: $L^p(U_\epsilon) \rightarrow L^p_1(U_\epsilon)$. In fact

$$(3.15) \quad K_1(t(\bar{x}, \bar{y})) = -(2/(n-2))\omega_n^{-1}Q(x, y)|\bar{x} - \bar{y}|^{-n+2},$$

where

$$(3.16) \quad Q(\bar{x}, \bar{y}) = 2^{(2-n)/2}(1-t)^{(2-n)/2}|\bar{x} - \bar{y}|^{n-2}$$

is C^∞ for $\bar{x} \neq \bar{y}$ and satisfies

$$(3.17) \quad \lim_{\bar{y} \rightarrow \bar{x}} Q(\bar{x}, \bar{y}) = x_n^{n-2}.$$

Now the function $Q^{2/(n-2)} = \frac{1}{2}|\bar{x} - \bar{y}|^2(1-t)^{-1}$ is for $|\bar{x} - \bar{y}| < \frac{1}{2}$ an analytic function of the variables $z_j = x_j - y_j$ ($j=1, \dots, n-1$). By using the Riemann removable singularities theorem for the function $\phi_j(z_j)$ obtained by setting all the z 's other than z_j equal to 0 we see that in fact $Q^{2/(n-2)}$ is an analytic function whose real part is bounded away from 0 for $|\bar{x}| < 1 - \delta$. Hence $Q(\bar{x}, \bar{y})$ is actually C^∞ for all \bar{x}, \bar{y} . Let us write

$$(3.18) \quad Q(\bar{x}, \bar{y}) = Q'(\bar{x}, \bar{x} - \bar{y}) = x_n^{n-2} + a_1(\bar{x})|\bar{x} - \bar{y}| + a_2(\bar{x})|\bar{x} - \bar{y}|^2 + \dots + a_{n-2}(\bar{x})|x - y|^{n-2} + b(x, y)|\bar{x}, \bar{y}|^{n-1},$$

where each a_j and b are bounded C^∞ functions with compact support. Substituting this expression into (3.15) and using (3.5) we see that since $\omega_n^{-1} = \frac{1}{2}\Gamma(\frac{1}{2}n)\pi^{-(1/2)n}$,

$$(3.19) \quad K_1(t(\bar{x}, \bar{y})) = x_n^{n-2}(2-n)^{-1}\Gamma(n/2)\pi^{-n/2}|\bar{x} - \bar{y}|^{-n+2+r},$$

where $r(\bar{x}, \bar{y})$ gives rise to an operator R that is bounded: $L^p(U_\epsilon) \rightarrow L^p_2(U_\epsilon)$. (See [15, p. 59].) The first term in (3.19) gives in coordinates \bar{x} for U_ϵ the operator $x_n^{n-2}\Lambda_E^{-1}$ defined by

$$(3.20) \quad \Lambda_E f = i \sum_{j=1}^{n-1} \left(\frac{\partial}{\partial x_j} \right) R_j f,$$

$$(R_j f)(\bar{x}) = -i\Gamma\left(\frac{n}{2}\right)\pi^{-n/2} \lim_{\epsilon \rightarrow 0} \int_{|\bar{x} - \bar{y}| > \epsilon} f(\bar{y})(\bar{x}_j - \bar{y}_j)|\bar{x} - \bar{y}|^{-n} dy.$$

(See [21, p. 662].)

Since U_ϵ has compact closure, Λ_E^{-1} is bounded: $L^p(V) \rightarrow L^p_1(V)$, where V is the domain of the coordinates \bar{x} for U_ϵ . This concludes the proof.

We remark that a simpler proof would show that R given by r in (3.19) is smoothing of order 1 on U_ϵ , but for the proof of Proposition 8 we need the fact that it is smoothing of order 2. Also, it is possible to arrive at a similar inequality

without using complex variable theory by suitable use of Taylor's theorem after transforming the expression (3.16) for $Q(\bar{x}, \bar{y})$.

The invariant Laplacian on S^{n-1} is given by $\Delta = \sum_{i < j} D_{ij}^2$. In terms of its action on spherical harmonics, $\Delta Y_s = -s(s+n-2) Y_s$ if $Y_s \in H_s$. (See [7, pp. 385-389 and p. 397] as well as [3] and [26, pp. 61-63].) Using Δ we can define the inverse to Λ^{-1} by

$$(3.21) \quad \Lambda = -\Delta \Lambda^{-1} + (n-4)I + (3-n)\Lambda^{-1}.$$

A simple computation shows that if $Y_s \in H_s$

$$(3.22) \quad \Lambda Y_s = -(s+1)Y_s,$$

so that Λ and Λ^{-1} are inverse to each other on $C^\infty(S^{n-1})$.

But it follows from Proposition 5 and the fact that Δ is bounded: $L_{k+2}^p \rightarrow L_k^p$ that Λ is bounded: $L_{k+1}^p \rightarrow L_k^p$ for $k \geq 0$. Since the dual of L_k^p is L_{-k}^q for $1 < p < \infty$ and $q = p/(p-1)$, we obtain analogous results for $k < 0$. We may summarize by stating

PROPOSITION 6. *For all integral k and $1 < p < \infty$, the map Λ^{-1} is an isomorphism: $L_k^p(S^{n-1}) \rightarrow L_{k+1}^p(S^{n-1})$. Its inverse is Λ given by (3.21) or (3.22).*

Thus Λ^{-1} bears a good analogy to $(1 + \Lambda_E)^{-1}$, where Λ_E^{-1} is defined by (3.20).

DEFINITION. The multiplier on spherical harmonics determined by the sequence $\{\lambda_s\}_{s=0}^\infty$ of complex numbers is the operator T defined by $TY_s = \lambda_s Y_s$ if $Y_s \in H_s$.

We shall often denote this operator by (λ_s) . Thus,

$$\Lambda^{-1} = -1/(s+1), \quad \Delta = (-s(s+n-2)).$$

PROPOSITION 7. *The multiplier on spherical harmonics $(1/(s+1) - 1/(s+2))$ is bounded: $L_k^p(S^{n-1}) \rightarrow L_{k+2}^p(S^{n-1})$ for all integral k and all complex λ .*

Proof. We consider only the case $k=0$. The case $k \geq 0$ follows from a duality argument. Expanding $1/(x+\lambda)$ in a power series in $1/(x+1)$ and setting $x=s$ gives

$$(3.23) \quad \frac{1}{s+2} - \frac{1}{s+1} = (s+1)^{-2} \sum_{j=0}^\infty (1-\lambda)^{j+1} (s+1)^{-j},$$

with convergence absolute and uniform for fixed λ and $s \geq 1$. We may restrict consideration to $s \geq 1$ since the operator sending f to $\int_{S^{n-1}} f d\sigma$ is smoothing of all orders. In view of Proposition 5 it is enough to show for $s \geq 1$ the infinite series in (3.23) gives rise to a bounded operator on $L^p(S^{n-1})$. It is sufficient to show that for some integer N (which may depend on λ) $\sum_{N+1}^\infty (1-\lambda)^{j+1} (1/(s+1))^j$ gives rise to a bounded operator on L^p .

Let $P_s(x \cdot y)$ be the zonal harmonic of degree s normalized so that $P_s(1) = 1$. (See [3] or [7] for a discussion.) Then the kernel $(1/\omega_{n-1}) \sum_0^\infty d_s c_s P_s(x \cdot y)$ gives rise to the multiplier on spherical harmonics determined by the sequence $\{c_s\}$. (Here d_s denotes the dimension of H_s .) For example, setting $c_s = r^s$ yields the Poisson kernel.

If $c_s = O(s^{-n/2})$, then the kernel written down above is an L^2 function of y for each fixed x . This follows from $\int P_s^2 dx = 1/d_s$, Parseval's theorem, and the estimate $d_s = O(s^{n-2})$. (See [5] for the last.) If we choose N so large that uniformly in $s \geq 1$

$$\left| \sum_{N+1}^{\infty} (1-\lambda)^{j+1}(s+1)^{-j} \right| \leq c_\lambda s^{-n/2},$$

then the operator it gives rise to will be given by a kernel $K_N(x \cdot y)$ that is an L^2 function of y for each x and invariant under rotations, and hence is bounded on L^p .

We are now in a position to obtain a more precise expression for Λ^{-1} in local coordinates \bar{x} for $U_{1/2} = \{x \in S^{n-1} \mid \frac{1}{2} < x_n \leq 1\}$.

PROPOSITION 8. *In local coordinates \bar{x} for $U_{1/2}$,*

$$(3.24) \quad (\Lambda^{-1}f)(\bar{x}) = x_n^{n-2}(\Lambda_E^{-1}f) + (Rf)(\bar{x}),$$

where for each integral k , R is bounded: $L_k^p(U_{1/2}) \rightarrow L_{k+2}^p(U_{1/2})$ and Λ_E^{-1} is defined by (3.20).

Proof. In view of the expression (3.19) for $K_1(t(\bar{x}, \bar{y}))$ and the remark made after (3.20), it is sufficient to prove that the last two terms in (3.12) give bounded operators: $L_k^p(U_{1/2}) \rightarrow L_{k+2}^p(U_{1/2})$. We shall show that they give bounded operators: $L_k^p(S^{n-1}) \rightarrow L_{k+2}^p(S^{n-1})$. As in the proof of Proposition 7 we may restrict consideration to the case $k=0$.

Integrating $A(t)$ in (3.12) against $Y_s \in H_s$ shows that $A(t)$ gives rise to the multiplier on spherical harmonics $(1/(s+n-3) - 1/(s+1))$, which has the desired boundedness by Proposition 7. The operator arising from $B(t)$ is treated by writing

$$(3.25) \quad (1-2rt+r^2)^{(2-n)/2} = \sum_{s=0}^{\infty} \frac{n-2}{2s+n-2} d_s r^s P_s(t),$$

where $P_s(t)$ is as in the proof of Proposition 7. (If $\lambda = (n-2)/2$, then

$$P_s(t) = P_s^\lambda(t)/P_s^\lambda(1),$$

where $(1-2rt+r^2)^{-\lambda} = \sum_0^\infty r^s P_s^\lambda(t)$. See [5, p. 903] and [3].) Multiplying (3.25) by r^{n-4} and integrating from 0 to 1 with respect to r gives an $L^1(S^{n-1})$ function of y for each fixed x . It has the expansion

$$\sum_0^\infty \frac{n-2}{2s+n-2} \frac{1}{s+n-3} d_s P_s(t).$$

Thus integration against $B(t)$ yields the multiplier on spherical harmonics

$$\frac{n-4}{n-2} \left(\frac{1}{2s+n-2} \right) \left(\frac{1}{s+n-3} \right),$$

which is bounded: $L_k^p(S^{n-1}) \rightarrow L_{k+2}^p(S^{n-1})$ by Propositions 7 and 5.

As a corollary of Proposition 8 we have

PROPOSITION 9. Let D be a differential operator of order s on S^{n-1} with C^∞ coefficients. Then

$$i^s D \circ \Lambda^{-s} = A + S,$$

where $A \in C^\infty(S^{n-1})$ and S is a smoothing operator.

Proof. We may assume the coefficients of D have support in an open set of S^{n-1} having arbitrarily small diameter. Since Λ^{-1} commutes with rotations, we may assume this set is U_ϵ as in Proposition 8. We write

$$(i^s D \circ \Lambda^{-s})f(\bar{x}) = \sum_{|\alpha| \leq s} a_\alpha(\bar{x}) [D^\alpha(x_n^{n-2} \Lambda_E^{-1})f](\bar{x}) + Rf(\bar{x}),$$

where the notation is as before and $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ is a multi-index. Using the Leibnitz rule on the terms in brackets we may express $i^s D \circ \Lambda^{-s}$ as the sum of a Euclidean singular integral operator and a remainder that smooths.

IV. In this chapter we prove the classification and decomposition theorems for singular operators on $L^p(S^{n-1})$. The main tool is a linear map $q \rightarrow T_q$ from the space of $SO(n)$ -harmonic polynomials to the $C^\infty(S^{n-1})$ operators. It is constructed in such a way that

$$(4.1) \quad \sigma(T_q) = q|_{CS}, \quad L_a T_q L_a^{-1} = T_{q_a},$$

where $q_a(x, \xi) = q(a^{-1}x, a^{-1}\xi)$. The fact that for each harmonic polynomial q the operator T_q is actually in $C^\infty(S^{n-1})$ follows from Proposition 9. Let us assume that this map $q \rightarrow T_q$ has already been defined. Using Theorem 3 we see that for each dominant weight of the form $m = (m_1, m_2, 0, \dots, 0)$ we obtain a family $\mathcal{O}_m = \{T_q \mid q \in \mathcal{Q}_m\}$. The equations (4.1) show that in fact $q \rightarrow T_q$ gives an equivalence between the action of $SO(n)$ on \mathcal{Q}_m and that given on \mathcal{O}_m by $T \rightarrow L_a T L_a^{-1}$. This representation of $SO(n)$ on \mathcal{O}_m is not irreducible, but we may use the decomposition (2.6) of \mathcal{Q}_m into irreducible subspaces to write

$$\mathcal{O}_m = \sum_{r+s=m_1+|m_2|; |m_2| \leq r \leq m_1} \oplus V_{r,s}^m,$$

where $V_{r,s}^m = \{T_q \mid q \in P_{r,s}^m\}$ is invariant and irreducible under the action of $SO(n)$ (with the suitable modification for $n=3$, using (2.15)).

DEFINITION. By a system of operators on $L^p(S^{n-1})$ (or some other Banach space) we shall mean a (complex) vector space V of such operators T . The systems $V_{r,s}^m$ defined above will be called the canonical systems of singular operators.

DEFINITION. We shall say that a system V transforms according to a representation R of $SO(n)$ on V if for each $a \in SO(n)$ and $T \in V$

$$(4.2) \quad L_a T L_a^{-1} = R_a T.$$

If $\{V_i\}$ is a (finite or infinite) collection of systems of $C^\infty(S^{n-1})$ operators transforming according to (possibly inequivalent) representations R_i of $SO(n)$, we shall

say that $\{V_i\}$ is a collection of *independent* systems if $T_1 + \dots + T_k$ (with $T_i \in V_i$) smooths implies that each T_i smooths.

THEOREM 5. *Any singular operator T in $C^\infty(S^{n-1})$ can be decomposed into a sum*

$$T = \sum_m \sum_{i=1}^{s(m)} T_{m,i},$$

where $T_{m,i} = T'_{m,i} + S_{m,i}$, with T'_i an element of one of the canonical systems $V_{r,s}^m$ and $S_{m,i}$ a smoothing operator.

The summation is over all dominant weight vectors of the form

$$m = (m_1, m_2, 0, \dots, 0);$$

for such m , $s(m) = m - |m_2| + 1$ if $n \geq 4$, $= 2m + 1$ if $n = 3$.

THEOREM 6. *If $n \geq 4$ and $m = (m_1, m_2, 0, \dots, 0)$ is a dominant weight there are $m_1 - |m_2| + 1$ independent systems of operators in $C^\infty(S^{n-1})$ transforming according to the irreducible representation R^m of $SO(n)$; if m is not of this form there are no such systems.*

If $n = 3$ the number for R^m is $2m + 1$.

REMARK. The first equation in (4.1) together with Theorem 2 shows that for fixed $m = (m_1, m_2, 0, \dots, 0)$ ($n \geq 4$) the systems $V_{r,s}^m \subset \mathcal{O}_m$ are in fact $m_1 - |m_2| + 1$ independent systems of $C^\infty(S^{n-1})$ operators. Of course, a different decomposition of \mathcal{Q}_m into irreducible subspaces could be used to define $m_1 - |m_2| + 1$ independent systems. (Suitable modifications hold for $n = 3$.) But the space \mathcal{O}_m can be invariantly characterized, modulo smoothing operators.

Before giving the proofs of Theorems 5 and 6 we construct the map $q \rightarrow T_q$ satisfying (4.1). We begin by defining the operators R_j and X_j and computing their symbols. For more details see §4 of [14].

The first-order differential operators D_1, \dots, D_n on S^{n-1} are defined by extension of functions to E_n : for $f \in C^\infty(S^{n-1})$

$$(4.3) \quad (D_j f)(x) = |x| \frac{\partial}{\partial x_j} f\left(\frac{x}{|x|}\right),$$

where $\partial/\partial x_j$ denotes differentiation in the j th direction in E_n . We note that the D_j inherit the transformation properties of the $\partial/\partial x_j$:

$$L_a D_j L_a^{-1} = \sum_{k=1}^n a_{kj} D_k.$$

The symbol of D_j as a first-order differential operator can be computed by the method given in [18, p. 63] as

$$(4.4) \quad \sigma(D_j)(x^0, \xi^0) = -i D_j \left[\sum_{k \neq j} (x_k \xi_k^0 - x_k^0 \xi_k^0) \right] = -i \xi_j^0.$$

This is consistent with the definitions to be adopted below.

DEFINITION. The operators R_j and X_j are defined on $C^\infty(S^{n-1})$ by

$$(4.5) \quad (X_j f)(x) = x_j f(x), \quad (R_j f)(x) = (iD_j \Lambda^{-1} f)(x).$$

One can easily compute the transformation laws

$$(4.6) \quad L_a X_j L_a^{-1} = \sum_k a_{kj} X_k, \quad L_a R_j L_a^{-1} = \sum_k a_{kj} R_k.$$

We shall want to speak of the symbols of such operators as $C^\infty(S^{n-1})$ singular operators, differential operators and powers of Λ^{-1} . It will be convenient to use the algebra $CZ(S^{n-1}) = \bigcup_r CZ_r(S^{n-1})$ discussed in [19] and, except for certain changes in sign, the definition of σ_r on $CZ_r(S^{n-1})/CZ_{r-1}(S^{n-1})$ given there. Briefly stated, an operator is in $CZ_r(S^{n-1})$ if in each coordinate neighborhood U the expression of A in local coordinates gives rise to an operator $A_U = \sum_{j=0}^\infty A_{r-j}$ satisfying $(A_{r-j} f)^\wedge(\xi) = a_{r-j}(x, \xi) \hat{f}(\xi)$ for $f \in L^2(E_{n-1})$ with support in the image Ω_U of U . We require that a_{n-j} be C^∞ for $\xi \neq 0$ and that $R_{k+1} = \sum_{j=0}^k A_{n-j}$ be bounded: $L_s^2(\Omega_U) \rightarrow L_{s+k+1}^2(\Omega_U)$. For $r < 0$ let \mathcal{F}_r denote the space of operators K that (1) are given by kernels $k(x, z)$ which are C^∞ for $z \neq 0$ and (2) are bounded: $L_s^p \rightarrow L_{s+r}^p$. By a modification of the proof of Lemma 4 in [21], we can extend the definition of σ_r for $r \leq 0$ to operators of the form $T + R$, where $T \in CZ_r(S^{n-1})$ and $R \in \mathcal{F}_{r+1}(S^{n-1})$, i.e., to $CZ_r(S^{n-1}) + \mathcal{F}_{r+1}(S^{n-1})$. With this extension

$$(4.7) \quad \sigma_r(A)\sigma_s(B) = \sigma_{r+s}(AB)$$

continues to hold. Also, the function σ_0 on $CZ_0(S^{n-1}) + \mathcal{F}_1(S^{n-1})$ agrees, except for sign, with the function σ defined on $C^\infty(S^{n-1})$ in [21]. In fact,

$$[CZ_0(S^{n-1}) + \mathcal{F}_1(S^{n-1})] / \mathcal{F}_1(S^{n-1}) = C^\infty(S^{n-1}),$$

and both σ_0 and σ vanish on $\mathcal{F}_1(S^{n-1})$. Here we shall use the sign conventions arising from the formula for operators H on $L^p(E_{n-1})$

$$(4.8) \quad \sigma(H)(x, \xi) = \lim_{\epsilon \rightarrow 0} \int_{\epsilon^{-1} > |z| > \epsilon} k(x, z) e^{iz\xi} dz.$$

(Here $k(x, z)$ is homogeneous of degree $-n+1$ in z and has mean value 0 for each x .)

If A is in $CZ_r(S^{n-1})$, then $\sigma_r(A)$ satisfies the transformation law for a function defined on $T^*(S^{n-1})$. More precisely, let U be a coordinate neighborhood for S^{n-1} and $\alpha: U \rightarrow \Omega_U \subseteq E_{n-1}$ a chart. If ϕ and ψ are in $C^\infty(S^{n-1})$, have their support in U , and are $\equiv 1$ on a neighborhood $V \subset U$, then there is an operator H in $CZ_r(E_{n-1})/CZ_{r-1}(E_{n-1})$ supported in Ω_U (which we may assume to have compact closure in E_{n-1}) such that

$$(4.9) \quad (\phi A \psi) f = H(f \circ \alpha^{-1}) \circ \alpha$$

holds for functions f supported in U . Moreover, for $x \in V$

$$(4.10) \quad \sigma_r(A)(x, \xi_x) = \sigma_r(H)(\alpha(x), (d\alpha_x^*)^{-1} \xi_x),$$

where ξ_x is any element in the fibre over x , and $d\alpha_x^*$ is the adjoint of the differential of α at x . This formula is given on p. 264 of [19], where slightly different notation is employed.

PROPOSITION 10. For $f \in C^\infty(S^{n-1})$ and $a \in SO(n)$ set $(L_a f)(x) = f(a^{-1}x)$, $l_a x = ax$. Then if

$$(4.11) \quad A \in CZ_r(S^{n-1})/CZ_{r-1}(S^{n-1}), \quad \sigma(L_a A L_a^{-1})(x, \xi) = \sigma_r(A)(a^{-1}x, a^{-1}\xi).$$

(Here we use (1.7) and the remarks following it.)

Proof. Let U be a coordinate neighborhood for S^{n-1} and let ϕ and ψ be two $C^\infty(S^{n-1})$ functions with support in U that are $\equiv 1$ on the neighborhood V of x . If $\alpha: U \rightarrow \Omega_U$ is a chart, then by (4.10)

$$\sigma(L_a A L_a^{-1})(x, \xi) = \sigma_r(H)(\alpha(x), (d\alpha_x^*)^{-1}\xi),$$

where H is an operator in $CZ_r(E_{n-1})$ such that

$$(4.12) \quad [(\phi L_a A L_a^{-1} \psi)f](x) = H(f \circ \alpha^{-1}) \circ \alpha.$$

But for $x \in V$ and f supported in $a^{-1}U$,

$$(4.13) \quad [(L_a^{-1} \phi)A(L_a^{-1} \psi)]f(a^{-1}x) = [(\phi L_a A L_a^{-1} \psi)(f \circ l_a^{-1}) \circ l_a](a^{-1}x) \\ = [H(f \circ l_a^{-1} \circ \alpha^{-1}) \circ (\alpha \circ l_a)](a^{-1}x),$$

on applying (4.12) to $f \circ l_a$. Now (4.10) can be used with α replaced by the chart $\beta = \alpha \circ l_a$ for $a^{-1}U$ to obtain

$$(4.14) \quad \sigma_r(A)(a^{-1}x, a^{-1}\xi) = \sigma_r(H)(\beta(a^{-1}x), (d\beta_{a^{-1}x}^*)^{-1}a^{-1}\xi) \\ = \sigma_r(H)(\alpha(x), (d\alpha_x^*)^{-1}(dl_a^*)_{a^{-1}x}^{-1}a^{-1}\xi) \\ = \sigma_r(L A L_a^{-1})(x, a a^{-1}\xi).$$

Proposition 10 is very useful when dealing with a system of operators transforming according to an irreducible representation R of $SO(n)$. For then if we know $\sigma(A)$ globally for one operator A we can compute $\sigma(R_a A)$ globally for every $a \in SO(n)$. Conversely, if for some open set U of S^{n-1} and each A we know $\sigma(A)(x, \xi)$ for $x \in U$, then (4.11) can be used to compute $\sigma(A)$ globally for each A . In particular we may use the results of §III together with the fact that the system $\{c\Lambda^{-1} \mid c \in C\}$ transforms according to the trivial representation of $SO(n)$ to compute $\sigma(\Lambda^{-1})$.

PROPOSITION 11. The operator Λ^{-1} is in $[CZ_{-1}(S^{n-1}) + \mathcal{F}_2(S^{n-1})]/\mathcal{F}_2(S^{n-1})$. Its -1 order symbol is

$$(4.15) \quad \sigma_{-1}(\Lambda^{-1})(x, \xi) = |\xi|^{-1},$$

where $(x, \xi) \in CS(S^{n-1})$ is as in (1.5).

Proof. It is enough to consider $\phi\Lambda^{-1}\psi$, where ϕ and ψ are arbitrary C^∞ functions that are $\equiv 1$ in $U_\epsilon = \{x \in S^{n-1} \mid 1 - \epsilon < x_n \leq 1\}$ and are supported in a slightly larger open set. By the remarks before (4.7) and Proposition 8,

$$\sigma_{-1}(\Lambda^{-1})(N, \xi_N) = (\xi_1^2 + \xi_2^2 + \dots + \xi_{n-1}^2)^{-1/2} = |\xi_N|^{-1}.$$

(See [21, p. 662] for $\sigma(\Lambda_E^{-1})(\xi)$.) Now (4.15) follows from (4.11) together with the fact that if $a\xi_N = (\zeta_1, \dots, \zeta_n)$, then $|a\xi_N|^2 = \zeta_1^2 + \dots + \zeta_n^2 = \xi_1^2 + \dots + \xi_{n-1}^2$. (Clearly, (4.11) continues to hold in the present situation.)

Since the symbol map (principal part) is an algebra homomorphism, it follows as a corollary of Proposition 11 that if D is a differential operator of order s on S^{n-1} with C^∞ coefficients, then

$$(4.16) \quad \sigma_0(D \circ \Lambda^{-s})(x, \xi) = \sigma_s(D)(x, \xi)|\xi|^{-s}.$$

In particular, for the operators R_j of (4.5)

$$(4.17) \quad \sigma_0(R_j)(x, \xi) = \sigma_0(iD_j \circ \Lambda^{-1}) = \xi_j|\xi|^{-1}.$$

Since the symbol of multiplication by a smooth function $a(x)$ is $a(x)$, we have

$$(4.18) \quad \sigma_0(X_j)(x, \xi) = x_j.$$

PROPOSITION 12. *Let $q(x, \xi)$ be any polynomial such that for real numbers t, u $q(tx, u\xi) = t^r u^s q(x, \xi)$. Then*

$$(4.19) \quad T_q = q(X, iD)\Lambda^{-s}$$

is a $C^\infty(S^{n-1})$ operator with

$$(4.20) \quad \sigma(T_q) = q|CS(S^{n-1}).$$

Moreover, for $a \in \text{SO}(n)$,

$$(4.21) \quad L_a T_q L_a^{-1} = T_{q_a}.$$

Proof. The transformation law (4.21) for a polynomial $q = \sum_{|\alpha| = r; |\beta| = s} c_{\alpha\beta} x^\alpha \xi^\beta$ follows from the transformation laws (4.6), which when rewritten become (4.21) for the polynomials x , and ξ_j . Also, (4.20) follows from (4.17) and (4.18), and the fact that σ_0 is a homomorphism.

DEFINITION. If q is an $\text{SO}(n)$ -harmonic polynomial, then T_q is the $C^\infty(S^{n-1})$ operator defined by (4.19).

From now on we shall only speak of T_q for $\text{SO}(n)$ -harmonic polynomials q . In view of Proposition 12 we have achieved the construction $q \rightarrow T_q$ required for the definitions at the beginning of this section.

Before passing to the proof of Theorem 5 we establish some preliminary estimates.

LEMMA 9. *Let $h \in C^\infty(\text{SO}(n)/\text{SO}(n-2))$ have the expansion $h = \sum_m h_m$, with $h_m \in \mathcal{H}_m$ (see (2.1)). Then as $|m| = m_1 + |m_2| \rightarrow \infty$*

- (1) $\|h_m\|_2 = O(|m|^{-k})$ for any integer $k > 0$;
- (2) $d_m = \text{degree } R^m = O(|m|^{2n})$.

Proof. For (1) we use the fact that \mathcal{H}_m is the sum of eigenspaces of the (elliptic invariant Laplacian (or Casimir operator) $\Delta = \sum_{i < j} D_{ij}^2$ on $SO(n)$). Since this operator is invariant under the right as well as the left action of $SO(n)$, the eigenvalue is the same for each summand in \mathcal{H}_m . This eigenvalue $\lambda(m)$ is of the form

$$(4.22) \quad \lambda(m) = - \sum a_{ij} m_i m_j + \sum b_k m_k,$$

where (a_{ij}) is a positive definite symmetric matrix (see [10, pp. 246–247]). From (4.22) and the projection formula

$$(4.23) \quad (\pi_m h)(g) = d_m \int_{SO(n)} h(a) \bar{\chi}_m(a^{-1}g) da$$

for π_m , the projection onto \mathcal{H}_m , we see that for any integer $k > 0$

$$\pi_m(\Delta^k h) = [\lambda(m)]^k \pi_m h = [\lambda(m)]^k h_m.$$

(We used the facts that Δ is selfadjoint and that $\chi(ab) = \chi(ba)$.)

Hence for any integer $k > 0$

$$\sum_m [\lambda(m)]^{2k} \|h_m\|_2^2 < \infty,$$

and the first part of the lemma follows.

The second assertion follows from an estimate using the degree formula for R^m given in [1, p. 250]. (A trivial estimate gives $d_m = O(|m|^{n^3})$, which is actually sufficient for our purposes.)

LEMMA 9. *Let us choose a finite set of differential operators on $CS(S^{n-1})$ whose linear span is the space of all differential operators on $CS(S^{n-1})$ of order $\leq 2k$. If D is any element of this set and $h_m \in \mathcal{H}_m$, then*

$$(4.24) \quad |Dh_m| \leq C_{n,k} |m|^{2n+2k}.$$

Proof. Any differential operator D can be expressed in a small open set as the sum with smooth coefficients of iterates of differentiations given by the standard basis elements of the Lie algebra of $SO(n)$. Since $CS(S^{n-1})$ is compact, it is enough to prove an estimate like (4.24) for D an iterate of order $\leq 2k$ of Lie algebra differentiations. Such operators D commute with the Laplacian on $SO(n)$, and thus preserve the spaces \mathcal{H}_m . Write

$$(Dh_m)(g) = d_m \int_{SO(n)} (Dh_m)(a) \bar{\chi}_m(a^{-1}g) da.$$

Then for each $g \in SO(n)$,

$$|Dh_m(g)| \leq d_m \|Dh_m\|_2 \|\chi_m\|_2.$$

But $\|\chi_m\|_2 = 1$ as a function of a for each g , and the other factor can be majorized by $C \|(\Delta - 1)^k h_m\|_2$, since $\Delta - 1$ provides an isomorphism between $L^2_{r+2}(SO(n))$ and

$L_r^p(\text{SO}(n))$ for $r \geq 0$ (see [21, pp. 684–685]). In view of (4.22) and the argument following it, together with Lemma 8, we have (4.24).

Proof of Theorem 5. Using (2.6) and Theorem 2, let us write \mathcal{H}_m as the (algebraic) direct sum of irreducible spaces:

$$\mathcal{H}_m = \sum_{i=1}^{s(m)} H_{m,i},$$

and choose an orthonormal basis $\{e_{ij}^m\}$, $j=1, \dots, d_m$, for each $H_{m,i}$. Then if $T \in C_\infty^\infty(S^{n-1})$,

$$(4.25) \quad \sigma(T) = \sum_m \sum_{i=1}^{s(m)} \sum_{j=1}^{d_m} c_{ij}^m e_{ij}^m.$$

Let $\{\phi_k\}$ be any C^∞ partition of unity for S^{n-1} . Let T_{ij}^m denote the $C_\infty^\infty(S^{n-1})$ operator constructed as in [21, p. 678] with the use of $\{\phi_k\}$ to have symbol $e_{ij}^m(x)$. Then

$$(4.26) \quad \|T_{ij}^m f\|_p \leq AB_m \|f\|_p,$$

where A is a constant depending on n, p and $\{\phi_k\}$, and B_m is a bound for $e_{ij}^m(x)$ and its derivatives of order $\leq 2n-2$ with respect to cosphere variables. See [21, p. 678]. By (4.25), (4.26) and Lemma 9 the series $\sum_m \sum_{i,j} T_{ij}^m$ converges in L^p -operator norm.

Now let $T_{m,i} = \sum_{j=1}^{d_m} c_{ij}^m T_{ij}^m$. Because of the way we have defined $H_{m,i}$ there is a pair (r, s) of indices such that $\sigma(T_{m,i}) = q_{m,i}|CS$, with $q_{m,i} \in P_{r,s}^m$. Setting $T'_{m,i} = T_{q_{m,i}}$ for each m, i we have

$$\sigma(T) = \sigma\left(\sum_m \sum_i T_{m,i}\right), \quad \sigma(T'_{m,i}) = \sigma(T_{m,i}),$$

with $T'_{m,i}$ lying in the canonical system $V_{r,s}^m$. The conclusion of Theorem 6 now follows from the fact that if S is a $C_\infty^\infty(S^{n-1})$ operator and $\sigma(S) = 0$, then S smooths.

Let T be a bounded operator on a Banach space B of measurable functions on S^{n-1} whose norm $\|\cdot\|$ satisfies

- (1) $\|f_a\| = \|f\|$;
- (2) $a \rightarrow \|f_a\|$ is a continuous function of $a \in \text{SO}(n)$ for each $f \in B$.

Then for each $a \in \text{SO}(n)$, $L_a T L_a^{-1}$ is bounded on B with norm equal to that of T . Moreover, if ϕ is continuous on $\text{SO}(n)$, the integral $\int_{\text{SO}(n)} (L_a T L_a^{-1}) f(x) \phi(a) da$ defines a bounded operator on B . These remarks apply in particular to $B = L_k^p(S^{n-1})$ for $1 < p < \infty$ and $k \geq 0$, and to any operator in $CZ(S^{n-1})$.

Proof of Theorem 6. Let V be any system of singular operators transforming according to the irreducible representation R^{m_0} of $\text{SO}(n)$. Fixing any element $T \in V$, we may write

$$(4.27) \quad T = \sum_m \sum_i (T'_{m,i} + S_{m,i}),$$

as in Theorem 5. Since this series converges in L^2 -operator norm, to compute

$$(4.28) \quad d_{m_0} \int_{\text{SO}(n)} (L_a T L_a^{-1}) f(x) \bar{\chi}_{m_0}(a) da$$

we may substitute (4.27) and integrate term by term. The resulting integrals involving $T'_{m,i}$ will be zero if $m \neq m_0$, while the integral involving $S_{m,i}$ yields a smoothing operator whose norm does not exceed that of $T'_{m,i} + S_{m,i}$. The first assertion can be justified by considering the integral

$$\int_{\text{SO}(n)} g(x) \int_{\text{SO}(n)} (L_a T L_a^{-1}) f(x) \bar{\chi}_m(a) da dx$$

for $g \in C^\infty(\text{SO}(n))$, and using Fubini's theorem and the Shur orthogonality relations. (See [1].) The second assertion follows from the L^p -operator convergence of $\sum T'_{m_0,i} + S_{m,i}$, and the remarks made just before this proof.

Thus the right-hand side of (4.28) becomes $\sum_i T'_{m_0,i} + S$, where $T'_{m_0,i}$ lies in one of the canonical systems $V_{r,s}^m$ and S smooths. By a similar argument the left-hand side of (4.28) is seen to be T . Thus $T = T_{q_{m_0}} + S$, with $q_{m_0} \in Q_{m_0}$. But for $n \geq 4$ each Q_m yields $m_1 - |m_2| + 1$ ($2m + 1$ for $n = 3$) independent systems, and no more. This concludes the proof of Theorem 6.

We remark that with the decomposition (2.6) of Q_m each canonical system $V_{r,s}^m$ is described by the numbers r and s of multiplications and compensated differentiations.

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