TRANSITIVE SEMIGROUP ACTIONS

BY

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Following Wallace [15], we define an act to be a continuous function $\mu: S \times X \to X$ such that (i) $S$ is a topological semigroup, (ii) $X$ is a topological space, and (iii) $\mu(s, \mu(t, x)) = \mu(st, x)$ for all $s, t \in S$ and $x \in X$. We call $(S, X, \mu)$ an action triple, $X$ the state space of the act, and we say $S$ acts on $X$. We assume all spaces are Hausdorff and write $sx$ for $\mu(s, x)$. $S$ is said to act transitively if $Sx = X$ for all $x \in X$ and effectively if $sx = tx$ for all $x \in X$ implies that $s = t$. The first section of this paper deals with transitive actions and especially with the case where the semigroup is simple. We obtain as a corollary that if $S$ is a compact connected semigroup acting transitively and effectively on a space $X$ that contains a cut point, then $K$, the minimal ideal of $S$, is a left zero semigroup and $X$ is homeomorphic to $K$.

A C-set is a subset, $Y$, of $X$ with the property that if $M$ is any continuum contained in $X$ with $M \cap Y \neq \emptyset$, then either $M \subseteq Y$ or $Y \subseteq M$. In the second section, we consider the position of C-sets in the state space and prove as a corollary that if $S$ is a compact connected semigroup with identity acting effectively on the metric indecomposable continuum, $X$, such that $SX = X$, then $S$ must be a group.

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Definitions and notation. The notation is generally that of Wallace [16] for semigroups and Stadtlander [12] for actions. Let $S$ be a topological semigroup then we denote by $K(S)$ the unique minimal ideal (if it exists) of $S$ and by $E(S)$ the set of idempotents of $S$. When the semigroup referred to is clear, the above will be shortened to $K$ and $E$ respectively. We recall that if $S$ is compact then $K(S)$ exists and is closed and $E(S) \neq \emptyset$. For each $e \in E(S)$, $H(e)$ denotes the maximal subgroup of $S$ containing $e$. $S$ is a left zero semigroup if $xy = x$ for all $x, y \in S$. A left group is a semigroup that is left simple and right cancellative; it is isomorphic to $E \times G$ where $E$ is a left zero semigroup, $G$ is a group and multiplication is coordinate wise [2]. An algebraic isomorphism that is simultaneously a topological homeomorphism is called an isomorphism.

The $Q$-set of the action triple $(S, X, \mu)$ is the set $Q = \{x \in X \mid Sx = X\}$, thus if $Q = X$ the action is transitive. The action triple $(S, X, \mu)$ is said to be equivalent to...
the action triple \((T, Y, \nu)\) if there is an isomorphism \(\phi: S \rightarrow T\) and a homeomorphism \(\psi: X \rightarrow Y\) such that the following diagram commutes:

\[
\begin{array}{ccc}
S \times X & \overset{\mu}{\longrightarrow} & X \\
\phi \times \psi \downarrow & & \downarrow \psi \\
T \times Y & \overset{\nu}{\longrightarrow} & Y
\end{array}
\]

We say that \(s \in S\) acts as a constant if \(sx\) is a point. Finally \(X^*\) denotes the topological closure of \(X\). Examples of actions include topological transformation groups, semigroups acting on their underlying space by multiplication and the following: let \(A_1\) be a locally compact space and \(M(X)\) the set of all continuous functions of \(X\) into \(X\). With the compact open topology and composition of maps as multiplication, \(M(X)\) is a topological semigroup. Defining \(\mu: M(X) \times X \rightarrow X\) by \(\mu(f, x) = f(x)\) makes \((M(X), X, \mu)\) an action triple.

**Transitive action.** It follows from a result of Stadtlander [10] that if a compact semigroup, \(S\), acts transitively on \(X\) then the restriction of the act to \(K(S) \times X\) is still a transitive action. Thus we use the transitive actions of compact simple \((K(S) = S)\) semigroups as a tool to study the transitive actions of arbitrary compact semigroups.

We first show that for compact simple semigroups transitive action results from a seemingly weaker assumption.

**Theorem 1.1.** Let \(S\) be a compact simple semigroup acting on \(X\) such that \(Q \neq \emptyset\). Then \(S\) acts transitively on \(X\).

**Proof.** Let \(x \in Q\) and \(y\) be any member of \(X\). Since \(S = \bigcup \{H(f) \mid f \in E\}\) [1], \(X = Sx = \bigcup \{H(f)x \mid f \in E\}\) so that \(x \in H(f)x\) for some \(f \in E\). Then \(X = Sx = Sfx = \bigcup \{H(e)x \mid e \in Sf \cap E\}\). Thus \(y \in H(e)x\) for some \(e \in Sf \cap E\); say \(y = px\) where \(p \in H(e)\). Then \(x = fx = fex = fp^{-1}px = fp^{-1}y \in Sy\) and we have \(X = Sx \subseteq Sy \subseteq X\), that is \(Sy = X\). Since \(y\) is arbitrary, the action is transitive.

The author wishes to thank the referee for pointing out the above proof which is more concise than the original one.

A band is a semigroup \(S\) such that \(E(S) = S\), that is, every element is an idempotent. We now characterize the transitive actions of a compact simple band.

**Theorem 1.2.** Let \(S\) be a compact simple band acting transitively and effectively on \(X\). Then \(S\) must be a left zero semigroup, \(X\) and \(S\) are homeomorphic and the action is equivalent to multiplication in \(S\).

Two lemmas are necessary to complete the proof.

**Lemma 1.3.** Let \(S\) be a compact simple band acting transitively on \(X\). Then every element of \(S\) acts as a constant.
Proof. It is shown in [10] that if \( T \) is a compact semigroup acting transitively on \( X \) and \( e \in E \cap K \), then \( (H(e), eX) \) is a topological transformation group which is transitive on \( eX \) and \( H(e)x = eX \) for each \( x \in X \). Since \( S \) is a band, \( S = E \cap K \) and \( H(e) = e \). Therefore \( eX \) is a point for each \( e \in S \).

The proof of Theorem 1.2 as stated could now follow from Lemma 1.3 and a result of Day and Wallace [4], however we choose to present the following lemma to cover the noneffective case. Let \( S \) be compact, \( \rho \) a closed left congruence on \( S \), and also let \( \rho \) denote the natural map from \( S \) onto \( S/\rho \). If \( \nu: S \times S/\rho \to S/\rho \) is defined by \( \nu(s, \rho(t)) = \rho(st) \), then \( \nu \) is an act called the canonical act [10]. Stadtlander has shown that if \( Y = Sx \) is an orbit of the action triple \( (S, X, \mu) \) such that \( SY = Y \) and if \( \rho \) is defined as \((s, t) \in S \times S \mid sx = tx \) then \((S, Y, \mu) \) is equivalent to \((S, S/\rho, \nu) \) where \( \nu \) is the canonical act.

Lemma 1.4. Let \( S \) be a compact simple band acting transitively on \( X \) by the function \( \mu \) and let \( x_0 \in X \) and define \( \rho = \{(s, t) \in S \times S \mid sx_0 = tx_0 \} \). Then \( \rho \) is a two-sided congruence, \((S, X, \mu) \) is equivalent to \((S, S/\rho, \nu) \) where \( \nu \) is the canonical action and \( S/\rho \) is a left zero semigroup.

Proof. By Lemma 1.3, every element of \( S \) acts as a constant, thus \( \rho \) is a two-sided congruence and since \( X = Sx_0 \) is an orbit, we know \((S, X, \mu) \) is equivalent to \((S, S/\rho, \nu) \) by Stadtlander's result. Because every element of \( S \) acts as a constant, we have \( \nu(s, \rho(t)) = \nu(s, \rho(s)) = \rho(s^2) = \rho(s) \) for all \( s, t \in S \). Now let \( t_1, t_2 \in S/\rho \) then \( t_1 = \rho(s_1) \), \( t_2 = \rho(s_2) \) for \( s_1, s_2 \in S \). But then \( t_1t_2 = \rho(s_1)\rho(s_2) = \rho(s_1s_2) = \nu(s_1, \rho(s_2)) = \rho(s_1) = t_1 \) which shows that \( S/\rho \) is a left zero semigroup.

Proof of Theorem 1.2. We have only to note, since every element acts as a constant and \( S \) acts effectively, that \( \rho = \Delta \) the diagonal of \( S \). Thus \( S = S/\rho \) and an application of Lemma 1.4 completes the proof.

The following lemma is a partial converse to Lemma 1.3 to be used in the proof of Corollary 1.9.

Lemma 1.5. Let \( S \) be a compact simple semigroup acting effectively on \( X \) such that some element of \( S \) acts as a constant then \( S \) is a band.

Proof. Since \( S \) is simple, we know \( S \) is isomorphic to \((Se \cap E) \times eSe \times (eS \cap E) \) when the latter is endowed with the Rees multiplication and \( e \in E \) [17]. We will show that \( eSe = e \) thus making \( S \) isomorphic to the band \((Se \cap E) \times \{e\} \times (eS \cap E) \). Since \( S \) is simple every element acts as a constant, thus \( e(SEX) = y \) for some \( y \in X \). Let \( g \in eSe \), then \( gx = egex = e(gex)y = ey \) for all \( x \in X \), but \( S \) acts effectively, therefore \( g = e \), thus \( eSe = e \).

We now investigate the effect a cut point in the state space has in a transitive action by a compact connected semigroup. First recall that if \( G \) is a compact connected group acting transitively on \( X \) then \( X \) is homogeneous [9], that is, for every \( x, y \in X \), there is a homeomorphism \( h: X \to X \) such that \( h(x) = y \). Furthermore \( X \) is a continuum and if nondegenerate must contain at least two noncut
points which together with the fact that $X$ is homogeneous implies that every point of $X$ is a noncut point. Thus in the group case $X$ cannot contain a cut point. This does not follow for semigroups however as the following example illustrates. Let $S=[-1,1]$ with the usual topology and for $s_1, s_2 \in [-1,0]$ and $t_1, t_2 \in [0,1]$ define multiplication in $S$ as follows: $s_1s_2 = s_1$, $s_1t_2 = s_1$, $t_1t_2 =$ the usual product of the real numbers $t_1$ and $t_2$, $t_1s_2 =$ the usual product of the real numbers $t_1$ and $s_2$. Then $S$ is a compact connected topological semigroup with identity. Now let $X = [0,1]$ with the usual topology. Define $\mu : S \times X \to X$ as follows where $s_1$ and $t_1$ are as above and $x \in X; \mu(s_1, x) = -s_1$ and $\mu(t_1, x) =$ the usual product of the real numbers $t_1$ and $x$, then $\mu$ is a transitive and effective act. Thus, the state space of a transitive act by a compact connected semigroup may contain cut points, however, in Corollary 1.9 already mentioned in the introduction, it is shown that this has a profound effect on the multiplication of $S$. We begin with the following lemma.

**Lemma 1.6.** Let $S$ be a compact connected simple semigroup acting transitively on $X$ such that no element of $S$ acts as a constant. Then $X$ has no cut points.

**Proof.** $Sx = X$ implies that $X$ is a continuum and since no element acts as a constant, $fX$ is a nondegenerate continuum for all $f \in E$. But then $fX$ contains at least two noncut points of $fX$ and since $(H(f), fX)$ is a transitive topological transformation group [10] making $fX$ homogeneous [9], we have that every element of $fX$ is a noncut point of $fX$. We now show for every $s \in S$, $sX = fX$ for some $f \in E$. Let $s \in S$. Because $S$ is simple, $S = \bigcup \{H(e) \mid e \in E\}$ [1], thus $s \in H(f)$ for some $f \in E$ and since $(H(f), fX)$ is a topological transformation group, $s(fX) = fX$. Hence, $fX = s(fX) \subseteq sX = (fSf)X \subseteq fX$, whence $fX = sX$. Thus, for each $s \in S$, no point of $sX$ is a cut point of $sX$.

Suppose $p \in X$ cuts $X$, then $X \setminus \{p\} = Y \cup Z$ where $Y$ and $Z$ are mutually separated. Let $A = \{s \in S \mid sX \subseteq Y \cup \{p\}\}$ and $B = \{s \in S \mid sX \subseteq Z \cup \{p\}\}$, then $S = A \cup B$. For let $s \in S$ and suppose $p \notin sX$, then since $sX$ is connected, $sX \subseteq Y$ or $sX \subseteq Z$, thus $s \in A \cup B$. Now suppose $p \in sX$, then since $p$ is a noncut point of $sX$, $sX \setminus \{p\}$ is connected which implies that $sX \setminus \{p\} \subseteq Y$ or $sX \setminus \{p\} \subseteq Z$ and $s \in A \cup B$. Therefore $S = A \cup B$. Now suppose that $t \in A \cap B$, then $tX = (Y \cup \{p\}) \cap (Z \cup \{p\}) = \{p\}$ which is impossible since no element acts as a constant, hence $A \cap B = \emptyset$. It is easy to show that $A$ and $B$ are both closed and thus contradict the fact that $S$ is connected. Therefore $X$ has no cut points.

Since a left group that is not left zero always acts transitively on itself with no element acting as a constant, we have the following corollary.

**Corollary 1.7.** A compact connected left group that is not a left zero semigroup contains no cut points.

It follows from a result of Stadtlander [10] that if $S$ acts transitively on $X$ then $K(S)$ acts transitively on $X$ and since $K(S)$ is connected whenever $S$ is [13] we can apply Lemma 1.6 to the action of $K(S)$ on $X$ to obtain the following theorem.
Theorem 1.8. Let $S$ be a compact connected semigroup acting transitively on $X$ such that no element of $K(S)$ acts as a constant. Then $X$ has no cut points.

It is easy to see that if $S$ acts effectively then $K(S)$ does also, thus we can put together Lemma 1.5 and Theorems 1.2 and 1.8 to obtain the following result, first proved for semigroups by Faucett [5].

Corollary 1.9. Let $S$ be a compact connected semigroup acting transitively and effectively on $X$. Then either (i) $X$ has no cut points or (ii) $K(S)$ is a left zero semigroup and $X$ is homeomorphic to $K(S)$.

C-sets in the state space. Let $Y = \{(0, y) \mid -1 \leq y \leq 1\}$ and let

$$X = \{(x, \sin (1/x)) \mid 0 < x \leq 1\} \cup Y,$$

then $Y$ is a C-set in $X$ and the complement of $Y$ is an open dense half line in $X$. C-sets of this type have been studied independently by Day and Wallace [4] and Stadtlander [19]. It follows from their results, for example, that a compact connected semigroup with identity cannot act on the space $X$ defined above such that $\emptyset \neq \partial X$. This also follows from the results to be given below.

In [8], Hunter has shown that if $S$ is a compact connected semigroup with identity and if $Y$ is a nondegenerate C-set contained in $S$, then $Y^* = K(S)$ and $K(S)$ is a group. We use the techniques of Hunter as an important tool in the proof of the following theorem.

Theorem 2.1. Let $S$ be a compact connected semigroup with identity acting on the continuum $X$ with $S X = X$ and suppose $Y$ is a nondegenerate C-set in $X$. Then $Y \subseteq eX$ for some $e \in E(S) \cap K(S)$.

We need the preliminary result that follows.

Theorem 2.2. Let $S$ be a compact connected semigroup with identity and zero acting on the continuum $X$ with $S X = X$ and such that zero acts as a constant. Then $X$ cannot contain a nondegenerate C-set.

Proof. Let $OX = \theta \in X$. Once it has been shown that $\theta$ cannot be an element of a nondegenerate C-set in $X$, the proof of Theorem 2.2 proceeds almost exactly the same as the proof of Theorem 1 of [8], thus we will show only that $\theta$ cannot be an element of a nondegenerate C-set in $X$. In order to do this we will use the notion of an ideal in $X$. If the semigroup $S$ acts on the space $X$ and $I$ is a subset of $X$ such that $S I \subseteq I$, then $I$ is called an ideal of $X$. For $A \subseteq X$, define

$$I_0(A) = \bigcup \{I \subset A \mid I \text{ is an ideal of } X\}.$$

If $S$ is compact and $A$ is an open set containing an ideal of $X$, then $I_0(A)$ is an open ideal of $X$. It is easy to see that under the conditions of this theorem, every ideal of $X$ is connected.
Now, suppose $\theta \in Y$ a nondegenerate C-set in $X$ and let $U$ be open in $X$ such that $\theta \in U$ and $Y \cap (X \setminus U) \neq \emptyset$. Let $V$ be open in $X$ such that $\theta \in V \subset V^* \subset U$ and let $W = I_0(V)$. Then $W$ is an open connected set, $W^*$ is a continuum and $\theta \in W^* \subset U$. But $W^* \cap Y \neq \emptyset$ and $W^* \neq Y$, hence $W \subset W^* \subset Y$, a contradiction since a $C$-set has empty interior.

Let $S$ be a compact connected semigroup with identity and let $T$ be a compact connected subsemigroup of $S$ such that (i) $T \cap K(S) \neq \emptyset$, (ii) $1 \in T$ and (iii) if $R$ is a compact connected subsemigroup of $T$ satisfying (i) and (ii) then $R = T$. $T$ is said to be algebraically irreducible from 1 to $K(S)$. In [7], Hofmann and Mostert show that if $S$ is a compact connected semigroup with identity then $S$ contains an algebraically irreducible semigroup and every algebraically irreducible semigroup is abelian.

We recall the Rees quotient [20]. Let $S$ be a semigroup, $I$ a closed ideal of $S$ and define $\rho = \{(s, t) \in S \mid s = t \text{ or } s, t \in I\}$ then $\rho$ is a closed congruence and we call the factor semigroup $S/\rho$ the Rees quotient and denote it by $S/I$. We now use Theorem 2.2 to prove Theorem 2.1.

**Proof of Theorem 2.1.** Let $T$ be a subsemigroup of $S$ algebraically irreducible from 1 to $K(S)$, then $T$ is a compact connected abelian semigroup with identity acting on $X$ with $TX = X$. Let $T' = T/K(T)$ be the Rees quotient and $X' = X/K(T)X$ be the ordinary topological quotient and let $\eta: T \rightarrow T'$ and $\beta: X \rightarrow X'$ be the canonical maps, then $T'$ acts on $X'$ by $\eta(t)\beta(x) = \beta(tx)$ [10] and satisfies the hypothesis of Theorem 2.2. It is routine to show that if $D$ is a continuum in $X'$ and $E = \beta^{-1}(D)$ then $E$ is a continuum in $X$.

We now show that $Y \subset K(T)X$. Suppose not then $\overline{Y} = \beta(Y)$ is a nondegenerate subset of $X'$ which is a $C$-set. For let $M$ be a continuum in $X'$ with $M \cap \overline{Y} \neq \emptyset$ and consider the two cases (i) $Y \cap K(T)X = \emptyset$ and (ii) $Y \cap K(T)X \neq \emptyset$. In case (i), $\beta^{-1}(\overline{Y}) = Y$ since $\beta|_{X(K(T)X)}$ is a homeomorphism, and $Y$ meets the continuum $\beta^{-1}(M)$, thus $\beta^{-1}(M) \subset Y$ or $Y \subset \beta^{-1}(M)$ which implies $M \subset \overline{Y}$ or $Y \subset M$. In case (ii), $Y \cap K(T)X \neq \emptyset$ implies $K(T)X \subset Y$ since $K(T)X$ is a continuum hence $\beta^{-1}(\overline{Y}) = Y$ and the same argument as in case (i) shows that $\overline{Y}$ is a $C$-set. But this contradicts Theorem 2.2, therefore $Y \subset K(T)X$.

Since $T$ is abelian, $K(T)$ is a group and $K(T) \subset K(S)$ which implies $K(T) \subset H(e)$ for some $e \in K(S) \cap E(S)$, thus $Y \subset K(T)X \subset H(e)X \subset eX$.

**Note.** We have actually proved a slightly stronger result than that stated since $Y$ is contained in the state space of the abelian topological transformation group $(K(T), eX)$.

As an application of Theorem 2.1, we prove the following corollary, which is a special case of a more general theorem in [18].

**Corollary 2.6.** Let $S$ be a compact connected semigroup with identity acting effectively on the metric indecomposable continuum $X$ with $SX = X$, then $S$ is a group.

**Proof.** Let $Y$ be a composant of $X$, then, as is well known, $Y$ is a $C$-set so
Y^eX for some e ∈ E ∩ K. But Y^* = X [6], thus X = eX and 1y = y = ey for all y ∈ X which implies 1 = e since S acts effectively. But 1 ∈ K implies K is a group and K = S.

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