SOME IMMERSION THEOREMS FOR PROJECTIVE SPACES

BY

A. DUANE RANDALL

1. Introduction. In this paper we obtain some results on the classical problem of immersing projective spaces into Euclidean space. Let \( a(n) \) denote the number of 1's appearing in the dyadic expansion of \( n \). We prove the following

**Theorem 1.1.** \( CP^n \) immerses in \( \mathbb{R}^{4n-5} \) for \( n \) odd and \( a(n) > 2 \).

Applying Theorem 4 of [6] with Theorem 1.1 gives

**Corollary 1.2.** \( CP^n \) has a best possible immersion in \( \mathbb{R}^{4n-5} \) for \( n = 2r + 2s + 1 \) with \( r > s > 0 \).

**Theorem 1.3.** \( RP^n \) immerses in \( \mathbb{R}^{2n+7} \) for \( n \equiv 4 \mod 8 \) and \( a(n) > 2 \).

We remark that the proof of (1.3) also shows \( RP^n \) does not immerse in \( \mathbb{R}^{2n-7} \) for \( n = 2r + 4 \) with \( r > 3 \), a result of [7].

**Theorem 1.4.** \( RP^n \) immerses in \( \mathbb{R}^{2n+9} \) for \( n \equiv 0 \mod 8 \) and \( n \) not a power of 2.

**Corollary 1.5.** \( RP^n \) immerses in \( \mathbb{R}^{2n+4a(n)-1} \) for \( n = 2r + 2s \) with \( r > s > 2 \).

It follows from [4] that \( RP^n \) does not immerse in \( \mathbb{R}^{2n-11} \) for \( n = 2r + 8 \) and \( r > 3 \).

**Theorem 1.6.** \( RP^n \) immerses in \( \mathbb{R}^{2n+8} \) for \( n \equiv 1 \mod 4 \) and \( a(n) > 3 \).

Adem and Gitler showed in [4] and [7] that \( RP^n \) has a best possible immersion in \( \mathbb{R}^{2n-4} \) for \( n \equiv 1 \mod 4 \) and \( a(n) = 3 \).

These results are interesting only for small values of \( a(n) \) due to Milgram's construction of linear immersions in [21]. The method of proof consists of expressing certain obstructions to the lifting of an appropriate map by Adams-Maunder operations and then evaluating these operations in projective space. The author wishes to express his gratitude to his advisor, Professor Emery Thomas, and to the Centro de Investigacion y de Estudios Avanzados del IPN, Mexico.

2. Preliminaries. The coefficient group for singular cohomology is understood to be \( \mathbb{Z}_2 \) when omitted. We let \( \alpha \in H^1(RP^n) \) and \( \beta \in H^2(CP^n) \) denote generators for the cohomology rings. Let \( A_k \) denote the vector subspace of the mod 2 Steenrod algebra \( A \) consisting of homogeneous elements of degree \( k \). If \( a(k+s) > a(s) \),
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A_{k}(a^{s})=0. A standard fact in number theory states that the highest power of 2 dividing a binomial coefficient \binom{r+s}{s} is 2^{a(r)+a(s)-a(r+s)}. Let \xi and \eta denote the Hopf line bundles over \(RP^\infty\) and \(CP^\infty\). The Thom complex \(T(m\xi)\) is homeomorphic to the stunted projective space \(RP^{m+r}/RP^m\) for \(m\xi\) based on \(RP^r\). \(T(r\eta)\) is homeomorphic to \(CP^{r+s}/CP^r\) for \(r\eta\) based on \(CP^s\). The Hopf map \(H: RP^\infty \to CP^\infty\) gives the real bundle equation \(H^{*}\eta = 2\xi\).

\[W(m\xi) = \sum_{s} \binom{m}{s} \alpha^s.\]


**Proposition 2.1.** \(RP^n\) immerses in \(R^{n+k}\) iff \((n+k+1)\xi\) has \(n+1\) independent nonzero sections iff \((2^{q(n)}-(n+1))\xi\) has \(2^{q(n)}-(n+k+1)\) independent nonzero sections.

### 3. Cohomology operations in projective space.

In [3] Adem and Gitler formulate an algorithm for computing a family of stable secondary cohomology operations in complex projective space. Let \(p(r, s)\) denote the following relation in \(A\) for any positive integers \(r\) and \(s\):

\[(3.1) \quad (Sq^{2}Sq^{1})Sq^{2s} + \sum_{i=0}^{r-1} Sq^{2i(s+1)+1} - 2^{i} Sq^{2i} + sSq^{1}Sq^{2(s+1)} = 0.\]

A straightforward generalization of Theorem 8.2 in [4] is the following

**Proposition 3.2** Let \(\Phi(r, s)\) denote any stable secondary operation associated to \(p(r, s)\). Let \(a = 2^t\) be such that \(a \leq 2^{r}(s+1) < 2a\). Let \(c\) be any integer such that \(c < s\) and \(a(c+s+1) > a(c)\). For \(m = ha + 2^{r-1}c\) with \(h > 0\), \(\Phi(r, s)\) is defined on \(\beta^m\) and with zero indeterminacy

\[\Phi(r, s)(\beta^m) = h\left(\frac{2^{r}c}{2^{r}(s+1)-a}\right)\beta^{m+2^{r-1}(s+1)}.\]

The proof of (3.2) is essentially given in [3] and [4] and so is omitted.

In [10] Gitler, Mahowald, and Milgram show that many secondary operations defined on the Thom class of a complex vector bundle measure the divisibility by 2 of its Chern classes. Applications of their argument yield the following results.

**Proposition 3.3.** Let \(\omega\) denote a complex bundle over a complex \(X\) such that \(c_{2t+1}(\omega) = 2x\) for \(x\) in \(H^{4t+2}(X; Z)\) and \(c_{2t+2}(\omega) = 2y\) for \(y\) in \(H^{4t+4}(X; Z)\). A secondary operation \(\varphi\) associated to the relation

\[(Sq^{2}Sq^{1})Sq^{4t+2} + Sq^{1}Sq^{4t+4} + Sq^{4t+4}Sq^{1} = 0\]

can be chosen independently of \(\omega\) so that

\[Sq^{2}(U_{\omega} \cdot x) + U_{\omega} \cdot y \in \varphi(U_{\omega}).\]
Proposition 3.4. Let $\rho$ denote a complex bundle over a complex $X$ such that $c_{4t+2}(\rho) = 2x$, $c_{4t+3}(\rho) = 2y$, $c_{4t+4}(\rho) = 2z$ for classes $x$, $y$, and $z$ in $H^*(X; \mathbb{Z})$. A secondary operation $\Gamma$ associated to the relation

$$(\Sigma^q \Sigma^1 \Sigma^q)^r + \Sigma^q \Sigma^8 + \Sigma^1 (\Sigma^8 + 6 \Sigma^q) + \Sigma^8 + 8 \Sigma^1 = 0$$

can be chosen independently of $\rho$ so that

$$U_\rho \cdot (z + y \cdot \Sigma^2(\rho)) + x \cdot \Sigma^2(\rho)) + \Sigma^q(U_\rho \cdot x) \in \Gamma(U_\rho).$$

Proposition 3.5. Let $\omega$ denote a complex bundle over a complex $X$ such that $c_4(\omega) = 0$, $c_4(\omega) = 0$, and $c_8(\omega) = 2x$ for $x$ in $H^{16t}(X; \mathbb{Z})$. Let $\Phi$ denote a secondary operation associated to the defining relation

$$(\Sigma^q \Sigma^1 \Sigma^q)^r + \Sigma^q + \Sigma^8 + \Sigma^1 (\Sigma^8 + 7 \Sigma^q) + \Sigma^1 (\Sigma^8 + 4 \Sigma^q) = 0 \quad \text{for} \ t > 0.$$

Then $\Phi$ can be chosen independently of $\omega$ so that $\Sigma^8(U_\omega \cdot x) \in \Phi(U_\omega)$.

Remark. mod 2 reduction of integral classes is understood whenever applicable in the above propositions. The proofs involve direct applications of the argument given in [10]. We give only the proof of (3.5).

Proof of 3.5. Consider the following diagram.

$$\begin{array}{ccc}
F & \xrightarrow{j} & E(2n) \\
\downarrow{g} & & \downarrow{p} \\
\Sigma^T(\omega) & \xrightarrow{T(f)} & MSU(n) \\
\downarrow{U_n} & & \downarrow{K(Z, 2n)} \\
& & \\
\end{array}$$

Here $p$ is the principal fibration induced from the universal example for the operation $\Phi$ on integral classes of dimension $2n$ for large $n$ by the Thom class $U_n: MSU(n) \to K(Z, 2n)$. Now $p^*[U_n; e_8] = 2e_1$ for some $e_1$ in $H^*(E(2n); \mathbb{Z})$ since $Sq^{16t}U_n = U_n \cdot w_{16t}$. One notes that $j^*e_1 \mod 2 = Sq^1t_1 \otimes 1 \otimes 1 \otimes 1$ where

$$F = K(Z_2, 2n + 16t - 1) \times K(Z_2, 2n + 16t + 7) \times K(Z_2, 2n + 1) \times K(Z_2, 2n)$$

is the fiber of $p$. (See [10].) Similarly,

$$p^*[U_n; (e_2e_{8t+2} + e_6e_{8t+4} + e_8^2e_{8t})] = 2e_2$$

for some integral class $e_2$ and $j^*e_2 \mod 2 = 1 \otimes Sq^1t_3 \otimes 1 \otimes 1$. Let $e_3$ and $e_4$ be classes in $H^*(E(2n))$ such that $j^*e_3 = 1 \otimes 1 \otimes t_3 \otimes 1$ and $j^*e_4 = 1 \otimes 1 \otimes 1 \otimes t_4$. We now choose $\Phi$ so that $\Phi$ vanishes on classes of dimension $\leq 16t - 2$. This is possible by [2] and [14]. It follows that

$$Sq^8e_1 + e_2 + Sq^{16t+7}e_3 + Sq^{16t+8}e_4$$

is the representative for this choice of $\Phi$ in $H^*(E(2n))$. Let $f: X \to BSU(n)$
classify the bundle \( \omega \oplus s \) where \( n - s \) is the fiber dimension of \( \omega \). \( T(f) \) is the natural map induced by \( f \) between the Thom complexes. Thus,
\[
\Sigma^2 \Phi(\Sigma^2 U_\omega) = \Phi(\Sigma U_\omega)
\]
\[
= \bigcup g^* (S^g e_1 + e_2 + S^{10}e_3 + S^{10}e_4)
\]
\[
= \bigcup g^* (S^g e_1 + e_2)
\]
where \( g \) ranges over all liftings of \( T(f) \). Since the Chern classes \( c_2(\omega) \), \( c_8(\omega) \), \( c_{8+1}(\omega) \), and \( c_{8+2}(\omega) \) are divisible by 2, it follows that \( S^g(U_\omega \cdot x) \in \Phi(U_\omega) \).

The proof of Theorem 1.4 uses a tertiary cohomology operation which we define here and evaluate in real projective space. Consider the following relations and associated secondary operations for \( s > 3 \):
\[
\Phi_1: (S^2S^q)^2 + S^{2q+2}S^q = 0,
\]
\[
\Phi_2: (S^4S^q)^2 + S^{2q+4}S^q + S^{2q+3}S^q = 0,
\]
\[
\Phi_3: S^2S^q + S^{q+1}S^q = 0,
\]
\[
\Phi_4: S^qS^q = 0.
\]
Let \( \Phi \) denote the 4-valued secondary operation \( (\Phi_1, \Phi_2, \Phi_3, \Phi_4) \).

**Proposition 3.6.** \( \Phi_1 \) and \( \Phi_2 \) can be chosen so \( (\Phi_1, \Phi_2) \) vanishes on classes having dimension \( < 2^t \). For these choices the following relation holds stably and with zero indeterminacy among the component operations \( \Phi_i \) of \( \Phi \).
\[
S^q \Phi_1 + S^q \Phi_2 + S^{2q+5} \Phi_3 + S^{2q+7} \Phi_4 + (\lambda S^{2q+4}) S^q = 0
\]
where \( \lambda \) is in \( Z_2 \).

**Proof.** The functional cohomology operations associated with the defining relations for \( \Phi_1 \) and \( \Phi_2 \) vanish on classes having dimension \( < 2^t \) by [2, Teorema 6.6]. Now \( \Phi_1 \) and \( \Phi_2 \) can be chosen trivial on classes in the domain of \( \Phi \) having dimension \( < 2^t \) by the Peterson-Stein formula [2, Teorema 5.2]. Consider the universal example for the operation \( \Phi \) on classes of dimension \( n \) for large \( n \).

The map \( p \) is the principal fibration with classifying map \( S^q \times S^q \times S^q \) and \( C \) is a product of Eilenberg-MacLane spaces. Let \( k_i^1 \) in \( H^q(E(n)) \) be the representative class for \( \Phi_i \) for \( 1 \leq i \leq 4 \). Let an arbitrary class \( x \) in \( H^q(X) \) in the domain of \( \Phi \) be
classified by a map \( f: X \to K(Z_2, n) \). By definition \( \Phi(x) = \bigcup_p g^*(k_1^p, k_2^p, k_3^p, k_4^p) \) where the union ranges over all liftings of \( f \). The Serre exact sequence applied to the map \( p \) gives

\[
Sq^*k_1 + Sq^*k_2 + Sq^{2s+5}k_3 + Sq^{2s+7}k_4 = \lambda \theta(p^*) \quad (\lambda \in Z_2)
\]

where \( \theta \) is a sum of admissible monomials in \( A \) each having degree \( 2s+8 \) and excess \( \geq 2s \). The Adem relations applied to \( Sq^8Sq^{2s} \) show that \( Sq^{2s+8}(p^*) = Sq^{2s+4}Sq^4(p^*) \) so \( \theta = Sq^{2s+4}Sq^4 \).

Let \( \psi \) be any stable tertiary operation associated to the relation given by (3.7). The indeterminacy subgroup \( \text{Indet}^\psi(X; \psi) \) arises in the following manner. The operation \( \psi \) determines a secondary operation \( \text{Indet}^\psi(X; \psi) \) of three variables. (See [20] and [28].) \( \text{Indet}^\psi(X; \psi) \) is defined on those classes \( x \in H^n(X), y \in H^{n+1}(X), \) and \( z \in H^{n+2s-1}(X) \) for which

\[
\begin{align*}
Sq^1x &= 0, \quad Sq^2y + Sq^3x = 0, \\
Sq^4Sq^1z + Sq^{2s+3}y + Sq^{2s+4}x &= 0, \\
Sq^2Sq^1z + Sq^{2s+2}x &= 0.
\end{align*}
\]

Then \( \text{Indet}^\psi(X; \psi) = \text{image} \text{Indet}^\psi(X; \psi) + \lambda Sq^{2s+4}H^{n+3}(X) \) in \( H^{n+2s+7}(X) \).

**Proposition 3.8.** \( \psi \) is defined on \( \alpha^{2s+1+8} \) in \( H^*(RP^\infty) \) and vanishes with zero indeterminacy.

**Proof.** Clearly \( \Phi_2(\beta^{2s+1+8}) = 0 \) and \( \Phi_4(\beta^{2s+1+8}) = 0 \) in \( H^*(CP^\infty) \). Let \( g: CP^\infty \to QP^\infty \) be a map such that \( g^*\gamma = \beta \) where \( \gamma \) generates \( H^*(QP^\infty) \). Now \( \Phi_1(\gamma_{2s+1+2}) = 0 \) for dimensional reasons so \( \Phi_1(\beta^{2s+4}) = 0 \) from naturality and zero indeterminacy. By Proposition 3.2

\[
\Phi_2(\beta^{2s+4}) = \Phi_2(2, 2s-2)(\beta^{2s+4}) = \binom{8}{4} \beta^{2s+2s-1+8} = 0.
\]

Thus \( \psi \) is defined on \( \beta^{2s+4} \) and so on \( \alpha^{2s+1+8} \). Clearly \( \psi \) vanishes on \( \beta^{2s+4} \) with zero indeterminacy so naturality under the Hopf map gives \( 0 \in \Phi(\alpha^{2s+1+8}) \).

One checks that \( \text{Indet}^\psi \) is defined with zero indeterminacy on

\[
(3.9) \quad H^{2s+1+8}(RP^\infty) \oplus H^{2s+1+9}(RP^\infty) \oplus H^{2s+1+2s+7}(RP^\infty).
\]

Let

\[
\begin{align*}
\Gamma_1 &: Sq^6Sq^{2s+4} + Sq^6Sq^{2s+2} + Sq^{2s+5}Sq^3 + Sq^{2s+7}Sq^1 = 0, \\
\Gamma_2 &: Sq^6Sq^{2s+3} + Sq^{2s+5}Sq^2 = 0, \\
\Gamma_3 &: Sq^6(Sq^4Sq^1) + Sq^6(Sq^2Sq^1) = 0
\end{align*}
\]

denote any stable secondary operations associated to the above relations. Clearly
\[ \Gamma_1(\beta^{2s+4}) = 0 \] so it follows that \( \Gamma_1(\alpha^{2s+1+b}) = 0 \) from naturality and zero indeterminacy. By [4, Theorem 5.1] there exists an \( S \)-map \( \lambda \) such that
\[ \lambda^* : H^{2d}(\sum^{2d+1} \mathbb{R}P^{2p+1}) \to H^{2d}(\mathbb{C}P^{d+1} / \mathbb{C}P^d) \]
is an isomorphism for all \( q \) where \( p = 2s + 2s-1 + 7 \) and \( d+1 = t2s+1 \) for some positive integer \( t \). By dimensionality \( \Gamma_2(\gamma^{2s+2s-1+2}) = 0 \) in \( H^*(\mathbb{Q}P^\infty) \) so naturality gives
\[ \lambda^* \Gamma_2(\sum^{2d+1} \alpha^{2s+1+b}) = \Gamma_2(\beta^{2s+b+5}) = 0. \]
The stability of \( \Gamma_2 \) and zero indeterminacy imply that \( \Gamma_2(\alpha^{2s+1+b}) = 0 \). Similarly,
\[ \lambda^* \Gamma_3(\sum^{2d+1} \alpha^{2s+1+b+7}) = \Gamma_3(\beta^{2s+1+b+7}) = \varphi \circ Sq^1(\beta^{2s+1+b+7}) = 0 \]
where \( \varphi \) is a secondary operation associated to the relation
\[ Sq^6Sq^2 + Sq^4Sq^4 + Sq^7Sq^1 = 0. \]
Such a \( \varphi \) can be chosen so that \( \Gamma_3 = \varphi \circ Sq^1 \) modulo total indeterminacies by [1]. Thus \( \Gamma_3(\alpha^{2s+1+2s+7}) = 0 \) from stability and zero indeterminacy. We conclude that \( \text{Ind} \) vanishes with zero indeterminacy on (3.9). Since \( Sq^{2s+1+2s+11} = 0 \), \( \text{Ind} \) is complete.

4. Generating class theorems. Thomas formulates a “generating class theorem” for lifting a second-order \( \kappa \)-invariant to the Thom complex and expressing it by means of a secondary operation applied to the Thom class in [Theorem 6.4, 29] and [Theorem 6.5, 30]. The proof of Theorem 1.4 in §5 uses the generating class theorem to express a third-order \( \kappa \)-invariant by a tertiary operation so we state the following versions to cover that application. Let \( B_m \) and \( B \) denote \( BSO(m) \) and \( BSO \), \( B \text{Spin} (m) \) and \( B \text{Spin} \), or \( BO[8](m) \) and \( BO[8] \) where \( BSO \), \( B \text{Spin} \), and \( BO[8] \) are the 1, 3, and 7-connective coverings of \( BO \). In the appendix Postnikov resolutions are constructed for the fiber map \( \pi : B_m \to B \) through dimensions \( \leq t \) where \( \pi^* \) is surjective and \( m < t < 2m \). Let \( T \) and \( U \) denote the Thom complex and Thom class of the universal bundle \( \xi \) over \( B \) and regard \( B \) as \( B_s \) for large \( s \). Following the notation of [28], [29], [30], we let \( T_y \) and \( U_y \) denote the Thom complex and Thom class of \( g^* \xi \) where \( g \) : \( Y \to B \) is any map. Consider the following commutative diagram.

\[ \Omega G \xrightarrow{j} E_2 \]
\[ K(Z_2, m) \xrightarrow{i} E_1 \xrightarrow{k_1 \times k_2 \times \cdots \times k_r} G \]
\[ B_m \xrightarrow{\pi} B_s \xrightarrow{w_m+1} K(Z_2, m+1) \]

The classes \( k_j \in H^*(E_1) \) for \( t_i \leq t \) and \( 1 \leq j \leq r \) are the second-order \( k \)-invariants in...
the resolution for \( \pi \). Now \( i^*k_1 = \alpha_1 \) and \( i^*k_2 = \beta_1 \) for elements \( \alpha_1, \beta_1 \) in \( A \). Suppose there are relations in \( A \)

\[
\begin{align*}
\alpha_1 \text{Sq}^{m+1} + \sum_{j=2}^{n} \alpha_j \theta_j &= 0, \\

\beta_1 \text{Sq}^{m+1} + \sum_{j=2}^{n} \beta_j \theta_j &= 0
\end{align*}
\]

(4.2)

where \( \theta_j U = 0 \) and degree \( (\alpha_j) > 0 \), degree \( (\beta_j) > 0 \) for \( 1 < j \leq n \). Let \( \Omega \) denote any secondary operation associated to the relations (4.2) and let \( C \) represent the coset in \( H^{s+t_1}(T_B) \oplus H^{s+t_2}(T_B) \) of the indeterminacy subgroup of \( \Omega \) such that \( (T\pi)^*C = \sum \Omega(U_{\beta_n}) \). Define \( K \) to be the coset of the pair \( (k_1, k_2) \) in \( H^i(E_1) \oplus H^i(E_1) \) with respect to the subgroup

\[
(\text{kernel } i^* \cap \text{kernel } q^*_T \cap H^i(E_1)) \oplus (\text{kernel } i^* \cap \text{kernel } q^*_T \cap H^i(E_1)).
\]

With these assumptions the generating class theorem states

**Proposition 4.3.** There is a class \( (k_1, k_2) \in K \) and a class \( (c_1, c_2) \in H^i(B) \oplus H^i(B) \) such that \( U^* (c_1, c_2) \in C \) and \( U_{E_1}^* ((k_1, k_2) + p_T^*(c_1, c_2)) \in \Omega(U_{E_1}) \).

The proof of (4.3) is essentially given in [27] and [29] and so is omitted. Let \( X \) be a complex of dimension \( \leq t \) and \( f: X \to B \) a map with \( f^*w_{m+1} = 0 \). Thus \( f \) classifies a bundle \( \rho \) over \( X \) and one defines \( (k_1, k_2)(\rho) = \bigcup_p (g^*k_1, g^*k_2) \), the union being over all liftings \( g: X \to E_1 \) of \( f \). Recall from [28] there are classes \( \hat{a}_1 \) and \( \hat{b}_1 \) in \( A(B) \) such that \( \mu(k_1) = \hat{a}_1 (\iota \otimes 1) \) in \( H^i(K(Z_2, m) \times E_1, E_1) \), \( \mu(k_2) = \hat{b}_1 (\iota \otimes 1) \) in \( H^i(K(Z_2, m) \times E_1, E_1) \). Thus \( (k_1, k_2)(\rho) \) is a coset of \( \text{Indet}^{i, t_1}(X; K) = (\hat{a}_1, \hat{b}_1) \cdot H^* (X) \cap (H^*(X) \oplus H^*(X)) \).

**Corollary 4.4.** Suppose the indeterminacy of \( \Omega(U_{\rho}) = U_{\rho} \cdot \text{Indet}^{i, t_1}(X; K) \). Suppose also that \( g^*(k_1, k_2) = g^*(k_1, k_2) \) for any \( (k_1, k_2) \) in \( K \) and any lifting \( g: X \to E_1 \) of \( f \). Then

\[
U_{\rho} \cdot ((k_1, k_2)(\rho) + f^*(c_1, c_2)) = \Omega(U_{\rho})
\]

as cosets in \( H^{s+t_1}(T_X) \oplus H^{s+t_2}(T_X) \).

In diagram (4.1) the map \( p_2: E_2 \to E_1 \) is a principal fibration classified by the cohomology vector \( (k_1, \ldots, k_r) \) where \( C = \times_{i=1}^{r} K(Z_2, t_i) \). Assume now that \( K \)

consists only of \( (k_1, k_2) \) and that \( \Omega \) can be chosen so \( U_{E_1} \cdot (k_1, k_2) \in \Omega(U_{E_1}) \). Let \( \iota_j \) denote the fundamental class of \( \Omega K(Z_2, t_j) \) in \( \Omega C \) for \( 1 \leq j \leq r \). Let \( k \in H^i(E_2) \) be a third-order \( k \)-invariant for \( \pi \) (so \( r \leq t \)) such that \( j^* k = \gamma_1 t_1 + \gamma_2 t_2 \) where \( \gamma_1 \) and \( \gamma_2 \) are in \( A \). Suppose \( \Omega' = (\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5) \) is a 5-valued stable secondary operation where \( \Omega_1 \) and \( \Omega_2 \) are the component operations belonging to \( \Omega \) and the
degree of \( \Omega_i \leq \) the connectivity of \( B \) for \( 3 \leq i \leq 5 \). Thus \( (k_1, k_2, 0, 0, 0) \in \Omega(U_{E_3}) \). Assume also the following relation holds.

\[(4.5) \quad \gamma_1 \Omega_1 + \gamma_2 \Omega_2 + \gamma_3 \Omega_3 + \gamma_4 \Omega_4 + \gamma_5 \Omega_5 = 0.\]

Let \( \psi \) be a tertiary operation associated to relation (4.5). Let \( D \) denote the coset of the indeterminacy subgroup of \( \psi \) in \( H^*(T_B) \) such that \( \Sigma^{k_1} \psi(U_{B_{m}}) = (Tm)^*D \) and let \( K' \) be the coset of \( k \) with respect to the subgroup kernel \( j^* \cap \text{kernel } q_2^* \cap H'(E_2) \).

Under these assumptions the generating class theorem states

**Proposition 4.6.** There is a class \( \bar{k} \) in \( K' \) and a class \( d \) in \( H(B) \) such that

\[U \cdot d \in D \text{ and } U_{E_2} \cdot (\bar{k} + (p_1 \circ p_2)^*d) \in \psi(U_{E_2}).\]

Proposition 4.6 is easily proved by applying the arguments of [27], [29], and [30]. See also [9]. An application of (4.6) is also given in [23].

5. Proofs of immersion theorems.

**Proof of Theorem 1.1.** Write \( 2n = 4t + 6 \) and refer to Postnikov resolution I in the appendix. Let \( v: CP^n \to B \text{ Spin} \) classify the stable normal bundle \( v \) of \( CP^n \). Now

\[\bar{w}_2(CP^n) = \binom{n+j}{j} \beta^j\]

so \( w_{4t+2}(v) = 0 \) and \( w_{4t+4}(v) = 0 \). The indeterminacy of \( k_4(v) = Sq^2 H^{4t+2}(CP^n) = H^{2n-2}(CP^n) \) so \( v \) lifts to \( B \text{ Spin} (4t+1) \) iff \( v \) lifts to \( E_2 \) iff \( k_2(v) = 0 \). Let \( \Omega \) denote a stable secondary operation associated to the relation

\[(Sq^2 Sq^4) Sq^{4t+2} + Sq^1 Sq^{4t+4} + Sq^{4t+4} Sq^1 = 0\]

and chosen so that it vanishes on classes of dimension \( <4t+2 \) (see [2]). Applying the generating class theorem [30, Theorem 6.5] gives \( U_v \cdot k_2(v) = \Omega(U_v) \) since the indeterminacy of \( k_2(v) = 0 \) is the indeterminacy of \( \Omega(U_v) \). Here \( U_v \) is the Thom class of the Thom complex \( T(v) \).

The order of \( J(v) \) in \( J(CP^n) \) is the Atiyah-Todd number \( M_{n+1} \) by [31]. Set \( s = M_{n+1} - (n+1) \). Since \( 2^s \) divides \( M_{n+1} \), it follows that

\[\binom{s}{2r} = 1 \text{ iff } \binom{n}{2r} = 0 \text{ for } 0 \leq r < n.\]

Write \( s = ha + c \) where \( c < n \) and \( a \) is the smallest power of 2 greater than \( n \). Atiyah-James duality for projective spaces in [5] states that an \( S \)-dual for \( CP^n \) is the space \( X = CP^m/CP^{m-1} = T(s\eta) \) for \( s\eta \) based on \( CP^m = CP^{m-s} \) where \( m = s + n - 1 \). Identify the generator of \( H^{2a}(X) \) with \( \beta^a \) under the collapsing map \( CP^m \to X \) and the standard embedding \( CP^m \to CP^n \). Since \( a(c + n - 1) > a(c) \), Proposition 3.2 states that

\[\Omega(\beta^a) = \Phi(1, 2t+1)(\beta^{ha+c}) = \binom{2c}{4t+4-a} \beta^{2t+n-1}.\]
But \((\frac{2^n}{a^2})=1\) so \((\frac{2^n}{a^2-a})=0\) for \(n(a)>2\). Thus \(k_a(v)=0\) and the result follows by Hirsch [12]. Note for \(n=2\) that

\[
\begin{pmatrix}
2c \\
4t + 4 - a
\end{pmatrix} = \begin{pmatrix}
2 \\
0
\end{pmatrix} = 1
\]

which gives a nonimmersion result of [3].

**Proof of Theorem 1.3.** Refer to Postnikov resolution II in the appendix. Write \(n=8t+12\) and let \(\gamma: \mathbb{R}P^n \to BSO\) classify the bundle \(\gamma=(16t+18)\xi\) over \(\mathbb{R}P^n\). It suffices to show \(\gamma\) lifts to \(BSO(8t+5)\) by Proposition 2.1. Note that \(w_{8t+6}(\gamma)=0\) and \(w_{8t+8}(\gamma)=0\) from §2 so \(\gamma\) lifts to \(E_3\). The indeterminacy of \(k_2^2(\gamma)=H^n(\mathbb{R}P^n)\) so \(\gamma\) lifts to \(BSO(8t+5)\) iff \(\gamma\) lifts to \(E_3\). \(Sq^1k_2^2\) occurs in the defining relation for \(k_2^2\) so \(0 \in k_2^2(\gamma)\). Likewise, \(Sq^1k_4^1\) occurs in the defining relation for \(k_2^2\) so \(0 \in k_2^2(\gamma)\). Thus any lifting of \(\gamma\) to \(E_2\) can be altered through indeterminacies to produce a lifting of \(\gamma\) to \(E_3\).

We apply the technique of factoring a classifying map for an even multiple of the Hopf bundle over \(\mathbb{R}P^n\) through complex projective space in order to determine the second-order \(k\)-invariants for \(\gamma\). Set \(m=4t+6\) and let \(v: \mathbb{C}P^m \to BSO\) classify the bundle \(v=(8t+9)\eta\). We regard \(\gamma=v \circ H: \mathbb{R}P^n \to BSO\) where \(H\) is the Hopf map in §2. Trivially \(k_2^1(v)=0\) and \(k_3^3(v)=0\). Note that both \(k_2^3(v)\) and \(k_1^4(v)\) have zero indeterminacy. Choose a stable secondary operation \(\varphi\) associated to the relation

\[
(Sq^2Sq^1)Sq^{8t+4} + Sq^1Sq^{8t+8} + Sq^{8t+8}Sq^1 = 0
\]

do that \(\varphi\) vanishes on classes of dimension \(\leq 8t+5\). The generating class theorem [29, Theorem 6.4] gives \(\varphi(U_v)=U_v \cdot k_2^3(v)\). But \(\varphi(U_v)=2^{n+1}U_v \cdot \beta^{8t+4}\) by Proposition 3.3. Thus \(k_2^3(v)\neq 0\) iff \(a(t+1)=1\) iff \(n=2t+4\) for \(r \geq 4\). Let \(g: \mathbb{C}P^m \to E_1\) be any lifting for \(v\) and set \(f=g \circ H\). Now \(f^*(k_2^1, k_2^2, k_3^3)=(0, a_2^2, 0)\) for \(n=2t+4\). The indeterminacy subgroup of \((k_2^1, k_2^2, k_3^3)(\gamma)\) is generated by \((a_2^{r+1}, 0, a_2^{r+1})\) and \((0, a_2^r, a_2^{r+1})\). So \(f\) cannot be altered to produce a lifting of \(\gamma\) to \(E_2\). That is, \(\mathbb{R}P^n \notin R_{2an-7}^n\) for \(n=2r+4\) and \(r \geq 3\).

Let \(h: \mathbb{C}P^{m+1} \to E_1\) be any lifting for the bundle \(v\) based on \(\mathbb{C}P^{m+1}\). The defining relation for \(k_2^3\) gives \(\beta^3 \cdot h^*k_1^1 + \beta \cdot h^*k_1^1 = 0\) in \(H^{2m+2}(\mathbb{C}P^{m+1})\). So \(h^*k_2^3=0\) iff \(h^*k_4^1=0\). It follows that \(g^*k_2^3=0\) iff \(g^*k_4^1=0\) for any lifting \(g: \mathbb{C}P^m \to E_1\) of \(v\). Thus \(f=g \circ H: \mathbb{R}P^n \to E_1\) lifts to \(E_2\) for \(n \neq 2t+4\).

**Proof of Theorem 1.4.** We consider the case \(n=8\) mod 16. The proof for \(n=0\) mod 16 is similar and so is omitted. Write \(n=16t+8\) and refer to Postnikov resolution III. Let \(v: \mathbb{C}P^n \to B Spin\) classify the bundle \(v=(16t+4)\eta\) for \(m=8t+4\). Let \(\gamma=v \circ H: \mathbb{R}P^n \to B Spin\) classify the bundle \(\gamma=(32t+8)\xi\) over \(\mathbb{R}P^n\). One checks easily that \(\gamma\) lifts to \(B Spin(16t+1)\) if \(v\) lifts to \(E_2\). Now

\[
w_{16t}(v) = \begin{pmatrix}
16t+4 \\
8t
\end{pmatrix} \beta^{8t} = 0
\]

so \(v\) lifts to \(E_1\). The defining relation for \(k_1^2\) gives \(Sq^2k_1^2(v)=0\) so \(k_1^2(v)=0\). The
defining relation for \( k_3^2 \) gives \( Sq^4k_3^2(v)=0 \). But \( Sq^4\beta^{8t+6} = \beta^{8t+8} \) so that \( k_3^2(v)=0 \).

It follows that \( v \) lifts to \( E_2 \) iff \( k_4(v)=0 \). We proceed to express by a secondary operation a class different from \( k_4^1 \) but equal to \( k_4^1 \) under pull-backs to \( CP^m \).

Consider the following commutative diagram.

\[
\begin{array}{ccc}
\Omega C & \xrightarrow{j} & E \\
\downarrow g & & \downarrow r \\
CP^m & \xrightarrow{v} & B \text{Spin}
\end{array}
\]

Here \( p: E \to B \text{Spin} \) is the principal fibration with classifying map

\[
w_{16t} \times W: B \text{Spin} \to C = K(Z_2, 16t) \times K(Z_2, 16t+8)
\]

where \( W \) represents the class \( w_4w_{16t+4} + w_6w_{16t+2} + w_8w_{16t} \) in \( H^*(B \text{Spin}) \). Let \( i_1 \) and \( i_2 \) denote the fundamental classes of the components of \( \Omega C \). The following exact sequence holds for \( j \leq 16t+15 \) from [26].

\[
0 \to H^j(E) \xrightarrow{\nu^*} H^j(\Omega C \times B \text{Spin}(16t-1)) \xrightarrow{\tau_1} H^j+1(B \text{Spin}).
\]

Regard \( k_4^1 \) as a class in \( H^*(E) \) via \( r^* \) and recall that

\[
\nu^*k_4^1 = Sq^8Sq^1i_1 \otimes 1 \otimes 1 + Sq^1i_1 \otimes 1 \otimes w_8 + Sq^2i_1 \otimes 1 \otimes w_4
\]

\[
+ Sq^1i_1 \otimes 1 \otimes w_6^2 + Sq^2i_1 \otimes 1 \otimes w_6 + Sq^2i_1 \otimes 1 \otimes w_7.
\]

Now

\[
\tau_1(1 \otimes Sq^1i_2 \otimes 1) = Sq^1W = \tau_1[Sq^1(Sq^4i_1 \otimes 1 \otimes w_4 + Sq^2i_1 \otimes 1 \otimes w_8)].
\]

Let \( z \) be the unique class in \( H^{16t+8}(E) \) for which

\[
\nu^*z = Sq^8Sq^1i_1 \otimes 1 \otimes 1 + Sq^1i_2 \otimes 1 \otimes w_8 + 1 \otimes Sq^1i_2 \otimes 1 + Sq^1i_1 \otimes 1 \otimes w_6.
\]

Let \( y \) be the unique class in \( H^{16t+7}(E) \) for which

\[
\nu^*y = 1 \otimes i_2 \otimes 1 + Sq^2i_1 \otimes 1 \otimes w_6 + Sq^4i_1 \otimes 1 \otimes w_4.
\]

Since \( \nu^*(Sq^1y) = \nu^*(z+k_4^1) \), it follows that \( k_4^1 = z + Sq^1y \). Choose a stable secondary operation \( \Phi \) associated to the relation \((Sq^8Sq^1)Sq^{16t} + Sq^{16t+8}Sq^1 + Sq^{16t+7}Sq^2 + Sq^1(Sq^{16t+4}Sq^4) = 0 \) such that \( \Phi \) vanishes on classes having dimension \( < 16t \).

Note that \( Sq^{16t+4}Sq^4U = U \cdot W \) in \( H^*(T_{B \text{Spin}}) \). By [29, Theorem 6.4] \( U_\nu \cdot (z+k') \in \Phi(U_\delta) \) for some class \( k' \) in \( H^{16t+8}(E) \cap \text{kernel} j^* \cap \text{kernel} q^* \). It follows that \( \Phi(U_\nu) = U_\nu \cdot z = U_\nu \cdot k_4^1(v) \). Let \( \delta \) denote the generator for \( H^*(QP^\infty) \) and \( p \) the Hopf line bundle over \( QP^\infty \). We regard \( \delta^{8t+2} \) as the Thom class of the bundle.
\( \zeta = (8t + 2)\rho \) based on \( QP^l \) for large \( l \). The highest power of 2 dividing the Chern class \( c_{8l}(\zeta) \) is \( 2^{\alpha(t)} \) from §2. By Proposition 3.5

\[ \Phi(U) = 2^{\alpha(t) - 1}Sq^8(U, \delta^t). \]

But \( Sq^8 \delta^{12t + 2} = \delta^{12t + 4} \) so \( \Phi(\delta^{8t + 2}) = 0 \) if \( \alpha(t) > 1 \) if \( n \neq 2^r + 8 \). Naturality under the Hopf map \( CP^\infty \to QP^\infty \) shows that \( \Phi(\delta^{16t + 4}) = 0 \) if \( n \neq 2^r + 8 \). Thus \( k_1^2(v) = 0 \) for \( n \neq 2^r + 8 \) from identifying \( \upsilon \) with \( \beta^{16t + 4} \). Since \( \gamma \) has a lifting \( f \circ H: RP^n \to E_2 \) where \( f: CP^m \to E_2 \) is a lifting for \( v \), clearly \( k_2^2(\gamma) = 0 \) for \( n \neq 2^r + 8 \). One checks indeterminacies and defining relations to show that \( \gamma \) lifts to \( BSpin(16t - 1) \) iff \( \gamma \) has a lifting to \( E_2 \) and \( k_2^2(\gamma) = 0 \). Thus \( \gamma \) lifts to \( BSpin(16t - 1) \) and the result follows from (2.1) for \( n \neq 2^r + 8 \). For \( n = 2^r + 8 \) we express the obstruction \( k_2^2(\gamma) \) by a tertiary operation.

We assume now that \( n = 2^r + 8 \) for \( r > 3 \). The natural map \( BO[8] \to B Spin \) induces a Postnikov resolution for the fiber map \( \pi': BO[8](2r - 1) \to BO[8] \) from Postnikov resolution III for the map \( \pi \). We denote the \( k \)-invariants for \( \pi' \) also by \( k_i^j \) and the spaces in the resolution by \( E_i \). Thus \( k_2^2 \) in \( H^*(E_1) \) has the defining relation \( Sq^4Sq^1w_2 = 0 \) in \( H^*(BO[8]) \) and \( k_2^2 \) has the defining relation \( Sq^6k_1^1 + Sq^4k_2^1 = 0 \) in \( H^*(E_1) \). Since \( Sq^2k_1^1 = 0 \) and \( BO[8] \) is 7-connected, the coset \( K \) of \( (k_1^1, k_2^1) \) defined in §4 contains only \( (k_1^1, k_2^1) \). Let \( \Phi = (\Phi_1, \Phi_2) \) be the double secondary operation with component operations \( \Phi_1 \) chosen in (3.6). By Proposition 4.3

\[ U_{E_1} \cdot (k_1^1, k_2^2) \in \Phi(U_{E_1}). \]

Let \( \psi \) be any tertiary operation associated to the relation (3.7).

By Proposition 4.6

\[ U_{E_2} \cdot (k_3^2 + (p_1 \circ p_2)P) \in \psi(U_{E_2}). \]

Here \( k_3^2 \) belongs to the coset \( K' \) in (4.6) determined by \( k_3^2 \), and \( P \) is a class in \( H^{2r + 7}(BO[8]) \) such that \( U \cdot P \in \psi(U') \) where \( U' \) denotes the Thom class of the universal bundle over \( BO[8](2r - 1) \).

Let \( h: RP^n \to E_2 \) be any lifting for the map \( \gamma: RP^n \to BO[8] \) classifying the bundle \( \gamma = (2^r + 1 + 8)z \). Now \( h^*k_3^2 = h^*k_3^2 \) since \( k_3^1(\gamma) = 0 \) and \( Sq^2Sq^1h^*k_3^2 = Sq^2h^*k_3^2 = 0 \). Clearly \( P(\gamma) = 0 \) so we conclude \( U \cdot k_3^2(\gamma) \in \psi(U_1) \).

Identify \( U \) with \( \alpha^{2r + 1 + 8} \) in \( H^*(RP^n) \) and apply Proposition 3.8 to give \( k_3^2(\gamma) = 0 \). Thus \( \gamma \) lifts to \( BO[8](2r - 1) \) and the result follows by (2.1).

**Proof of Theorem 1.6.** Let \( \gamma: RP^n \to BSO \) classify the bundle \( \gamma = 2p_1 \xi = (2^{\alpha(n)} - (n + 1))\xi \). The argument that \( \gamma \) lifts to \( BSO(n - 8) \) for \( n \equiv 5 \) mod 8 and \( \alpha(n) > 3 \) is similar to the proof of Theorem 1.3 and so is omitted. We consider the case \( n \equiv 1 \) mod 8 and \( \alpha(n) > 3 \). Write \( n = 8t + 9 \) and refer to Postnikov resolution IV. By (2.1) it suffices to show \( \gamma \) lifts to \( BSO(8t + 1) \). Let \( v: CP^n \to BSO \) classify the bundle \( v = p_1 \) where \( m = 4t + 4 \). One checks easily that \( \gamma \) lifts to \( BSO(8t + 1) \) if \( k_3^2(\gamma) = 0 \). Clearly \( k_3^2(\gamma) = 0 \) if \( v \) lifts to \( E_2 \). Note that \( v \) and hence \( \gamma \) lift to \( E_1 \) by §2.
Let $h: CP^{m+1} \to E_1$ be a lifting for the bundle $\nu$ based on $CP^{m+1}$. The defining relation for $k_2^2$ gives $\beta \cdot Sq^4(h^*k_2^2) = 0$ in $H^{2m+2}(CP^{m+1})$. But $Sq^4\beta^{4t+2} = \beta^{4t+4}$ so $h^*k_2^2 = 0$. Thus $k_2^2(\nu) = 0$ and $\nu$ lifts to $E_1$ iff $k_2^2(\nu) = 0$. Consider the following commutative diagram.

Here $p: E \to BSO$ is the principal fibration with classifying map

$$w_{8t+2} \times w_{8t+4} \times w_{8t+8} \times W: BSO \to C$$

where $W$ represents the class $w_2w_{8t+6} + w_3w_{8t+5} + w_4^2 w_{8t+4}$ in $H^*(BSO)$. The following exact sequence from [26] holds for $j \leq 8t+15$

$$0 \to H^j(E) \to H^j(\Omega C \times BSO(8t+1)) \to H^{j+1}(BSO).$$

Let $\iota_j$ for $1 \leq j \leq 4$ denote the fundamental classes of the components of $\Omega C$. Now

$$v^*k_3^2 = 1 \otimes 1 \otimes Sq^1\iota_3 \otimes 1 \otimes 1$$

$$+ 1 \otimes Sq^4 Sq^1\iota_2 \otimes 1 \otimes 1 \otimes 1 + 1 \otimes Sq^3 \iota_2 \otimes 1 \otimes 1 \otimes w_4$$

$$+ 1 \otimes Sq^2 Sq^1\iota_2 \otimes 1 \otimes 1 \otimes w_2 + Sq^1(1 \otimes Sq^2 \iota_2 \otimes 1 \otimes 1 \otimes w_2).$$

Let $y$ be the unique class in $H^{8t+7}(E)$ such that

$$v^*y = 1 \otimes 1 \otimes 1 \otimes \iota_4 \otimes 1 + 1 \otimes Sq^2 \iota_2 \otimes 1 \otimes 1 \otimes w_2$$

$$+ 1 \otimes Sq^1 \iota_2 \otimes 1 \otimes 1 \otimes w_3.$$ 

Define $z$ in $H^*(E)$ so that

$$v^*z = 1 \otimes 1 \otimes Sq^1 \iota_3 \otimes 1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes Sq^1 \iota_4 \otimes 1$$

$$+ 1 \otimes Sq^4 Sq^1 \iota_2 \otimes 1 \otimes 1 \otimes 1 + 1 \otimes Sq^3 \iota_2 \otimes 1 \otimes 1 \otimes w_4$$

$$+ 1 \otimes Sq^2 Sq^1 \iota_2 \otimes 1 \otimes 1 \otimes w_2.$$ 

Thus $k_2^3 = z + Sq^3y$ since $v^*(Sq^3y) = v^*(z + k_2^3)$.

By [2] we can select a secondary operation $\Gamma$ associated to the relation

$$(Sq^4 Sq^1)Sq^{8t+4} + Sq^3 Sq^{8t+6} + Sq^1(Sq^{8t+6} Sq^2) + Sq^{8t+8} Sq^1 = 0$$

so that $\lambda Sq^2 x \cdot Sq^3 x \in \Gamma(x)$ for any class $x$ of dimension $8t+3$ in the domain of $\Gamma$. 

\[\text{\ast\ast} \]
and such that $\lambda \in \mathbb{Z}_2$ is independent of $x$. Note that $Sq^{4t+6} Sq^2 U = U \cdot W$ in $H^*(T_{BSO})$.

The generating class theorem [30, Theorem 6.5] gives the result

$$U_2 \cdot (z + k') \in \Gamma(U_e)$$

for some class $k'$ in $H^{4t+6}(E) \cap \text{kernel } J^* \cap \text{kernel } g^*$. It follows that $\Gamma(U_v) = U_v \cdot z(v) = U_v \cdot k_3(v)$. One checks from §2 that the highest power of 2 dividing $c_i(v)$ is 2, dividing $c_{4t+2}(v)$ is $2^{2(n-2)}$, and dividing $c_{4t+4}(v)$ is $2^{2(n-1)}$. By Proposition 3.4

$$\Gamma(U_v) = 2^{2(n-3)} U_v \cdot \beta^{4t+4}.$$ 

Thus $k_3(v) = 0$ for $\alpha(n) > 3$ so $v$ lifts to $E_2$ and the proof is complete.

6. Appendix. These Postnikov resolutions for the fiber map $\pi: B_n \to B$ are constructed by the techniques of [26]. We refer the reader also to [17] and [8] for the theory and construction of modified Postnikov resolutions. The homotopy groups of the fibers for $\pi$ appear in [13] and [22]. The tower of spaces is displayed only for resolution I. The $k$-invariant $k'_I$ represents a class in $H^*(E_1)$ whose defining relation is a relation in $H^*(E_{1-1})$ where $E_0 = B$.

6.1. Postnikov resolution I for the fibration $\pi: B Spin(4t+1) \to B Spin$ for stable spin bundles over complexes of dimension $\leq 4t + 6$ for $t > 1$. $K(n)$ denotes $K(Z_2, n)$.

$$\begin{array}{c}
B Spin(4t+1) \\
\downarrow q_3 \\
E_3 \\
\downarrow p_3 \\
E_2 \rightarrow K(4t+4) \\
\downarrow p_2 \\
E_1 \rightarrow K(4t+3) \times K(4t+4) \times K(4t+5) \\
\downarrow p_1 \\
B Spin \rightarrow K(4t+2) \times K(4t+4).
\end{array}$$

Defining relations for $k$-invariants:

$$k_1: Sq^2 w_{4t+2} = 0,$$
$$k_2: (Sq^3 Sq^1) w_{4t+2} + Sq^3 w_{4t+4} = 0,$$
$$k_3: (Sq^4 + w_4) w_{4t+2} + t Sq^2 w_{4t+4} = 0,$$
$$k_4: Sq^3 k_1 + Sq^1 k_2 = 0.$$ 

6.2. Postnikov resolution II for the fibration $\pi: BSO(8t+5) \to BSO$ for stable
orientable bundles over complexes of dimension $\leq 8t+13$ for $t > 0$. Defining relations for $k$-invariants:

$$k_1^0 = w_{8t+6},$$
$$k_2^0 = w_{8t+8},$$
$$k_1^1: (Sq^2 + w_2)w_{8t+6} = 0,$$
$$k_2^1: (Sq^2 + w_2)Sq^1w_{8t+6} + Sq^1w_{8t+8} = 0,$$
$$k_3^1: (Sq^4 + w_4)w_{8t+6} + Sq^2w_{8t+8} = 0,$$
$$k_4^1: (Sq^4 + w_4 + w_2^2)Sq^1w_{8t+8} + Sq^1(w_2 \cdot Sq^2)w_{8t+8} = 0,$$

$t$ even

$$k_1^5: (Sq^8 + w_8)w_{8t+6} + w_6 + w_{8t+8} = 0,$$
$$k_2^5: (Sq^8 + w_8)k_1^4 + Sq^1k_2^4 = 0,$$
$$k_3^5: Sq^1k_1^4 + (Sq^2Sq^3 + w_8)k_2^4 + Sq^2(w_2 \cdot Sq^1k_2^4) = 0,$$
$$k_4^5: (Sq^2 + w_2)k_1^4 + (Sq^2Sq^3 + w_2 \cdot Sq^2 + w_8)k_3^4 + \cdot w_5)k_3^4 + \cdot w_5)k_3^4 + \cdot w_5)k_3^4,$$
$$k_2^5: Sq^2k_1^4 + (Sq^4 + w_4)k_2^4 = 0,$$
$$k_3^5: Sq^2k_1^4 + (Sq^4 + w_4)k_3^4 = 0,$$
$$k_4^5: k_1^4: Sq^1k_1^4 + Sq^1Sq^1k_1^4 = 0,$$
$$k_2^4: Sq^2k_1^4 = 0,$$
$$k_3^4: Sq^2k_1^4 = 0,$$
$$k_4^4: Sq^2k_1^4 = 0.$$

6.3. Postnikov resolution III for the fibration $\pi: B\text{Spin}(16t-1) \to B\text{Spin}$ for stable spin bundles over complexes of dimension $\leq 16t+8$ for $t > 0$.

Defining relations for $k$-invariants:

$$k_1^0 = w_{16t},$$
$$k_2^0 = w_{16t},$$
$$k_1^1: Sq^2Sq^1w_{16t} = 0,$$
$$k_2^1: (Sq^4 + w_4)Sq^1w_{16t} = 0,$$
$$k_3^1: (Sq^4 + w_4)Sq^2w_{16t} = 0,$$
$$k_4^1: (Sq^8 + w_8)Sq^1w_{16t} + Sq^1(w_4 \cdot Sq^4 + w_8 \cdot Sq^2 + w_2^2)w_{16t} = 0,$$
$$k_1^2: Sq^2k_1^1 = 0,$$
$$k_3^2: Sq^2k_1^1 + Sq^1k_2^1 = 0,$$
$$k_4^2: (Sq^6 + w_6)k_1^1 + (Sq^4 + w_4)k_2^1 = 0,$$
$$k_2^2: (Sq^4 + Sq^3Sq^2 + w_4)k_1^1 + (Sq^6 + w_8)Sq^1k_1^1 + (w_4 \cdot Sq^1)k_2^1 = 0,$$
$$k_3^2: Sq^1k_1^1 + (Sq^7 + Sq^4Sq^2)k_1^1 = 0,$$
$$k_4^2: Sq^2k_1^2 + Sq^1k_2^2 = 0,$$
$$k_3^3: Sq^1k_1^2 + Sq^3k_1^2 = 0,$$
$$k_4^3: Sq^2k_1^2 + Sq^3k_1^2 = 0.$$

6.4. Postnikov resolution IV for the fibration $\pi: BSO(8t+1) \to BSO$ for stable orientable bundles over complexes of dimension $\leq 8t+9$ for $t > 1$.
Defining relations for \( k \)-invariants:
\[
\begin{align*}
\kappa^1_1 &= w_{8t+2}k^0_2 = w_{8t+4},
\kappa^1_2 &= (Sq^2 + w_2)w_{8t+2} = 0,
\kappa^2_1 &= (Sq^2 + w_2)w_{8t+2} + Sq^1w_{8t+4} = 0,
\kappa^2_2 &= (Sq^4 + w_4)w_{8t+2} + w_2 + w_{8t+4} = 0,
\kappa^3_1 &= (Sq^4 + w_6)w_{8t+4} = 0,
\kappa^3_2 &= Sq^1w_{8t+8} + (Sq^4 + w_4)w_{8t+4} + (w_2 + Sq^2w_{8t+4} + w_4 + w_6)w_{8t+4} = 0,
\end{align*}
\]

\( t \) even \( k^3_3 = (Sq^4 + w_6)w_{8t+2} + w_2 + w_{8t+4} + w_4 + w_6 + w_2w_4)w_{8t+4} = 0,

\( t \) odd \( k^3_3 = (Sq^4 + w_4)w_{8t+2} + Sq^2w_{8t+4} + w_2 + w_{8t+4} = 0,

\kappa^3_4 &= (Sq^4 + w_6)w_{8t+4} = 0,
\kappa^3_5 &= Sq^1k^1_1 + Sq^1w_2 = 0,
\kappa^3_6 &= Sq^2k^3_1 + Sq^2w_1 + Sq^1k^1_1 = 0,
\kappa^3_7 &= Sq^2k^3_1 + (Sq^4 + w_4)k^1_1 = 0,
\kappa^3_8 &= Sq^4k^1_1 + Sq^4w_2 + Sq^2k^1_1 = 0,
\kappa^3_9 &= Sq^4k^1_1 + Sq^4w_2 + Sq^2k^1_1 + Sq^7w_{8t+4} = 0,
\kappa^3_{10} &= Sq^4k^1_1 + Sq^4w_2 + Sq^2k^1_1 + Sq^7w_{8t+4} = 0,
\kappa^3_{11} &= Sq^4k^1_1 + Sq^4w_2 + Sq^2k^1_1 + Sq^7w_{8t+4} = 0,
\kappa^3_{12} &= Sq^4k^1_1 + Sq^4w_2 + Sq^2k^1_1 + Sq^7w_{8t+4} = 0.
\]

6.5. Postnikov resolution \( V \) for the fibration \( \pi: B \text{Spin}(16t+7) \to B \text{Spin} \) for stable spin bundles over complexes of dimension \( \leq 16t+16 \) for \( t > 0 \).

Defining relations for \( k \)-invariants:
\[
\begin{align*}
k^0_1 &= w_{16t+8}, \quad k^0_0 = w_{16t+16},
k^1_1 &= Sq^2w^1_{16t+8} = 0,
k^1_2 &= (Sq^4 + w_4)w_{16t+8} = 0,
k^1_3 &= (Sq^4 + w_4)w_{16t+8} = 0,
k^1_4 &= (Sq^4 + w_4)w_{16t+8} + Sq^1w_{16t+16} = 0,
k^1_5 &= Sq^4w_4 + w_6 + Sq^2 + w_3w_4)k^1_1 = 0,
k^1_6 &= Sq^4w_4 + w_6 + Sq^2 + w_3w_4)k^1_1 = 0,
k^1_7 &= Sq^4w_4 + w_6 + Sq^2 + w_3w_4)k^1_1 = 0,
k^1_8 &= Sq^4w_4 + w_6 + Sq^2 + w_3w_4)k^1_1 = 0,
k^1_9 &= Sq^4w_4 + w_6 + Sq^2 + w_3w_4)k^1_1 = 0,
k^1_{10} &= Sq^4w_4 + w_6 + Sq^2 + w_3w_4)k^1_1 = 0.
\end{align*}
\]
REFERENCES


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University of Notre Dame, Notre Dame, Indiana