

TAMING EMBEDDINGS OF CERTAIN POLYHEDRA IN CODIMENSION THREE⁽¹⁾

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1. **Introduction.** In [18], [19], by using techniques of Homma developed in [24] and of Bing and Kister developed in [2], Gluck showed that a locally tame embedding of a polyhedron into a combinatorial manifold is ε -tame in the trivial range. It also followed that a locally flat embedding of a closed combinatorial manifold into a combinatorial manifold is ε -tame in the trivial range. Greathouse obtained similar results in [21]. Both Bryant in [3] and Dancis in [14] showed that if the set of points on which an embedding is not known to be locally tame is mapped into a tame polyhedron in the trivial range with respect to the ambient combinatorial manifold, then the embedding must be tame, in the trivial range. Černavskiĭ climbed out of the trivial range in [10] and showed that if an embedding of a polyhedron into a combinatorial manifold is locally flat on open simplexes, then it is ε -tame, in the metastable range. Finally, Bryant and Seebeck show in [5] that if an embedding of a combinatorial manifold into another combinatorial manifold is locally nice, then it is ε -tame, in the metastable range. The work of Bryant and Seebeck has the added advantage that if the work of Homma in [25] and [23] can be corrected (since it is said to be incorrect by Berkowitz, [1]), then it can be brought up to codimension three. (Homma's techniques simplify a great deal in the metastable range, and his work is easily corrected in this range. Also, Weber [39] has established an approximation theorem in the metastable range similar to Homma's.)

We begin this paper by showing that an allowable embedding of one combinatorial manifold into another which is PL on the inverse of the boundary of the ambient manifold and locally flat on the rest of the embedded manifold can be ε -tamed around the boundary of the ambient manifold in codimension three (Theorem 3.10). Another principal result of this paper is a taming theorem (Theorem 5.5) from which it follows that embeddings of certain nice polyhedra that are locally flat on open simplexes are tame in codimension three (Corollary

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5.6). Theorem 5.5 also yields a criterion for determining when spheres are unlinked (Corollary 5.8). A useful result employed in the proof of Theorem 5.5 is an unknotting of codimension three ball pairs theorem, Theorem 4.4, which generalizes [17]. We also prove a “crushing lemma” (Lemma 3.6) that applies to a variety of situations. Included at the end of the paper is a “best possible” metastable range taming theorem (Theorem 6.4) which generalizes the work of Bryant and Dancis mentioned above. The proof of this theorem however depends heavily on the work of Bryant and Seebeck and of Černavskii. In a later paper, we will employ results of this paper and prove taming theorems for embeddings of combinatorial manifolds into combinatorial manifolds in codimension three. Throughout this paper we assume that $n > 5$ unless otherwise specified.

2. Basic definitions and notation. We let E^n , E^n_+ , S^n and I^n denote euclidean n -space, n -dimensional euclidean half-space, the n -sphere and the n -cube, respectively. (I^1 will be denoted by I .) By a *polyhedron* we mean the underlying space of a finite simplicial complex unless specified otherwise. A *piecewise linear (PL) n -manifold* is the polyhedron of a locally finite simplicial complex in which the star of each vertex is PL homeomorphic to the standard n -simplex. The word manifold will be used for topological manifolds and we will indicate which manifolds are PL.

We denote the interior of a manifold M by either $\text{Int}(M)$ or M° and the boundary of M by either $\text{Bd}(M)$ or \bar{M} . A k -manifold M contained in the interior of an n -manifold N is *locally flat* at $x \in M^\circ$ if there is a neighborhood \mathcal{U} of x in N such that $(\mathcal{U}, \mathcal{U} \cap M)$ is homeomorphic to (E^n, E^k) ; and, is *locally flat* at a point $x \in \bar{M}$ if there is a neighborhood \mathcal{U} of x in N such that $(\mathcal{U}, \mathcal{U} \cap M)$ is homeomorphic to (E^n, E^k_+) . An embedding $f: M \rightarrow N^\circ$ is *locally flat* at a point $x \in M$ if $f(M)$ is locally flat at $f(x)$. Let C be a closed subset of the manifold M . Then, $M - C$ is 1-LC at $x \in C$ if for each neighborhood \mathcal{U} of x there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of x such that the injection $i: \pi_1(\mathcal{V} - C) \rightarrow \pi_1(\mathcal{U} - C)$ is trivial for any base point. A closed embedding $f: M^k \rightarrow Q^n$, $n - k \geq 3$, of a k -manifold M into an n -manifold Q is *locally nice* at $x \in M$ if $Q - f(M)$ is 1-LC at $f(x)$; (see [22]). Let $f: P^k \rightarrow Q^n$ be an embedding of the k -polyhedron P into the PL n -manifold Q . Then, f is a *locally tame* embedding if there is a triangulation of P such that for each point $x \in P$ there is an open neighborhood \mathcal{U} of $f(x)$ in Q and a triangulation of \mathcal{U} as a combinatorial manifold in terms of which f is PL on some neighborhood of x ; (see [19]).

The closure of a set X contained in the manifold Q is denoted by $\text{Cl}(X)$. If $x \in Q$, then $N_\varepsilon(x) = \{y \in Q \mid d(y, x) < \varepsilon\}$. Let $\varepsilon(x)$ be a nonnegative, continuous function defined on $X \subset Q$. By the $\varepsilon(x)$ -neighborhood of X , $N_{\varepsilon(x)}(X)$, we mean $\bigcup_{x \in X} N_{\varepsilon(x)}(x)$. (Notice that $N_{\varepsilon(x)}(X)$ is not necessarily a neighborhood of X .) If $\varepsilon(x)$ is defined on all of Q , we define an $\varepsilon(x)$ -isotopy of Q to be any isotopy $e_t: Q \rightarrow Q$ such that $e_0 = 1$ and $d(x, e_t(x)) \leq \varepsilon(x)$ for all $x \in Q$ and $t \in I$. An $\varepsilon(x)$ -isotopy of Q is called

an $\varepsilon(x)$ -push of (Q, X) if $e_t = \text{identity}$ outside $N_{\varepsilon(x)}(X)$ for each $t \in I$. An embedding f of a (possibly infinite) polyhedron P into the PL manifold Q is $\varepsilon(x)$ -tame if for each nonnegative, continuous function $\varepsilon(x)$ defined on Q , there is an $\varepsilon(x)$ -push e_t of $(Q, f(P))$ such that $e_1 f$ is PL in some neighborhood of any point $x \in P$ such that $\varepsilon(f(x)) > 0$. We define ε -neighborhood, ε -isotopy, ε -push and ε -tame similarly, where the function $\varepsilon(x)$ in the above definitions is the function $\varepsilon(x) = \varepsilon$. An embedding f of a polyhedron P into a PL manifold Q is tame if there is an isotopy, e_t , of Q such that $e_0 = \text{identity}$ and $e_1 f: P \rightarrow Q$ is PL. A map of one manifold into another will be called proper if it takes boundary to boundary and interior to interior. A topological embedding $f: M^k \rightarrow Q^n$ of the PL manifold M into the PL manifold Q is said to be allowable if $f^{-1}(\dot{Q})$ is a PL $(k-1)$ -submanifold (possibly empty) of M .

We say that k is in the metastable range with respect to n , if $k < \frac{2}{3}n - 1$, and that k is in the trivial range with respect to n , if $k \leq \frac{1}{2}n - 1$.

3. Taming embeddings of manifolds close to the boundary in codimension three.

In this section as well as later sections we will assume familiarity with the techniques of Černavskii developed in [9] as modified by Cantrell and Lacher in [7]. Before proving the main result of this section, Theorem 3.10, we will list some definitions and lemmas. I would like to thank Ralph Tindell for pointing out to me that the idea of "the natural parameterization" as defined by M. Cohen in [12] is useful in the proof of Lemma 3.6.

DEFINITION 3.1. If X and Y are subpolyhedra of the polyhedron Z then

$$\text{Cl}(X - Y) = X_{\natural} \quad \text{and} \quad \text{Cl}(X - Y) \cap Y = Y_{\natural}.$$

DEFINITION 3.2. If K and L are subcomplexes of some larger complex, we say that K is link-collapsible on L if $\text{lk}(\sigma, K_{\natural})$ is collapsible for each simplex σ in L_{\natural} . If X and Y are subpolyhedra of some larger polyhedron we say that X is link-collapsible on Y if for some triangulation K, L of X, Y we have K link-collapsible on L . (It is shown in [28] that the definition of link-collapsibility for polyhedra is independent of the triangulation.)

DEFINITION 3.3. If S is a set (which may or may not be contained in the polyhedron $|J|$) then

$$N(S, J) = \{\sigma \in J \mid \sigma \text{ is a face of a simplex of } J \text{ which meets } S\},$$

$$C(S, J) = \{\sigma \in J \mid \sigma \cap S = \emptyset\}, \quad \text{and} \quad \dot{N}(S, J) = N(S, J) \cap C(S, J).$$

In [12], it is established that if K and L are full subcomplexes of the complex J , then every simplex σ of $N(K-L, J)$ is uniquely expressible as $\alpha * \beta * \gamma$ where $\alpha \in L_{\natural}, \beta \in C(L_{\natural}, K_{\natural}) = C(L, K)$ and $\gamma \in \dot{N}(K_{\natural}, N(K-L, J))$. Furthermore, if σ is principal, then $\beta \neq \emptyset$. This leads to the following definition.

DEFINITION 3.4. Let $a = (0, 0), b_0 = (1, 0)$ and $b_1 = (1, 1) \in E^2$. Let $\Delta = a * b_0 * b_1$. If K and L are full subcomplexes of J , then the natural parameterization of

$N(K-L, J)$ is the unique simplicial mapping $\eta: N(K-L, J) \rightarrow \Delta$ such that $\eta(\alpha) = a$, $\eta(\beta) = b_0$ and $\eta(\gamma) = b_1$, for every simplex $\alpha * \beta * \gamma \in N(K-L, J)$.

DEFINITION 3.5 (COHEN). If X, Y and V are subpolyhedra of the polyhedron Z , then V is a *regular neighborhood of $X \bmod Y$ in Z* if there exists a full triangulation $(J, K, L; h)$ of (Z, X, Y) and a first derived J' such that $V = hN(K-L, J')$.

LEMMA 3.6 (THE CRUSHING LEMMA). *Let K and L be full subcomplexes of the complex J , let $N = N(K-L, J')$ and let $C \subset |J|$ be a compact set such that $C \cap K_{\natural} \subset L_{\natural}$. Then, there is a regular neighborhood N_* of $K \bmod L$ in J and an ϵ -push e_t of $(|J|, L_{\natural})$ keeping*

$$K_{\natural} \cup \text{Cl}(|J| - |N|)$$

fixed such that $[e_t(C) \cap N_] \subset L_{\natural}$.*

Proof. (We will just get an isotopy; however, it will be clear from the proof that we can get an ϵ -push.) Let $\eta: N(K-L, J') \rightarrow \Delta$ be the natural parameterization of $N(K-L, J')$. Consider the set $\eta(C \cap N) \subset \Delta$. Since $\eta(C) \cap ((a * b_0) - a) = \emptyset$, if $b_{\gamma} = (1, \gamma)$, then there is a $\gamma > 0$ such that $\eta(C) \cap b_{\delta}^* b_{\gamma} = \emptyset$. Let $\delta = a * b_{\delta}^* b_{\gamma}$. Then, $N_* = \eta^{-1}(\delta)$ is a regular neighborhood of $K \bmod L$ in J ; however, C may intersect N_* off of L_{\natural} . We denote by S_{ϵ} the segment $(\epsilon, 0) * b_1, 0 \leq \epsilon \leq 1$. Let ψ be a continuous function defined on $[0, 1]$ such that $\psi(0) = 0, \psi(\epsilon) > 0$ for $\epsilon > 0$, and if $x_{\epsilon} \in S_{\epsilon}$ is the point such that $d((\epsilon, 0), x_{\epsilon}) / d((\epsilon, 0), b_1) = \psi(\epsilon)$, then $(\epsilon, 0) * x_{\epsilon} \cap \eta(C) = \emptyset$ for $\epsilon > 0$. (We can assume that $b_{\gamma} = x_1$.) Let A denote the arc consisting of the points $x_{\epsilon}, 0 \leq \epsilon \leq 1$. Then, there is an obvious way to define an isotopy h_t of Δ which is the identity on $\hat{\Delta}$ such that $h_0 = \text{identity}$ and which when restricted to a segment S_{ϵ} slides the point x_{ϵ} "linearly" to the point $S_{\epsilon} \cap a * b_{\gamma}$. Thus, if W is the region bounded by the simple closed curve $a * b_0 \cup b_{\delta}^* b_{\gamma} \cup A$, we see that the isotopy h_t takes W onto δ .

We are now ready to define an isotopy e_t of $|J|$ that takes $\eta^{-1}(W)$ onto $N_* = \eta^{-1}(\delta)$. We define e_t to be the identity on $K_{\natural} \cup \text{Cl}(|J| - |N|)$ and so it remains only to define e_t on the rest of $N = N(K-L, J')$. Let σ be an arbitrary simplex of $N(K-L, J')$. We will show how to define e_t on σ . Recall that $\sigma = \alpha * \beta * \gamma$ as defined above. Let l be a segment in the join structure between $\alpha * \beta$ and γ . Then l is mapped isomorphically onto a unique S_{ϵ} under η . Thus, we define e_t on l by taking η^{-1} of the isotopy h_t restricted to S_{ϵ} . This completes the proof.

LEMMA 3.7. *Let R^{k-1} be a PL manifold of dimension $k-1$ which is contained in the boundary of a k -dimensional PL manifold M^k and let $\epsilon > 0$ be given. Then, there is a PL homeomorphism $\lambda: R \times I \rightarrow M$ such that*

1. $\lambda(r, 0) = r, r \in R,$
2. $\text{diam}(\lambda(r \times I)) < \epsilon, r \in R,$ and
3. $\lambda(R \times I)$ is a neighborhood of R in M .

Proof. Let N be a regular neighborhood of R in M . Let $\eta: \dot{M} \times I \rightarrow M$ be a

PL-collar of \dot{M} in M , i.e., $\eta(x, 0) = x$ for all $x \in \dot{M}$. It is easy to see that N and $\eta(R \times I)$ are both regular neighborhoods of R modulo \dot{R} in the sense of Hudson and Zeeman [28] as modified by Husch [30]. Hence, by uniqueness of relative regular neighborhoods, there is a PL homeomorphism $h: \eta(R \times I) \rightarrow N$ such that $\eta(x, 0) = x$ for all $x \in R$. Thus, $\lambda: R \times I \rightarrow M$ defined by $\lambda(x, t) = h\eta(x, t)$ is the desired PL homeomorphism. (Condition 2 of the conclusion follows by choosing the regular neighborhoods small and by uniform continuity.)

The next lemma states that every PL manifold can be decomposed as a ‘‘handlebody’’ and is well known.

LEMMA 3.8. *Let M^k be a k -dimensional PL manifold. Then $M = \bigcup_{i=0}^p H_i$ where*

$$\left(H_j, H_j \cap \left(\bigcup_{i=0}^{j-1} H_i \right) \right)^{PL} \approx (I^k, I^l \times I^{k-l})$$

for some $l \leq k, j = 0, 1, 2, \dots, p$.

Let T be a combinatorial triangulation of M . If $\Delta_0, \Delta_1, \dots, \Delta_p$ are the simplexes of T ordered in increasing dimension and $\hat{\Delta}_0, \hat{\Delta}_1, \dots, \hat{\Delta}_p$ are their barycenters, then one can show that $H_i = St_T \hat{\Delta}_i$ is the desired decomposition, where T'' is the second barycentric subdivision of T .

The final lemma is an unpublished result of M. Cohen the proof of which we omit.

LEMMA 3.9 (COHEN). *If $Y \subset X$ are polyhedra, then a regular neighborhood of $X \times 0 \text{ mod } Y \times 0$ in $X \times I$ is topologically homeomorphic to*

$$X_{\natural} \times I / [(y, t) = (y, 0) \text{ if } y \in Y_{\natural}, 0 \leq t \leq 1].$$

THEOREM 3.10. *Let $f: M^k \rightarrow N^n, n - k \geq 3, n \geq 5$, be an allowable embedding of the PL k -manifold M^k into the PL n -manifold N^n such that $f|f^{-1}(\dot{N})$ is PL and $f|(M - f^{-1}(\dot{N}))$ is locally flat and let $\epsilon > 0$ be given. Then, there exists a neighborhood \mathcal{U} of $f^{-1}(\dot{N})$ in M and an ϵ -push e_t of $(N, f(f^{-1}(\dot{N})))$ which is fixed on \dot{N} such that $e_1 f|_{\mathcal{U}}: \mathcal{U} \rightarrow N$ is PL.*

Proof. We will not worry about getting an ϵ -push, but will simply get a push. We could get an ϵ -push by making small choices for our triangulations, neighborhoods, collars, etc. First we will get a push e_t which satisfies the conclusion except for being the identity on \dot{N} , although it will be the identity on $f(f^{-1}(\dot{N}))$ and outside a small neighborhood of $f(f^{-1}(\dot{N}))$ in \dot{N} . Then, we will show how to modify the proof so as to have e_t be the identity on \dot{N} .

Let $R = f(f^{-1}(\dot{N}))$ and suppose that $R = \bigcup_{i=0}^p H_i^{k-1}$ is a decomposition of R as a ‘‘handlebody’’ assured by Lemma 3.8. By Lemma 3.7 there is a PL homeomorphism $\lambda: R \times I \rightarrow M$ such that $\lambda(r, 0) = r$, $\text{diam}(\lambda(r \times I))$ is small, and $\lambda(R \times I)$ is a neighborhood of R in M . Now, consider the collection $\lambda(H_0 \times I), \lambda(H_1 \times I), \dots, \lambda(H_p \times I)$. This is a covering of $\mathcal{U} = \lambda(R \times I)$ with k -balls. Note that it follows from Lemma 3.8 that $\lambda(H_i \times I)$ meets $\dot{M} \cup [\bigcup_{j=0}^{i-1} \lambda(H_j \times I)]$ in a $(k-1)$ -ball.

We will get a sequence of isotopies e_i^t , $i=0, 1, \dots, p$, of N onto itself such that $e_0^t=1$, $e_i^t|_R=1$, and $e_1^t e_1^{t-1} \dots e_1^1 f|_{\bigcup_{j=0}^i \lambda(H_j \times I)}$ is PL. We will construct the e_i^t so that $e_i^t|_{e_1^{t-1} \dots e_1^1 f(\bigcup_{j=0}^{i-1} \lambda(H_j \times I))}$ is the identity. Let $\lambda_*: \dot{N} \times I \rightarrow N$ be a PL-collaring of \dot{N} . Then, $D_0 = \lambda_*(f(H_0) \times I)$ is a PL k -ball in N such that $D_0 \cap \dot{N} = f(H_0)$. Let \mathcal{V} be the interior of a regular neighborhood of $f(H_0)$ in \dot{N} . Then, $(\mathcal{V}, f(H_0)) \approx (E^{n-1}, I^{k-1})$ and $(\lambda_*(\mathcal{V} \times [0, 1]), f(H_0)) \approx (E_+^n, I^{k-1})$. Let $g: \lambda(H_0 \times I) \rightarrow D_0$ be a PL homeomorphism such that $g|_{\lambda(H_0 \times 0)} = f|_{\lambda(H_0 \times 0)}$. Then, by applying the methods of [7], we can get an isotopy $e_i^0: N \rightarrow N$ such that $e_0^0=1$, $e_1^0|\dot{N}=1$ and $e_1^0 f|_{\lambda(H_0 \times I)} = g$. Thus, $e_1^0 f|_{\lambda(H_0 \times I)}$ is PL.

Now, we will show how to construct e_i^1 and then it will be clear how to construct e_i^t , $i=2, 3, \dots, p$. Let K^0 be a regular neighborhood of

$$e_1^0 f(\lambda(H_0 \times I)) \text{ mod } \text{Cl} (\lambda_*(f(R) \times I) - \lambda_*(f(H_0) \times I)).$$

Since N is a PL manifold and $e_1^0 f(\lambda(H_0 \times I))$ is link-collapsible on

$$\text{Cl} (\lambda_*(f(R) \times I) - \lambda_*(f(H_0) \times I)),$$

it follows from [28] that K^0 is a PL manifold which collapses to $e_1^0 f(\lambda(H_0 \times I))$ and so is an n -ball. Furthermore, it follows from [28] that K^0 meets the boundary of N regularly, i.e., in an $(n-1)$ -ball. Now we apply Lemma 3.6 where the compact set C of that lemma is $e_1^0 f(\bigcup_{j=1}^p \lambda(H_j \times I))$ and get a regular neighborhood K_*^0 of $e_1^0 f(\lambda(H_0 \times I)) \text{ mod } \text{Cl} (\lambda_*(f(R) \times I) - \lambda_*(f(H_0) \times I))$ and an isotopy \tilde{e}_1^1 which is the identity on $\lambda_*(f(R) \times I)$ such that $\tilde{e}_1^1 e_1^0 f(\bigcup_{j=1}^p \lambda(H_j \times I)) \cap K_*^0 = \lambda_*(f(H_0) \times I) \cap (\bigcup_{j=1}^p \lambda_*(f(H_j) \times I))$. We now form a new PL manifold $N_1 = \text{Cl} (N - K_*^0)$. Then, we have the embedding $\tilde{e}_1^1 e_1^0 f|_{\lambda(H_1 \times I)}: \lambda(H_1 \times I) \rightarrow N_1$ and we have the PL k -ball $D_1 = \lambda_*(f(H_1) \times I) \subset N_1$ such that $\tilde{e}_1^1 e_1^0 f(\lambda(H_1 \times I)) \cap \dot{N}_1 = D_1 \cap \dot{N}_1$ is a $(k-1)$ -ball. Thus, by applying the methods of [7], similarly to the way we did in the construction of e_i^0 , we get an isotopy \hat{e}_1^1 of N_1 which is the identity on \dot{N}_1 and is such that $\hat{e}_1^1 \tilde{e}_1^1 e_1^0 f|_{\lambda(H_1 \times I)}$ is PL. Now, extend \hat{e}_1^1 to N by way of the identity. Then $e_1^1 = \hat{e}_1^1 \tilde{e}_1^1$ is the desired isotopy and we see our way clear.

It remains only to show why we can assume e_i is the identity on \dot{N} . Let K be a regular neighborhood of $\dot{N} \text{ mod } f(R)$. Then, by Lemma 3.6 there is a regular neighborhood K_* of $\dot{N} \text{ mod } f(R)$ and an isotopy h_i of N keeping \dot{N} fixed such that $h_1 f(M) \cap K_* = f(R)$, (the compact set C of Lemma 3.6 is $f(M)$). Now, $N_* = \text{Cl} (N - K_*)$ is a PL manifold and $h_1 f: M \rightarrow N_*$ is an allowable embedding. Thus, we can go through the preceding proof and get an isotopy \bar{e}_i of N_* such that $\bar{e}_i h_1 f|_{\mathcal{U}}: \mathcal{U} \rightarrow N_*$ is PL and $\bar{e}_i|_f(R) = 1$. But, K_* is a collar of \dot{N} in N pinched at $f(R)$ by Lemma 3.9, and so we can use K_* to extend \bar{e}_i to N so that $\bar{e}_i|\dot{N} = 1$. Then, $e_i = \bar{e}_i h_i$ is the desired isotopy.

4. Unknotting locally flat embeddings by isotopy. The problem which motivated most of the work of this paper was the attempt to bring Theorem 1' of [10] up to codimension three. Partial results in this direction are obtained in the next section.

A slightly less ambitious problem is to try to show that if an allowable embedding $f: M^k \rightarrow N^n$ of the PL k -manifold M into the PL n -manifold N , $n - k \geq 3$, is PL on $f(f^{-1}(\dot{N}))$ and locally flat on $M - f(f^{-1}(\dot{N}))$, then it can be tamed keeping \dot{N} fixed. As mentioned in the introduction, we will consider this problem in a later paper. It follows from Theorem 3.10, that this problem would be a consequence of the first problem mentioned. In this section, we solve this question for the special case of proper embeddings of I^k into I^n in codimension three.

Let us digress a moment to show that we get an answer (Theorem 4.1) to the second problem if we weaken it to showing that a locally flat embedding of one PL manifold into the interior of another can be carried to a PL embedding by homotopy. Theorem 4.1 essentially says that if one restricts the maps of Irwin's Embedding Theorem (Theorem 1.1 of [31] or Theorem 23 of [42]) to locally flat embeddings, then the connectivity condition on the ambient manifold can be dropped. This theorem is obviously weaker than Theorem 2.1 of [23], the proof of which, as we have mentioned, is in error above the metastable range.

THEOREM 4.1. *Let $f: M^k \rightarrow N^n$, $n - k \geq 3$, be a locally flat embedding (equivalently, locally flat on open simplexes) of the closed $(2k - n + 1)$ -connected PL k -manifold M into the interior of the PL n -manifold N^n . Then, given $\varepsilon > 0$, f is homotopic to a PL embedding such that the track of the homotopy is contained in an ε -neighborhood of $f(M)$.*

Proof. First observe that $N - f(M)$ is 1-LC at $f(M)$ since $f(M)$ being locally flat in N implies that $N - f(M)$ is 1-LC at each point of $f(M)$ which, by Theorem 1 of [22], implies $N - f(M)$ is 1-LC at $f(M)$. Now, by Theorem 4.6 of [15], $f(M)$ has a $(2k - n + 1)$ -connected ε -neighborhood. Theorem 6.1 follows by applying Theorem 1.1 of [31], where the ambient space of that theorem is our ε -neighborhood.

We make the following definition so that the statement of Theorem 4.4 might generalize Corollary 1 of [17] as much as possible.

DEFINITION 4.2. Let $f: M^k \rightarrow N^n$ be an embedding (M and N topological manifolds) and let $K^m \subset M^k$, $m \leq k$, be an abstract locally finite m -complex each open simplex of which is either locally flat in \dot{M} or in M° . Furthermore, suppose that f is locally flat on each open simplex of K and that $|K|$ has a neighborhood \mathcal{U} in M such that f is locally flat at each point of $\mathcal{U} - |K|$. Then, $|K|$ is called a *polyhedron of singularity*. (This generalizes the notion of singular points defined in [11] and of cells of singularity introduced in [35].)

The following two theorems are the main results of this section. These theorems can be brought up to codimension three, if someone is able to correct the work of Homma mentioned in the introduction.

THEOREM 4.3. *Suppose $X^k \in \{S^k, E^k, E_+^k, I^k\}$ and let $f: X^k \rightarrow E^n$, $k < \frac{2}{3}n - 1$, be a closed embedding such that either*

- (1) *f is locally flat modulo polyhedra of singularity, or*
- (2) *f is locally nice and $n \geq 5$.*

Then, there is an isotopy $e_t: E^n \rightarrow E^n$ such that $e_0 = \text{identity}$ and $e_1 f$ is the inclusion $i: X \subset E^n$.

THEOREM 4.4. *Suppose $(Y^n, X^k) \in \{(I^n, I^k), (E_+^n, E_+^k)\}$ and let $f: X^k \rightarrow Y^n$, $k < \frac{2}{3}n - 1$, be a proper, closed embedding such that*

- (1) $f|X^\circ: X^\circ \rightarrow Y^\circ$ is either (i) locally flat modulo polyhedra of singularity or, (ii) locally nice, and,
- (2) $f|\dot{X}: \dot{X} \rightarrow \dot{Y}$ is either (i) locally flat modulo polyhedra of singularity and $n \geq 5$ or, (ii) locally nice and $n \geq 6$.

Then, there is an isotopy $e_t: Y \rightarrow Y$ such that $e_0 = \text{identity}$ and $e_1 f$ is the inclusion $i: X \subset Y$.

Furthermore, if $f|\dot{X}$ is the identity then e_t is the identity on \dot{Y} , and IF $f|\dot{X}$ IS PL THEN THE THEOREM HOLDS IN CODIMENSION THREE WITH THE LOCALLY FLAT HYPOTHESES.

Before proving the above theorems we will need a few lemmas.

LEMMA 4.5. *Suppose that X is a closed subset of a topological manifold M^n (n arbitrary) such that $\dim X \leq n - 2$ and $M - X$ is 1-LC at each point of X . Then, for every closed subset C of X , $M - C$ is 1-LC at each point of C .*

We omit the proof which is straightforward using the fact that subsets of codimension two do not separate.

LEMMA 4.6. *If $f: M^k \rightarrow N^{\circ n}$ is a locally tame embedding of the PL k -manifold M into the PL n -manifold N , $n - k \neq 2$, (n arbitrary) then f is locally flat.*

Proof. This follows easily using the fact that PL spheres and balls unknot in S^n when the codimension is not two; (see [40]).

By examining the first local group at the origin as defined in [38], one can see that the PL-disk in E^4 which is the cone from the origin over a trefoil knot in I^4 would be a counterexample to this lemma in codimension two.

LEMMA 4.7. *Suppose that $f: M^k \rightarrow E^n(S^n)$, $k < \frac{2}{3}n - 1$, $n \geq 5$, is a closed embedding of the topological manifold M^k . Then, f is locally flat if and only if it is locally nice.*

Proof. A standard general position argument shows that if f is locally flat, then f is locally nice. Let x be any point of M and let $g: I^k \rightarrow M$ be an embedding such that $g(I^k)$ is a closed neighborhood of x . Then, $fg: I^k \rightarrow E^n$ is locally nice by Lemma 4.5; hence, by Theorem 1 of [5], fg is ϵ -tame. (This follows in the trivial range by Theorem 4.2 of [4].) Lemma 4.6 now implies that $f|g(I^k)$ is locally flat, from which it follows that f is locally flat at x .

LEMMA 4.8. *An embedding $f: M^k \rightarrow N^{\circ n}$, $n - k \geq 3$, (n arbitrary) of a topological k -manifold M into a topological n -manifold N which is locally flat modulo polyhedra of singularity is locally flat.*

This lemma follows from Theorem 6.1 of [8].

LEMMA 4.9. *Let $f: I^k \rightarrow I^n, n - k \geq 3, n \geq 5$, be a proper embedding such that $f|I^k =$ identity and $f|I^{\circ k}$ is locally flat. Then, there is an ε -push e_ε of (I^n, I^k) which is fixed on I^n and is such that $e_1 f|N =$ identity for some neighborhood N of I^k relative to I^k .*

Lemma 4.9 can be derived from the proof of Theorem 3.10 by using a standard collaring of I^n in that proof rather than an arbitrary PL-collaring. (Lemma 4.9 will also follow from the statement of Theorem 3.10 by making a double application of Theorem 4 of [26].)

LEMMA 4.10. *Let $f: E^k \rightarrow E^n, n - k \geq 3, n \geq 5$, be a closed, locally flat embedding such that $f|I^k =$ identity and $f(E^k) \cap I^n = I^k$. Then, for any n -cell $I_*^n \subset I^{\circ n}$ which is concentric with I^n , we can extend f to a homeomorphism $\tilde{f}: E^n \rightarrow E^n$ such that $\tilde{f}|I_*^n =$ identity.*

This lemma can be proved by making slight modifications in [37] and these modifications were essentially made in [17]. Černavskii suggests that one might do the "horizontal" engulfing (i.e., Lemma 2 of [17] or Lemma 5.3 of [37]) by using the techniques of proof of his Proposition C of [10].

Proof of Theorem 4.3. (1) ($X^k = S^k$): It follows from Lemma 4.7, Lemma 4.8 and Theorem 1 of [5], that for $n \geq 5, f: S^k \rightarrow E^n$ is ε -tame. (This follows for $n > 5$ from Theorem 1 of [10].) For the case $n = 4, f: S^k \rightarrow E^n$ is ε -tame by Theorem 8.1 of [18]. Thus, there is an isotopy $e'_\varepsilon: E^n \rightarrow E^n$ such that $e'_0 =$ identity and $e'_1 f: S^k \rightarrow E^n$ is PL. But, now we can use [40] or the Unknotting Theorem of [26], to get an isotopy $e''_\varepsilon: E^n \rightarrow E^n$ such that $e''_0 =$ identity and $e''_1 e'_1 f: S^k \rightarrow E^n$ is the inclusion $i: S^k \subset E^n$. Thus, $e_\varepsilon = e''_\varepsilon e'_\varepsilon$ is the required isotopy.

(2) ($X^k = E^k$): We first observe that by the above proof Theorem 4.3 is true when $X^k = S^k$ and we replace E^n by S^n . Now suppose we have the closed embedding $f: E^k \rightarrow E^n$. Then, we can extend f to $\tilde{f}: S^k \rightarrow S^n$ by defining $\tilde{f}(\infty) = \infty$. By either Lemma 4.7 or Lemma 4.8, we know that \tilde{f} is locally flat. Thus, by our first observation there is an isotopy $e_\varepsilon: S^n \rightarrow S^n$ such that $e_0 =$ identity and $e_1 \tilde{f}: S^k \rightarrow S^n$ is the inclusion. Then, since $e_0(\infty) = \infty$ and $e_1(\infty) = e_1 f(\infty) = \infty$, it follows from Theorem 2.1 of [20], that $e_0|E^n$ and $e_1|E^n$ are isotopic and the conclusion follows.

(3) ($S^k = E^k_+$): By either Lemma 4.7 or Lemma 4.8 we know that $f: E^k_+ \rightarrow E^n$ is locally flat. Consequently, Theorem 2.3 of [32] gives us a homeomorphism $h: (E^n, f(E^k_+)) \rightarrow (E^n, E^k_+)$. Now, $hf: E^k_+ \rightarrow E^k_+$ can be extended to $g: (E^n, E^k_+) \rightarrow (E^n, E^k_+)$. Notice that $h^{-1}g$ is an extension of f to E^n since $h^{-1}g|E^k_+ = h^{-1}hf = f$. Clearly, $h^{-1}g|E^k: E^k \rightarrow E^n$ is a closed, locally flat embedding; hence, we get the desired isotopy by applying Theorem 4.3 when $X^k = E^k$.

(4) ($X^k = I^k$): This proof is similar to the preceding one.

Proof of Theorem 4.4. (1) ($(Y^n, X^k) = (I^n, I^k)$): Consider $f|I^k: I^k \rightarrow I^n$. By Theorem 4.3, there is an isotopy $e'_\varepsilon: I^n \rightarrow I^n$ such that $e'_0 =$ identity and

$$e'_1 f|I^k: I^k \rightarrow I^n$$

is PL. Then by [40], or the Unknotting Theorem of [26], we can get an isotopy

$e''_i: E^n \rightarrow E^n$ such that $e''_0 = \text{identity}$ and $e''_1 e'_1 f: S^k \rightarrow E^n$ is the inclusion $i: S^k \subset E^n$. Now, extend the isotopy $e''_i e'_i$ conewise to get an isotopy $\bar{e}_i: I^n \rightsquigarrow I^n$. Then, $\bar{e}_1 f: I^k \rightarrow I^n$ is a proper embedding which is the identity on \dot{I}^k . It follows from either Lemma 4.7 or Lemma 4.8, that $\bar{e}_1 f|_{I^{\circ k}}$ is locally flat.

Thus, by Lemma 4.9, there is a homeomorphism $h: I^n \rightsquigarrow I^n$ which is the identity on \dot{I}^n and such that $h\bar{e}_1 f = \text{identity}$ on some neighborhood N of \dot{I}^k relative to I^k . Next, connect $h\bar{e}_1 f(0)$ and 0 with a polygonal arc $A \subset I^{\circ n} - N$. Let $B \subset I^{\circ n} - N$ be a regular neighborhood of A . Let $h': I^n \rightsquigarrow I^n$ be the identity on $I^n - B^{\circ}$ and in B map $h\bar{e}_1 f(0)$ to 0 and extend conewise over the identity on \dot{B} . Define $\bar{h} = h'h$ and notice that $\bar{h}\bar{e}_1 f$ is the identity on $\dot{I}^n \cup N \cup \{0\}$.

Now the embedding $p: E^k \rightarrow E^n$ which is defined to be the identity on $E^k - I^k$ and $\bar{h}\bar{e}_1 f$ on I^k is locally flat. Since p is the identity on N , we can get a cube $I^n_* \subset I^{\circ n}$ which is concentric with I^n and is such that $p(I^{\circ n}_* \cap I^k) \subset I^{\circ n}_*$ and $p|(E^k - I^{\circ n}_*) = \text{identity}$. After one point compactifying E^n with ∞ and removing 0 , we can consider the resulting space to be E^n where ∞ corresponds to 0 , $(E^k - 0) \cup \{\infty\}$ corresponds to E^k , $(E^n - I^{\circ n}) \cup \{\infty\}$ corresponds to I^n_* , $(E^n - I^{\circ n}_*) \cup \{\infty\}$ corresponds to I^n , and $I^{\circ n}_* - \{0\}$ corresponds to $E^n - I^n$. Thus, by applying Lemma 4.10 and then compactifying by 0 and removing ∞ , we obtain an extension \bar{p} of p to E^n which is the identity on $E^n - I^{\circ n}$. Therefore, $\bar{p}^{-1}\bar{h}\bar{e}_1 f: I^n \rightarrow I^n$ is the identity on I^k . Since $\bar{p}^{-1}\bar{h}|_{\dot{I}^n} = \text{identity}$, it is easy to see that there is an isotopy $\hat{e}_i: I^n \rightsquigarrow I^n$ which is the identity on \dot{I}^n and satisfies $\hat{e}_0 = \text{identity}$ and $\hat{e}_1 = \bar{p}^{-1}\bar{h}$. Then, $e_i: I^n \rightsquigarrow I^n$ defined by $e_i = \hat{e}_i \bar{e}_i$ is the desired isotopy.

(2) $((Y^n, X^k) = (E^n_+, E^k_+))$: Since $f|_{\dot{E}^k_+}: \dot{E}^k_+ \rightarrow \dot{E}^n_+$ is essentially a closed embedding of E^{k-1} into E^{n-1} , we can apply Theorem 4.3 to unknot this embedding with an isotopy e'_i in the metastable range. In codimension three, for the case $f|_{\dot{E}^k_+}$ is PL we can one point compactify \dot{E}^k_+ and \dot{E}^n_+ with ∞ and extend f to $\bar{f}: \dot{E}^k_+ \cup \{\infty\} \rightarrow \dot{E}^n_+ \cup \{\infty\}$ and then apply Theorem 1 of [36], and then [40] to unknot \bar{f} . Therefore, we observe that it follows from Theorem 2.1 of [20], that we can unknot $f|_{\dot{E}^k_+}$ with an isotopy e'_i . Now, extend this isotopy to an isotopy e''_i of E^n_+ . Next, one point compactify (E^n_+, E^k_+) with ∞ and we have (I^n, I^k) . By defining $e''_1 f(\infty) = \infty$, we have $e''_1 f: I^k \rightarrow I^n$ is a proper embedding which is the identity on \dot{I}^k . Hence, by applying Theorem 4.4 when $(Y^n, X^k) = (I^n, I^k)$ and then removing ∞ , we obtain the desired isotopy e_i .

5. Taming embeddings of nice polyhedra in codimension three.

DEFINITION 5.1. If $P \subset Q$ are polyhedra, we say there is an elementary $C(X)$ -construction from P to Q , written $P \overset{C_e(X)}{\nearrow} Q$, if there exists a pair $(\mathcal{C}(\chi), \chi)$ which is PL homeomorphic to $(C(X), X)$, where $C(X)$ denotes the cone over a polyhedron X , such that

- (1) $Q = P \cup \mathcal{C}(\chi)$,
- (2) $\chi = P \cap \mathcal{C}(\chi)$, and
- (3) P is link-collapsible on χ and $\text{Cl}(P - \chi) = P$.

DEFINITION 5.2. A polyhedron Q is said to be $C(B)$ -constructible from the spine P , written $P \overset{C(B)}{\nearrow} Q$, if there is a finite sequence,

$$P \overset{C_e(X_1)}{\nearrow} P_1 \overset{C_e(X_2)}{\nearrow} P_2 \overset{C_e(X_3)}{\nearrow} \dots \overset{C_e(X_p)}{\nearrow} Q,$$

of elementary constructions from P to Q where each X_i is a PL ball. Similarly, Q is said to be $C(S)$ -constructible if there is such a finite sequence where each X_i is a PL sphere.

DEFINITION 5.3. A P -nice polyhedron Q has the property that $P \overset{C(B)}{\nearrow} L \overset{C(S)}{\nearrow} Q$.

DEFINITION 5.4. A B -nice polyhedron, B a ball, is simply called a nice polyhedron. The main result of this section is the following theorem.

THEOREM 5.5. Let Q be a P -nice polyhedron such that $\dim \text{Cl}(Q - P) \leq n - 3$, $n \geq 5$, and let $f: Q \rightarrow M^{\circ n}$ be an embedding of Q into the interior of the PL n -manifold M which is locally flat on the open simplexes of some triangulation T and is such that $f|P: P \rightarrow M^n$ is tame. Then, $f: Q \rightarrow M^n$ is tame and if $f|P$ is PL, the taming isotopy is fixed on $f(P)$.

COROLLARY 5.6. Let Q be a nice polyhedron such that $\dim \text{Cl}(Q - B) \leq n - 3$, $n \geq 5$, and let $f: Q \rightarrow S^n(E^n)$ be an embedding which is locally flat on open simplexes. (If $\dim B > n - 3$, we also require that f be locally flat on B .) Then, there is a homeomorphism $h: S^n(E^n) \rightarrow S^n(E^n)$ such that $hf: Q \rightarrow S^n(E^n)$ is PL.

COROLLARY 5.7. Let $f_i: I^{k_i} \rightarrow I^n$, $n - k_i \geq 3$, $n \geq 5$, $i = 1, 2, \dots, r$ be proper embeddings such that

- (1) $f_i|I^{k_i}$ is PL,
- (2) $f_i|I^{\circ k_i}$ is locally flat, and
- (3) $f_i(I^{k_i}) \cap f_j(I^{k_j}) = \emptyset$ when $i \neq j$.

Then, there is an isotopy $e_i: I^n \rightarrow I^n$ such that

- (1) $e_0 = 1$,
- (2) $e_t|I^n = 1$, $t \in I$, and
- (3) $e_1 f_i: I^{k_i} \rightarrow I^n$ is PL for $i = 1, 2, \dots, r$.

The next corollary follows from Corollary 5.7 and Theorem 7 of [33].

COROLLARY 5.8. Piecewise linear spheres $S_i^{k_i}$, $i = 1, 2, \dots, r$, contained in S^n , $n - k_i \geq 3$, $n \geq 5$, are unlinked (i.e., are contained in disjoint n -balls) if and only if regarding S^n as I^{n+1} , $S_i^{k_i}$, $i = 1, 2, \dots, r$, bound disjoint cells D^{k_i+1} , $i = 1, 2, \dots, r$, respectively, whose interiors are contained in the interior of I^{n+1} and are locally flat there.

Before proving Theorem 5.5, we will digress to formulate an embedding theorem and then state a preliminary lemma. It is easy to show that any collapsible n -dimensional polyhedron can be embedded in E^{2n} and it is easy to find examples of collapsible n -dimensional polyhedra that cannot be embedded in E^{2n-1} . We can justifiably call an n -polyhedron Q^n such that $D \overset{C(B)}{\nearrow} Q^n$, where D is a ball, a nice

collapsible polyhedron, and it follows from the following theorem that any nice collapsible n -polyhedron can be embedded in E^n .

THEOREM 5.9. *Suppose that Q^n (n arbitrary) is an n -polyhedron such that $P \overset{C(B)}{\nearrow} L \overset{C_e(X)}{\nearrow} Q$ where P is a collapsible polyhedron and X is an arbitrary polyhedron. Then, any PL embedding $f: P \rightarrow E^m$ can be extended to a PL embedding of Q into E^m .*

The above theorem follows easily by using [28] and conewise extensions.

LEMMA 5.10. *Let $f: M^k \rightarrow Q^n$ (n arbitrary) be a proper map (i.e., boundary to boundary, interior to interior) of the compact, k -dimensional PL k -manifold M into the topological n -manifold Q^n (whose interior has a PL-structure) such that $f|_{\mathcal{V}}$ is an embedding and $f|_{(\mathcal{V} - \dot{M})}$ is PL, where \mathcal{V} is a closed PL-collar of \dot{M} in M . Let $\mathcal{U} \subset Q^\circ$ be an open set such that either*

- (1) $M - f^{-1}(\mathcal{U}) \subset \mathcal{V}^\circ \cup \dot{M}$ and is a collar of \dot{M} , or
- (2) $M - f^{-1}(\mathcal{U}) = \dot{M}$. Then, f is homotopic keeping $M - f^{-1}(\mathcal{U})$ fixed, to a proper embedding f' which is PL on M° , where the track of the homotopy lies inside $\mathcal{U} \cup f(M)$, provided
 - (1) $n - k \geq 3$,
 - (2) M is $(2k - n)$ -connected, and
 - (3) \mathcal{U} is $(2k - n + 1)$ -connected.

Furthermore, if Q is a PL manifold and if f is PL on \mathcal{V} , then f' is PL on M .

The proof of this lemma follows by using the techniques of proof of one of Irwin's embedding theorems (see [42, Chapter 8], or Theorem 1.1 of [31]). This lemma can be proved in the metastable range by using the techniques of proof of the Penrose, Whitehead, Zeeman Theorem (see [41, p. 66], which handles a special case of [34]).

Proof of Theorem 5.5. Since Q is a P -nice polyhedron, we know that

$$P \overset{C(B)}{\nearrow} L \overset{C(S)}{\nearrow} Q.$$

More explicitly, we know that

$$P = P_0 \overset{C_e(B_1)}{\nearrow} P_1 \overset{C_e(B_2)}{\nearrow} \dots \overset{C_e(B_k)}{\nearrow} P_k = L = L_0 \overset{C_e(S_1)}{\nearrow} L_1 \overset{C_e(S_2)}{\nearrow} \dots \overset{C_e(S_p)}{\nearrow} L_p = Q$$

where the B_i are balls and the S_i are spheres. Let T' be a triangulation of Q that contains the $C_e(B_i)$ and $C_e(S_i)$ as subcomplexes and let T'' be a common subdivision of T and T' . (Here we are abusing our notation slightly by identifying $C_e(X)$ with $\mathcal{C}(X)$.) Since the property of being locally flat on open simplexes is preserved under subdivisions, f is locally flat on the open simplexes of T'' . It follows now from Corollary 6.3 of [8], that f is locally flat on each $C_e(B_i) = \text{Cl}(P_i - P_{i-1})$ and on each $C_e(S_i) = \text{Cl}(L_i - L_{i-1})$. Let e_i^0 denote the taming isotopy for P , i.e., $e_0^0 = \text{identity}$ and $e_1^0 f: P \rightarrow M$ is PL.

Our next step is to show that $e_1^0 f|L$ can be tamed keeping $e_1^0 f(P)$ fixed. The proof proceeds by induction on the number of elementary $C(B)$ -constructions, i.e., by induction on k . Thus, we may suppose that we have $f_i: Q \rightarrow M^n$ such that $f_i|P_i: P_i \rightarrow M^n$ is PL and we want to get an isotopy $e_i^{t+1}: M \rightarrow M$ such that $e_0^{t+1} = \text{identity}$, $e_i^{t+1}|P_i = \text{identity}$ and $e_1^{t+1} f_i|P_{i+1}: P_{i+1} \rightarrow M$ is PL.

Let N_i be a regular neighborhood of $f_i(P_i) \bmod f_i(P_i) \cap f_i(C_e(B_{i+1})) = f_i(B_{i+1})$. Now, it follows from Lemma 3.6 that there is a regular neighborhood N_i^* of $f_i(P_i) \bmod f_i(B_{i+1})$ and an isotopy \bar{e}_i^{t+1} of M keeping $f_i(P_i) \cup \text{Cl}(M - N_i)$ fixed such that $\bar{e}_1^{t+1} f_i(C_e(B_{i+1})) \cap N_i^* = f_i(B_{i+1})$. But, from [28], we know that N_i^* is a PL manifold; hence, it follows easily from the Alexander-Newman Theorem that $M_i = \text{Cl}(M - N_i^*)$ is a PL manifold. Thus, $\bar{e}_1^{t+1} f_i|C_e(B_{i+1}): C_e(B_{i+1}) \rightarrow M_i$ is an embedding such that $(\bar{e}_1^{t+1} f_i)^{-1}(\dot{M}_i) = B_{i+1}$, $\bar{e}_1^{t+1} f_i|B_{i+1}$ is PL and

$$\bar{e}_1^{t+1} f_i|[C_e(B_{i+1}) - B_{i+1}]$$

is locally flat. Now, by applying the techniques of [7], similarly to the way we did in the construction of the isotopy e_i^0 of Theorem 3.10, we can get an isotopy \hat{e}_i^{t+1} of M_i such that $\hat{e}_0^{t+1} = \text{identity}$, $\hat{e}_i^{t+1}|M_i = \text{identity}$ and $\hat{e}_1^{t+1} \bar{e}_1^{t+1} f_i|C_e(B_{i+1})$ is PL. Extend \hat{e}_i^{t+1} to M by way of the identity. Then, $e_i^{t+1} = \hat{e}_1^{t+1} \bar{e}_1^{t+1}$ is the desired isotopy and we can conclude that $e_1^0 f|L$ can be tamed keeping $e_1^0 f(P)$ fixed by an isotopy e_i^t of M .

We will complete the proof by showing that $g_0 = f_k = e_k^0 f: Q \rightarrow M$ can be tamed keeping $g_0(L)$ fixed. The proof will be by induction on the number of elementary $C(S)$ -constructions, i.e., by induction on p . Hence, we may suppose that we have $g_i: Q \rightarrow M$ such that $g_i|L_i: L_i \rightarrow M$ is PL and we want to get an isotopy $e_i^{t+1}: M \rightarrow M$ such that $e_0^{t+1} = \text{identity}$, $e_i^{t+1}|L_i = \text{identity}$ and $e_1^{t+1} g_i|L_{i+1}: L_{i+1} \rightarrow M$ is PL.

Let N_i be a regular neighborhood of $g_i(L_i) \bmod g_i(L_i) \cap g_i(C_e(S_{i+1})) = g_i(S_{i+1})$. Now, it follows from Lemma 3.6 that there is a regular neighborhood N_i^* of $g_i(L_i) \bmod g_i(S_{i+1})$ and an isotopy \bar{e}_i^{t+1} of M keeping $g_i(L_i) \cup \text{Cl}(M - N_i)$ fixed such that $\bar{e}_1^{t+1} g_i(C_e(S_{i+1})) \cap N_i^* = g_i(S_{i+1})$. But, again $M_i = \text{Cl}(M - N_i^*)$ is a PL manifold and $\bar{e}_1^{t+1} g_i|C_e(S_{i+1}): C_e(S_{i+1}) \rightarrow M_i$ is a proper embedding which is PL on $\text{Bd}(C_e(S_{i+1})) = S_{i+1}$ and is locally flat on $\text{Int}(C_e(S_{i+1}))$. Thus, by Theorem 3.10, there is a PL-collar \mathcal{V} of S_{i-1} in $C_e(S_{i+1})$ and an isotopy \hat{e}_i^{t+1} of M_i which is fixed on \dot{M}_i and is such that $\hat{e}_0^{t+1} = \text{identity}$ and $\hat{e}_1^{t+1} \bar{e}_1^{t+1} g_i|\mathcal{V}: \mathcal{V} \rightarrow M_i$ is PL. Now, extend \hat{e}_i to M by the identity.

Suppose $C_e(S_{i+1})$ is a k_i -ball and let $h_i: I^{k_i} \rightarrow C_e(S_{i+1})$ be a homeomorphism. Then, by using the lemma in the appendix of [32], we can extend

$$\hat{e}_1^{t+1} \bar{e}_1^{t+1} g_i h_i: I^{k_i} \rightarrow M$$

to an embedding $\bar{h}_i: I^n \rightarrow M$. Now, consider $\bar{h}_i^{-1}(N_i^*)$. We can get a homeomorphism $\eta_i: I^n \rightarrow I^n$ such that $\eta_i|I^{k_i} = \text{identity}$ and $\eta_i(I^n) \cap \bar{h}_i^{-1}(N_i^*) = I^{k_i}$. Consider the open set $\mathcal{U} = \bar{h}_i \eta_i(I^n) \subset M_i^0$. Apply, Lemma 5.10, where $(f, M, Q, \mathcal{U}, \mathcal{V})$ of that lemma is replaced by $(\hat{e}_1^{t+1} \bar{e}_1^{t+1} g_i|C_e(S_{i+1}), C_e(S_{i+1}), M_i, \mathcal{U}, \mathcal{V})$. We then

get a PL embedding $\lambda_i: C_e(S_{i+1}) \rightarrow M_i$ such that $\lambda_i(\text{Int}(C_e(S_{i+1}))) \subset \text{Int}(\bar{h}_i\eta_i(I^n))$ and $\lambda_i|\text{Bd}(C_e(S_{i+1})) = \bar{e}_1^{i+1}\bar{e}_1^{i+1}g_i|\text{Bd}(C_e(S_{i+1}))$. Thus, by Theorem 4.4, there is an isotopy \bar{e}_i^{i+1} of $\bar{h}_i\eta_i(I^n)$ such that $\bar{e}_0^{i+1} = 1$, $\bar{e}_i^{i+1}|\text{Bd}(\bar{h}_i\eta_i(I^n)) = 1$ and

$$\bar{e}_1^{i+1}\bar{e}_1^{i+1}\bar{e}_1^{i+1}g_i|C_e(S_{i+1}) = \lambda_i.$$

Now, extend \bar{e}_i^{i+1} to M by way of the identity. Then, $e_i^{i+1} = \bar{e}_i^{i+1}\bar{e}_i^{i+1}\bar{e}_i^{i+1}$ is the desired isotopy and this completes the proof.

6. Taming embeddings of polyhedra in the metastable range. In this section we will prove a theorem (Theorem 6.4) which generalizes results of [3], [14], and [10]. First we will establish a couple of lemmas.

LEMMA 6.1. *Let D be an m -cell in the manifold M^n , $n - m \geq 3$ (n arbitrary), and let B^k , $k \leq m - 1$, be a k -cell in \dot{D} such that $D - B$ is locally flat in M^n and B is locally flat in M^n . Then, D is locally flat in M^n .*

Proof. Since locally flat implies 1-LC, the lemma follows from Lemma 5.1 of [4] for $n \geq 5$, and from [6] for $n = 4$.

LEMMA 6.2. *Let f be a closed embedding of a k -dimensional space S into an m -hyperplane of E^n where $m < n$, (n arbitrary) and $k \leq n - 3$. Then, $E^n - f(S)$ is 1-LC at each point p of $f(S)$.*

Proof. We may assume that $m = n - 1$ and that the m -hyperplane is E^{n-1} . Let \mathcal{U} be any neighborhood of p in E^n . Then, there is an open ball B about p in E^{n-1} such that $\bar{B} \subset \mathcal{U} \cap E^{n-1}$. Also, there exists an $\epsilon > 0$ such that $\mathcal{V} = B \times (-\epsilon, \epsilon) \subset \mathcal{U}$. To show $E^n - f(S)$ 1-LC at p , it will suffice to show $\mathcal{V} - f(S)$ simply connected. Let $\mathcal{V}_1 = \mathcal{V} - [(f(S) \cap B) \times [0, \epsilon)]$ and let $\mathcal{V}_2 = \mathcal{V} - [(f(S) \cap B) \times [-\epsilon, 0)]$. Clearly, \mathcal{V}_1 and \mathcal{V}_2 are open and arcwise connected. Now, notice that

$$\mathcal{V}_1 \cap \mathcal{V}_2 = \mathcal{V} - [(f(S) \cap B) \times (\epsilon, -\epsilon)].$$

Therefore, \mathcal{V} is an open set of dimension n , and $\dim [(f(S) \cap B) \times (\epsilon, -\epsilon)] \leq n - 2$. Hence, by Corollary 1 of Theorem IV.4 of [28], $\mathcal{V}_1 \cap \mathcal{V}_2$ is arcwise connected. Thus, by Van Kampen's Theorem (see [13] for a proof), $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V} - f(S)$ is 1-connected as desired.

COROLLARY 6.3. *Let f be a closed embedding of a k -dimensional space S into a locally flat m -cell D^m contained in the interior of the n -dimensional manifold Q^n where, $m < n$, and $k \leq n - 3$. Then, $Q - f(S)$ is 1-LC at each point of $f(S)$.*

THEOREM 6.4. *Suppose that g is an embedding of the p -dimensional polyhedron P^p into the interior of the n -dimensional PL manifold Q^n , $p < n$, which is locally flat on the open simplexes of some locally finite triangulation T . Let f be an embedding of a k -dimensional polyhedron X^k into the interior of Q^n , $k < \frac{2}{3}n - 1$, which is locally flat on the open simplexes of some locally finite triangulation T_* , except possibly at points*

of $f^{-1}(g(P))$. Suppose also that f is PL on some polyhedron $X_1 \subset X$. Then, f can be $\varepsilon(x)$ -tamed keeping $f(X_1)$ fixed.

Proof. The conclusion will follow from Theorem 1' of [10], if we can show that f is locally flat on closed simplexes (hence on open simplexes). Let Δ be a simplex of T_* . By applying Lemma 6.1 locally at $f(\Delta) - g(P)$, we see that $f|\Delta$ is locally flat on $\Delta - f^{-1}(g(P))$. Thus, $Q - f(\Delta)$ is 1-LC at every point of $f(\Delta) - g(P)$.

We now want to show that $Q - f(\Delta)$ is 1-LC at points of $f(\Delta) \cap g(P)$. This will be established for points of $f(\Delta) \cap g(P)$ contained in the interior of the image under g of top dimensional simplexes of T . (One can show that $Q - f(\Delta)$ is 1-LC at the other points of $f(\Delta) \cap g(P)$ by repeating this argument, going down inductively on the dimension of simplexes in T .) Let x be a point of $f(\Delta) \cap g(P)$ contained in $g(\delta^\circ) \cap g(P)$, where δ is a top dimensional simplex of T . Let δ_* be a simplex contained in δ° which is concentric with δ and which contains $g^{-1}(x)$ in its interior. Then, $g(\delta_*) \cap f(\Delta)$ is a closed subset of $g(\delta_*)$ of dimension less than $\frac{2}{3}n - 1$. Since $g(\delta_*)$ is locally flat, it follows from Corollary 6.3 that $Q - [g(\delta_*) \cap f(\Delta)]$ is 1-LC at x . We can find a small neighborhood \mathcal{U} of x in Q such that we know $Q - f(\Delta)$ to be 1-LC at every point of $\mathcal{U} \cap f(\Delta)$ except possibly at the points of $g(\delta_*) \cap f(\Delta)$. Since $Q - [g(\delta_*) \cap f(\Delta)]$ is 1-LC at x , there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of x such that any loop in $\mathcal{V} - [g(\delta_*) \cap f(\Delta)]$ can be shrunk to a point in $\mathcal{U} - [g(\delta_*) \cap f(\Delta)]$. Let l be a loop in $\mathcal{V} - f(\Delta)$. Then, l bounds a singular ball, B , in $\mathcal{U} - [g(\delta_*) \cap f(\Delta)]$. We can move this ball off of $f(\Delta)$ staying inside \mathcal{U} and fixed on \dot{B} by a standard argument which uses the fact that subsets of codimension two do not separate and which uses Theorem 2 of [16]. Hence, $Q - f(\Delta)$ is 1-LC at each point of $f(\Delta)$ and we can apply Theorem 1 of [5] to $f|\Delta$ and conclude that $f|\Delta$ is tame. But now Lemma 4.6 implies that $f|\Delta$ is locally flat and this completes the proof.

REMARK 6.5. An analogous theorem holds where one lets a locally flat topological manifold of codimension at least one play the role of $g(P)$ in Theorem 6.4, and the proof is almost the same.

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