I. In the theory of differentiable dynamical systems, the natural actions on coset spaces are of particular interest, not least because the general situation is so intractable. Let $G$ be a Lie group and $H$ a closed subgroup such that $G/H$ is compact. A natural action on $G/H$ is a group of transformations of $G/H$ of the form $xH \mapsto a(x)H$, where $a$ is an element of $G$ and $\alpha$ is an automorphism of $G$ such that $\alpha(H) = H$. The space of right-invariant vector fields on $G$ is carried by the natural projection $\pi, x \mapsto xH$, onto a space of vector fields on $G/H$, and this space is invariant under each natural action. Suppose now that $H$ is discrete. Then $G$ is unimodular. A Haar measure on $G$ determines a finite Borel measure on $G/H$ invariant under each action. Analogously, a translation invariant $n$-form $\omega$ on $G$, where $n$ is the dimension of $G$, determines an $n$-form $\eta$ on $G/H$, and $T^*\eta$ equals $\eta$ for each natural transformation $T$.

It is such a phenomenon that we examine in this paper: A dynamical system in which a certain finite-dimensional linear space of vector fields and a certain differential form are (essentially) invariant under the action.

In the next few paragraphs we define several expressions, and then state our main result. The following section is devoted to a proof of this theorem. Additional results are presented in the third section.

Let $M$ be a differentiable manifold and let $\mathcal{X}$ be a finite-dimensional linear space of vector fields on $M$. We say that $\mathcal{X}$ is spanning if, for each $p$ in $M$, the evaluation at $p$, $X \mapsto X_p$, maps $\mathcal{X}$ onto the tangent space of $M$ at $p$. If the dimension of $\mathcal{X}$ equals the dimension of $M$, in which case these mappings are each one-to-one, then $\mathcal{X}$ is said to be simply spanning. If $\mathcal{X}$ is simply spanning, an affine connection $\nabla$ on $M$ is determined by the condition:

$$\nabla_Y X = 0$$

for $X$ in $\mathcal{X}$ and $Y$ any vector field on $M$. We call $\nabla$ the connection associated with $\mathcal{X}$.

Let $\mathcal{F}$ be a group of diffeomorphisms of $M$. For each $T$ in $\mathcal{F}$, let $T_*$ be the associated mapping of vector fields, and let $T^*$ be the associated mapping of differential forms. A linear space $\mathcal{X}$ of vector fields on $M$ is said to be invariant under $\mathcal{F}$, if $T_*(\mathcal{X}) = \mathcal{X}$ for each $T$ in $\mathcal{F}$. In this case, the restriction of $\{T_* : T \in \mathcal{F}\}$ to $\mathcal{X}$ is a group of linear transformations of $\mathcal{X}$.

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Let $\mathcal{F}$ be a group of homeomorphisms of a space $S$. An eigenfunction for $\mathcal{F}$ on $S$ is a function $f$ on $S$, with values in the unit circle, such that, for each $T$ in $\mathcal{F}$, there exists a complex number $\lambda_T$ satisfying $fT = \lambda_T f$.

**Theorem 1.** Let $M$ be a compact differentiable manifold of dimension $n$, $\omega$ an $n$-form on $M$, $\mathcal{X}$ a finite-dimensional linear space of vector fields on $M$, and $\mathcal{F}$ a group of diffeomorphisms of $M$. Suppose the following statements are true.

(I) $\omega$ is not identically zero, and $T^*\omega = \pm \omega$ for each $T$ in $\mathcal{F}$.

(II) $\mathcal{X}$ is spanning and invariant under $\mathcal{F}$.

(III) $\mathcal{F}$ is abelian and each differentiable eigenfunction for $\mathcal{F}$ on $M$ is constant. Then the following statements are also true.

(I) Except for multiplication by a scalar, $\omega$ is the only $n$-form on $M$ satisfying assumption (I) above.

(II) $\mathcal{X}$ is simply spanning.

(III) If $\mathcal{Y}$ is a finite-dimensional linear space of vector fields on $M$ invariant under $\mathcal{F}$, then $\mathcal{Y} \subseteq \mathcal{X}$.

(IV) $\mathcal{X}$ is a Lie algebra.

(V) The system $(M, \omega, \mathcal{X}, \mathcal{F})$ is isomorphic to $(G/\Gamma, \eta, \mathcal{Y}, \mathcal{P})$ where: $G$ is a simply connected Lie group whose Lie algebra is isomorphic to $\mathcal{X}$; $\Gamma$ is a discrete subgroup of $G$; $\eta$ corresponds to a translation-invariant $n$-form on $G$; $\mathcal{Y}$ corresponds to the space of right-invariant vector fields on $G$; and, each transformation, $x\Gamma \mapsto S(x\Gamma)$, in $\mathcal{F}$ is of the form

$$x\Gamma \mapsto a\alpha(x)\Gamma,$$

$a$ being in $G$ and $\alpha$ being an automorphism of $G$ such that $\alpha(\Gamma) = \Gamma$.

**Corollary 1.** Let $M$ be a compact differentiable manifold, $\nabla$ an affine connection on $M$ associated with a simply spanning space of vector fields, and $\mathcal{F}$ an abelian group of diffeomorphisms of $M$ which preserve the affine structure. If each differentiable eigenfunction for $\mathcal{F}$ on $M$ is constant, then $\mathcal{F}$ preserves no other such affine structure, and $(M, \mathcal{F})$ is isomorphic to a natural action on a coset space $G/\Gamma$ with $\Gamma$ discrete.

II.

**Lemma 1.** Let $\mathcal{E}$ be a finite-dimensional real vector space, $\mathcal{L}$ an abelian group of linear transformations of $\mathcal{E}$, and $C$ a compact subset of $\mathcal{E}$ invariant under $\mathcal{L}$. If $C$ has more than one point, then there exists a differentiable function $f: \mathcal{E} \to C$ such that $f$ is an eigenfunction for $\mathcal{L}$ on $C$.

**Proof.** Let $\mathcal{F}$ be the space of all complex-valued real-linear functionals on $\mathcal{E}$; $\mathcal{F}$ is a complex vector space. For each $L$ in $\mathcal{L}$ define $L^*: \mathcal{F} \to \mathcal{F}$ by $L^*f = fL$. Thus, $\mathcal{L}^* = \{L^* : L \in \mathcal{L}\}$ is an abelian group of linear transformations of $\mathcal{F}$. These transformations can be put simultaneously in triangular form. That is, there exists a basis $\{f_1, f_2, \ldots, f_n\}$ of $\mathcal{F}$ such that for each $k$, the linear span of $\{f_1, \ldots, f_k\}$
is invariant under $\mathcal{L}^*$. Let $m$ be the smallest integer $k$ such that $f_k$ is not constant on $C$. Set $g = f_m$. Hence, for each $L$ in $\mathcal{L}$, there exists $\alpha_L$ in $C$ such that $L^*g - \alpha_Lg$ is constant on $C$; let the constant be $\beta_L$; on $C$ then, $gL = \alpha_Lg + \beta_L$.

Let $B = g(C)$. Define $A_L : C \rightarrow C$ by $A_Lz = \alpha_Lz + \beta_L$. Since $A_L(B) = B$, $|\alpha_L| = 1$. Let $K$ be the convex hull of $C$; then $A_L(K) = K$. Thus there exists a point $\gamma$ in $K$ left fixed by each $A_L$:

$$\gamma = \alpha_L\gamma + \beta_L$$

for $L$ in $\mathcal{L}$. Define $h : \mathcal{E} \rightarrow \mathcal{E}$ by $h(x) = g(x) - \gamma$. For $x$ in $C$,

$$h(Lx) = \alpha_Lg(x) + \beta_L - \gamma = \alpha_Lh(x).$$

Thus $hL = \alpha_Lh$ on $C$ for each $L$.

If $|h|$ is constant on $C$, let the constant be $c$ (which cannot equal 0) and define $f$ by $f = h/c$.

If $|h|$ is not constant on $C$, there exists an $\varepsilon > 0$ such that

$$f = e^{i\varepsilon|h|}$$

is not constant on $C$, and this $f$ is an eigenfunction for $\mathcal{L}$ on $C$, each eigenvalue being 1 in this case.

**Lemma 2.** Given the assumptions of Theorem 1, conditions (I) and (II) are valid.

**Proof.** Let $\mathcal{E}$ be the space of all multilinear alternating mappings $\eta : \mathcal{X}^n \rightarrow R$; here $\mathcal{X}$ is regarded as a real vector space. Thus $\mathcal{E}$ is a finite-dimensional real vector space. For each $T$ in $\mathcal{T}$, define $T' : \mathcal{E} \rightarrow \mathcal{E}$ by

$$(T'\eta)(X_1, \ldots, X_n) = \eta(T_\ast X_1, \ldots, T_\ast X_n).$$

Each $T'$ is linear. For each $T$, let $e_T$ in $\{1, -1\}$ be such that $T_\ast o = e_T o$. Thus

$$\omega(T\ast X_1, \ldots, T\ast X_n) = e_T\omega(X_1, \ldots, X_n)T.$$ 

For each $p$ in $M$, define $\omega_p : \mathcal{X}^n \rightarrow R$ by

$$\omega_p(X_1, \ldots, X_n) = (\omega(X_1, \ldots, X_n))(p).$$

Clearly $\omega_p$ is in $\mathcal{E}$ and

$$T'\omega_p = e_T\omega_{Tp}.$$ 

Define $L_T = e_T T'$. Thus $\mathcal{L} = \{L_T : T \in \mathcal{T}\}$ is an abelian group of linear transformations of $\mathcal{E}$ leaving the set

$$C = \{\omega_p : p \in M\}$$

invariant. The mapping $p \mapsto \omega_p$ of $M$ into $\mathcal{E}$ is differentiable. By Lemma 1 and assumption (III), $C$ has but one point and it is not 0.
Conclusion (I) is valid because we have proved that \( \omega \) is everywhere nonzero. To verify (II), suppose that \( X \) is in \( \mathcal{X} \), \( p \) is in \( M \), and \( X_p = 0 \). Then
\[
\omega_p(X, X_2, \ldots, X_n) = 0
\]
for \( X_2, \ldots, X_n \) in \( \mathcal{X} \). Thus, for each \( q \) in \( M \),
\[
\omega_q(X, X_2, \ldots, X_n) = 0
\]
for \( X_2, \ldots, X_n \) in \( \mathcal{X} \); \( X_q = 0 \) for each \( q \); \( X = 0 \).

**Lemma 3.** Let \( \mathcal{X} \) be simply spanning on the compact differentiable manifold \( M \) of dimension \( n \), and invariant under the group \( \mathcal{F} \). Then \( M \) is orientable and there exists an everywhere nonzero \( n \)-form \( \omega \) on \( M \) with the following property: For each \( T \) in \( \mathcal{F} \), \( T^*\omega = \pm \omega \) and the determinant of \( T_\ast \) on \( \mathcal{X} \) equals \( \pm 1 \), according as (for both) \( T \) preserves or reverses the orientation of \( M \).

**Proof.** Let \( \{X_1, X_2, \ldots, X_n\} \) be a basis for \( \mathcal{X} \). Let \( \{\omega_1, \omega_2, \ldots, \omega_n\} \) be a dual system of 1-forms on \( M \). That is, \( \omega_i(X_j) = \delta_{ij} \). Set \( \omega = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n \). For each \( T \) in \( \mathcal{F} \), \( T^*\omega = d_T\omega \) where \( d_T \) is the determinant of \( T_\ast \) on \( \mathcal{X} \). The finite nonnegative Borel measure on \( M \) corresponding to \( \omega \) is invariant under \( \mathcal{F} \). Hence \( |d_T| = 1 \).

**Lemma 4.** Let \( \mathcal{X} \) be simply spanning on the differentiable manifold \( M \). Then there exists a unique affine connection \( \nabla \) on \( M \) such that \( \nabla_YX = 0 \) for \( X \) in \( \mathcal{X} \) and \( Y \) any vector field on \( M \). Moreover, a diffeomorphism \( T \) of \( M \) is affine with respect to \( \nabla \) if and only if \( T_\ast(\mathcal{X}) = \mathcal{X} \).

**Proof.** The connection \( \nabla \) is given by
\[
\nabla_Y \left( \sum_{k=1}^{n} f_k X_k \right) = \sum_{k=1}^{n} (Yf_k) X_k,
\]
where \( \{X_1, X_2, \ldots, X_n\} \) is a basis for \( \mathcal{X} \). Uniqueness is clear.

Let \( T \) be an affine transformation of \( M \). Thus \( T \) carries geodesics to geodesics, and the geodesics are just the integral curves for members of \( \mathcal{X} \). Hence \( T_\ast(\mathcal{X}) = \mathcal{X} \).

Let \( T \) be a diffeomorphism such that \( T_\ast(\mathcal{X}) = \mathcal{X} \). Let \( \nabla' \) be the connection on \( M \) induced by \( T \). That is,
\[
\nabla'_{T_*Y} T_*X = T_\ast(\nabla_YX).
\]
Clearly, \( \nabla'_{Y}X = 0 \) for \( X \) in \( \mathcal{X} \). Hence \( \nabla' = \nabla \).

**Lemma 5.** Let \( \mathcal{X} \) be a finite-dimensional linear space of vector fields on the compact differentiable manifold \( M \). Let \( V \) be a fixed vector field on \( M \) and let \( \mathcal{F} = \{T_t : t \in \mathbb{R}\} \) be the flow on \( M \) corresponding to \( V \). For each vector field \( X \) on \( M \), let \( LX = [V, X] \).

Then \( \mathcal{X} \) is invariant under \( \mathcal{F} \) if and only if \( L(\mathcal{X}) \subseteq \mathcal{X} \). If \( \mathcal{X} \) is so invariant, then
\[
T_t = e^{-tv}
\]
on \( \mathcal{X} \).

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Proof. It is known [3, p. 15] that
\[ -L = \lim_{t \to 0} \frac{1}{t} (T_t - I) \]
pointwise, on the space of vector fields, and that \( L \) commutes with each \( T_t \).

If \( \mathcal{X} \) is invariant under \( \mathcal{T} \), the conclusion is clear.

Suppose now that \( L(\mathcal{X}) \subseteq \mathcal{X} \). Each space \( T_t(\mathcal{X}) \) is invariant under \( L \). Let \( \mathcal{Y} \) be the linear span of the union of these spaces. Each point in \( \mathcal{Y} \) is contained in a finite-dimensional subspace of \( \mathcal{Y} \) invariant under \( L \). Define operators \( K_t \) on \( \mathcal{Y} \) by \( K_t = T_t e^{\lambda t} \). It is readily shown that the derivative of the mapping \( t \mapsto K_t \) is 0 on \( \mathcal{X} \). Thus \( K_t \) is constant in \( t \) and equals the identity transformation on \( \mathcal{Y} \).

Proof of Theorem 1. Conclusions (I) and (II) have been verified.

(III) Let \( \mathcal{Y} \) be as given. Let \( \mathcal{E} \) be the space of linear mappings \( f: \mathcal{Y} \to \mathcal{X} \). For each \( T \) in \( \mathcal{T} \), define \( L_T: \mathcal{E} \to \mathcal{E} \) by
\[ (L_T f)(x) = T_* f(T_* x) = T_* f(T_* x). \]
Clearly, each \( L_T \) is linear, and \( \mathcal{L} = \{ L_T : T \in \mathcal{T} \} \) is an abelian group of linear transformations of \( \mathcal{E} \). For each \( p \) in \( \mathcal{M} \), define \( f_p: \mathcal{Y} \to \mathcal{X} \) by requiring that \( f_p(Y) \) be that element of \( \mathcal{X} \) which agrees with \( Y \) at \( p \). That is, \( (f_p Y)_p = Y_p \). The mapping \( p \mapsto f_p \) is differentiable. Since \( T_* (X_p) = (T_* X)_p \), we have
\[ [(L_T f_p) Y]_p = T_* [(f_p T_* Y)_p] = T_* [(T_* Y)_p] = Y_p = (f_p Y)_p. \]
Thus, \( L_T f_p = f_T p \). The set \( \{ f_p : p \in \mathcal{M} \} \) is invariant under \( \mathcal{L} \) and, by Lemma 1, it must consist of but one point. Therefore, \( \mathcal{Y} \subseteq \mathcal{X} \).

(IV) The set \( \{ [X, Y] : X, Y \in \mathcal{X} \} \) is invariant under \( \mathcal{T} \). Its linear span is finite-dimensional and invariant under \( \mathcal{T} \). By (III), the set is contained in \( \mathcal{X} \).

(V) Let \( G \) be a simply connected Lie group with Lie algebra isomorphic to \( \mathcal{X} \). It is well known that there exists a transitive action, \( (x, p) \mapsto x \cdot p \), of \( G \) on \( \mathcal{M} \) which leaves \( \mathcal{X} \) invariant. Since \( G \) has dimension \( n \), there exists a discrete uniform subgroup \( \Gamma \) of \( G \) and a diffeomorphism \( U: G/\Gamma \to \mathcal{M} \) such that
\[ U(x y \Gamma) = x \cdot U(y \Gamma). \]
Let \( \mathcal{Y} \) be the image under \( \pi_* \) of the space of right-invariant vector fields on \( G \). Since \( \mathcal{Y} \) generates the action of \( G \) on \( G/\Gamma \) and \( \mathcal{X} \) generates the action of \( G \) on \( M \), \( U_* (\mathcal{X}) = \mathcal{Y} \). By Lemma 3, \( \omega \) is invariant under the action of \( G \) on \( M \). Hence, \( \eta = (U^{-1})_* \omega \) is invariant under the action of \( G \) on \( G/\Gamma \), and \( \eta \) must correspond to a translation-invariant \( n \)-form on \( G \).

That the diffeomorphisms \( U^{-1} T U \) on \( G/\Gamma \) are of the stated form follows from Theorem 2 of the next section.

Proof of Corollary 1. By Lemma 3, there exists an appropriate \( n \)-form on \( M \) and Theorem 1 applies. The transformations are affine by Lemma 4, and the structure is unique by (III) of Theorem 1.
III.

Corollary 2. Let $M$ be a compact differentiable manifold of dimension $n$, $\omega$ an $n$-form on $M$, $\mathcal{X}$ a finite-dimensional linear space of vector fields on $M$. Let there be given a differentiable action of $\mathbb{R}^m$ on $M$. Suppose that:

(I) $\omega$ is not identically zero, and is invariant under $\mathbb{R}^m$;

(II) $\mathcal{X}$ is spanning and invariant under $\mathbb{R}^m$;

(III) Either (i) $M$ is simply connected, or (ii) the action has a fixed point in $M$.

Then there exists a nonconstant differentiable function on $M$ which is invariant.

Proof. Let $\mathcal{T}$ be the group of diffeomorphisms. Let $f$ be a differentiable eigenfunction for $\mathcal{T}$ on $M$. Choose $\alpha_T$ in $\mathbb{R}$ such that $f^T = e^{\alpha_T}f$.

If there exists a fixed point, $f^T = f$ for all $T$, and $f$ is invariant. If $M$ is simply connected, there exists a differentiable function $g: M \to \mathbb{R}$ such that $f = e^{\theta}$; then $e^{\theta T} = e^{\theta + \alpha_T}$ and $gT - g$ is constant, hence zero; $f$ is invariant.

Suppose now that there does not exist a nonconstant differentiable invariant function on $M$. Then Theorem 1 applies. Let $\mathcal{Y}$ be the space of vector fields on $M$ corresponding to the action of $\mathbb{R}^m$. Since $\mathcal{Y}$ is invariant under $\mathcal{T}$, $\mathcal{Y} \subseteq \mathcal{X}$. Thus the system $(M, \mathcal{T})$ is isomorphic to $(G/\Gamma, \mathcal{P})$ where $\mathcal{P}$ consists of translations, $x\Gamma \mapsto ax\Gamma$. If $\mathcal{T}$ has a fixed point, this is surely a contradiction. If $M$ is simply connected, then $\Gamma = \{e\}$, since $\Gamma$ is the fundamental group of $G/\Gamma$. In this case $(M, \mathcal{T})$ is isomorphic to $(G, A)$ where $G$ is a compact simply connected group and $A$ is a connected abelian subgroup of $G$. Let $\bar{A}$ be the closure of $A$. This is a torus (or just $\{e\}$) and hence not equal to $G$. But any differentiable function on $G/\bar{A}$ would determine an invariant function on $G$. This is a contradiction.

Theorem 2. For $k = 1, 2$: Let $G_k$ be a simply connected Lie group, $\Gamma_k$ a discrete subgroup of $G_k$, and $\mathcal{X}_k$ the space of vector fields on $G_k/\Gamma_k$ corresponding to the right-invariant vector fields on $G_k$. Let $T: G_1/\Gamma_1 \to G_2/\Gamma_2$ be a diffeomorphism and suppose that $T_* (\mathcal{X}_1) = \mathcal{X}_2$. Then there exists a point $a$ in $G_2$ and an isomorphism $\alpha: G_1 \to G_2$ such that $\alpha (G_1) = G_2$, $\alpha (\Gamma_1) = \Gamma_2$, and $T(x\Gamma_1) = a\alpha(x)\Gamma_2$ for $x$ in $G_1$. Moreover, if $a'$ and $a'$ also have this property, then there exists $\gamma$ in $\Gamma_2$ such that $a' = a\gamma$ and $a'(x) = \gamma^{-1}\alpha(x)\gamma$.

Proof. Let $\mathcal{Y}_k$ be the space of right invariant vector fields on $G_k$. Let $\pi_k: G_k \to G_k/\Gamma_k$ be the natural projection. Then $\mathcal{X}_k = \pi_k_* (\mathcal{Y}_k)$. Thus $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are isomorphic as Lie algebras, and $T_*$ lifts to an isomorphism $L: \mathcal{Y}_1 \to \mathcal{Y}_2$ such that $\pi_2 L = T_* \pi_1$. Since $G_1$ and $G_2$ are simply connected, there exists an isomorphism $\alpha$ of $G_1$ onto
Let $G_2$ such that $L = a_\ast$. Let $e_k$ be the identity element of $G_k$. Choose $a$ in $G_2$ such that $T(e_1 \Gamma_1) = a \Gamma_2$. The two mappings of $G_1$ onto $G_2/\Gamma_2$ given by

$$x \mapsto T(x \Gamma_1), \quad x \mapsto a \alpha(x) \Gamma_2,$$

are equal at $e_1$ and have equal differentials. Hence, these two mappings are equal. It is clear that $\alpha(\Gamma_1) = \Gamma_2$.

Now suppose that $\alpha'$ and $\alpha$ are such that

$$\alpha'a(x) \Gamma_2 = a \alpha(x) \Gamma_2$$

for all $x$ in $G_1$. Letting $x = e_1$ shows that $\alpha' \Gamma_2 = a \Gamma_2$. There exists $\gamma$ in $\Gamma_2$ such that $\alpha' = \gamma a$. Thus

$$\alpha'(x) \Gamma_2 = \gamma^{-1} a(x) \Gamma_2 = \gamma^{-1} \alpha(x) \gamma \Gamma_2.$$

The two isomorphisms, $x \mapsto \alpha'(x)$ and $x \mapsto \gamma^{-1} \alpha(x) \gamma$, are equal on a neighborhood of $e_1$ in $G_1$. Thus they are equal on $G_1$.

Example 1. Let $C$ denote the unit circle. There exists a two-dimensional spanning space of vector fields on $C$ and a flow $\mathcal{F} = \{T_t : t \in \mathbb{R}\}$ on $C$ such that:

(I) The only invariant Borel measures on $C$ are concentrated on the set $\{1, -1\}$;  
(II) $\mathcal{F}$ is invariant under $\mathcal{F}$;  
(III) There exist no nonconstant continuous eigenfunctions for $\mathcal{F}$ on $C$.

To see this, let $V$ denote the vector field $d/d\theta$ on $C$, with respect to the parametrization $\theta \mapsto e^{i\theta}$. Define the functions $x, y, z$ on $C$ by

$$x(p) = \frac{i}{2}(p + \bar{p}), \quad y(p) = (1/2i)(p - \bar{p}), \quad z(p) = p.$$

Set $Y = y V$ and let $\mathcal{F} = \{T_t : t \in \mathbb{R}\}$ be the flow corresponding to the vector field $Y$. By considering at which points $Y$ is positive, negative, or zero, one sees that:

$$T_t 1 = 1, \quad T_t(-1) = -1$$

for all $t$; and

$$\lim_{t \to \infty} T_t p = -1, \quad \lim_{t \to -\infty} T_t p = 1$$

for $p$ not in $\{1, -1\}$. Conditions (I) and (III) are thus satisfied.

Define vector fields $X_1$ and $X_2$ on $C$ by

$$X_1 = (1 + x)V, \quad X_2 = (1 - x)V.$$

It is readily seen that

$$Vx = -y, \quad Vy = x, \quad [Y, V] = -xV, \quad [Y, xV] = -V, \quad [Y, X_1] = -X_1, \quad [Y, X_2] = X_2.$$
Note that $Y$ is real-analytic on $C$. Thus each $T_t$ is a fractional linear transformation leaving 1 and $-1$ fixed. Such transformations are of the form

$$p \mapsto (p - a)/(1 - ap)$$

for $a$ real and not equal to $\pm 1$. In our case, $-1 < a < 1$. The system $(C, \mathcal{F})$ is thus isomorphic to a natural action on $G/H$, where $G$ is the group of all fractional linear transformations, and $H$ is not discrete.

**Example 2.** Let $C$ denote the unit circle. There exists a simply spanning space of vector fields $\mathcal{X}$ on the two-dimensional torus $C \times C$ which is invariant under all translations, but is not a Lie algebra.

To see this, define functions $u$ and $v$ on $C \times C$ by

$$u(p, q) = \frac{1}{2}(p + \bar{p}), \quad v(p, q) = (1/2i)(p - \bar{p}).$$

Define the vector fields $U$ and $V$ on $C \times C$ by $U = \partial/\partial \alpha$ and $V = \partial/\partial \beta$, with respect to the parametrization

$$(\alpha, \beta) \mapsto (e^{i\alpha}, e^{i\beta}).$$

Now define $X_1$ and $X_2$ on $C \times C$ by

$$X_1 = uU + vV, \quad X_2 = vU - uV.$$

One can readily see that

$$[U, X_1] = -X_2, \quad [U, X_2] = X_1$$
$$[V, X_1] = 0, \quad [V, X_2] = 0$$
$$[X_1, X_2] = U.$$

Let $\mathcal{X}$ be the linear span of $\{X_1, X_2\}$. This is simply spanning. It is not a Lie algebra. Moreover, by Lemma 5, $\mathcal{X}$ is invariant under flows corresponding to vector fields of the form $\alpha U + \beta V$. But these are just the flows of translations.

**References**


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