A 1-LINKED LINK WHOSE LONGITUDES LIE IN THE SECOND COMMUTATOR SUBGROUP

BY

H. W. LAMBERT

1. Introduction. In this paper we give an example of a link $L$ of two polygonal simple closed curves in $S^3$ such that the longitudes of $L$ lie in the second commutator subgroup, $G''$, of its link group $G=\pi_1(S^3-L)$, but $L$ is 1-linked, that is the two simple closed curves of $L$ do not bound disjoint orientable surfaces in $S^3$. The question of the existence of such a link was raised by Eilenberg in [3] and again by Smythe in [5] and one motivation for this question is the observation that the longitudes of any boundary link lie in the second commutator subgroup of its link group. (A link is a boundary link if and only if it is not 1-linked, see Smythe [5]). In [2], [3], and [5] examples of 1-linking are given. In all these examples the authors proved 1-linking by showing that at least one of the longitudes was not in the second commutator subgroup. It is clear then that in our example $L$ we must invent some other argument to show it is 1-linked.

2. The example $L$. In $E^3$ let $T$ be the solid torus obtained by rotating the disk $x=0$, $(y-2)^2+z^2 \leq 1$ about the $z$-axis. Let $l_1$ be the simple closed curve in $S^3$ consisting of the $z$-axis and the point at infinity. Figure 1 pictures the oriented

![Figure 1](image-url)

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simple closed curve \( l_2 \) in \( T \) as seen by looking down upon it from a point on the positive \( z \)-axis. (All the remaining figures in this paper will be drawn from the same viewpoint.) Our example is \( L = l_1 \cup l_2 \). Let \( G = \pi_1(S^3 - L) \). To say the longitudes of \( L \) belong to \( G'' \), the second commutator subgroup of \( G \), we mean that on the boundary of each component of a regular neighborhood of \( L \) there is a simple closed curve (scc) lying in \( G'' \) (a loop in \( S^3 - L \) actually determines a class of conjugate elements in \( G \)); for short we may say \( l_1 \) and \( l_2 \) belong to \( G'' \) or \( L \in G'' \). (See [4, p. 123] for a discussion of longitudes.)

Since \( l_2 \) is rather complicated some further sets will be needed. Let \( V \) be the neighborhood of \( l_2 \) in \( T \) illustrated by Figure 2; \( V \) is a cube with four handles \( a_1, a_2, a_3 \) and \( a_4 \). We may think of the \( a_i \)'s as disjoint annuli such that \( \text{Bd} (\bigcup_{i=1}^{4} a_i) \subset \text{Bd} V \) and \( \text{Int} (\bigcup_{i=1}^{4} a_i) \subset \text{Int} T - V \). Let \( a_i^+ \) denote the side of \( a_i \) facing the same direction as the small arrow next to \( a_i \) and \( a_i^- \) its opposite side. Define similarly \( a_1^+, a_2^+, a_3^+, a_4^+ \), and \( a_1^-, a_2^-, a_3^-, a_4^- \). (See Figure 2.) Let \( D_l = T \cap \{(x, y, z) : x = 0, y < 0\} \) and \( D_r = T \cap \{(x, y, z) : x = 0, y > 0\} \). \( D_l \) and \( D_r \) are meridional disks of \( T \). Choose the positive side of \( D_l \), \( D_l^+ \) (negative side \( D_l^- \)) to be the side of \( D_l \) intersecting the component \( V^+ (V^-) \) of \( V - D_l \) containing the handles \( a_1 \) and \( a_2 \) (the handles \( a_3 \) and \( a_4 \)).

The next observation is important. Suppose \( l \) is a loop in \( T - V \) and as we go around \( l \) we count 1 or \(-1\) (\( \pm 2 \) in the case of \( a_3 \)) each time we pass through a handle of \( V \), the sign being determined by whether we went through the handle with or against the direction of the small arrow next to the handle, for instance if we pass from \( a_3^+ \) to \( a_3^- \) on \( l \) add \(-2 \). Using the right-hand rule we see that the sum total of the number of times \( l \) goes through the handles of \( V \) gives the algebraic linking number of \( l \) with respect to \( l_2 \), denoted by \( \text{Lk} (l, l_2) \). (See [1, p. 81] for a definition of algebraic linking.) Obviously the number of times \( l \) passes through \( D \).
We first show \( L \in G' \) and then show \( L \) is 1-linked.

3. \( L \in G' \). Let \( S_0 \) be the surface (compact, orientable) having one boundary component and of genus 2. To show \( L \in G' \) we show that there is a map \( f \) of \( S_0 \) into \( T-V \) such that \( f|\text{Bd} \, S_0 \) is a homeomorphism onto \( \text{Bd} \, D_t \) and the image of the homology generators of \( S_0 \) belong in \( G' \). We do not build \( f \) but rather we construct the desired image \( S \) of \( S_0 \) in \( T-V \). We start with a disk with four holes \( S' \) as illustrated in Figure 3 (in Figure 3, \( V \) appears as a dotted 1-dimensional object except at the right-hand side of the illustration). We add a handle \( H_1 \) (\( H_1 \) is an annulus) to \( S' \) by starting at \( \gamma_2 \), following \( V \) up through its handle \( a_1 \), down through the hole bounded by \( \gamma_1 \), through handle \( a_3 \), then, like a fountain, the annulus widens out, reverses direction in \( T \) and goes back through \( S' \) in a simple closed curve parallel to \( \text{Bd} \, S' \cap \text{Bd} \, T \); then narrows down again around \( V \), reverses direction in \( T \), goes through \( a_2 \) and ends at \( \gamma_4 \). (See Figure 3.) Notice the figure eight \( \alpha_1\beta_1 \) in \( S' \cup H_1 \) is such that \( (\alpha_1\beta_1) \cap D_r = \emptyset \), hence \( \text{Lk} (\alpha_1, l_1) = \text{Lk} (\beta_1, l_1) = 0 \) and, again using the comment of §2, \( \text{Lk} (\alpha_1, l_2) = 1-2+1=0 \) and \( \text{Lk} (\beta_1, l_2) = 0 \). Hence \( \alpha_1, \beta_1 \in G' \).

Since it would complicate Figure 3 too much to add the second handle \( H_2 \), we just imagine it in Figure 3 as an annulus attaching \( \gamma_1 \) to \( \gamma_3 \) by going down through \( a_3 \), up through \( S' \) in a simple closed curve parallel to \( \text{Bd} \, S' \cap \text{Bd} \, T \), down through \( a_2, \gamma_4 \) and up through \( a_4 \). Again \( S' \cup H_2 \) contains a figure eight \( \alpha_2\beta_2 \) disjoint from \( \alpha_1\beta_1 \) such that \( \text{Lk} (\alpha_2, l_1) = \text{Lk} (\beta_2, l_1) = \text{Lk} (\alpha_2, l_2) = \text{Lk} (\beta_2, l_2) = 0 \) (where \( \text{Lk} (\alpha_2, l_2) = -2+1+1=0 \). Hence \( \alpha_2, \beta_2 \in G' \). It can be checked that \( S = S' \cup H_1 \cup H_2 \) is the image of \( S_0 \). It then follows that \( \text{Bd} \, S \) is homotopic in \( T-V \) to \((\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}) \cdot (\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}) \). Since \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in G' \), \( \text{Bd} \, S \in G'' \) and it follows that \( l_1 \in G'' \). Let
Q be the disk indicated in the right side of Figure 3, \( Q \) bounds a portion of \( l_2 \).
The four scc’s making up \( Cl \) \( [(l_2—Q) \cup (Bd Q—l_2)] \) are homotopic to \( Bd S \) in \( T—I_2 \) and it follows that \( l_2 \in G^* \).

It is interesting to note that the singularities of \( f \) on \( S_0 \) can be made to consist of two disjoint scc’s parallel to \( Bd S_0 \) and two disjoint scc’s in each of the two handles. Under \( f \) one meridian curve is sewed to the other and the remaining two meridian curves are sewed to the two curves parallel to \( Bd S_0 \).

4. \( L \) is 1-linked. The following nine lemmas combined with the assumption that \( l_1, l_2 \) bound disjoint orientable surfaces \( S_1, S_2 \), respectively, (i.e. \( L \) is not 1-linked) will be shown to lead to a contradiction. It should be noted that the linking properties developed in the following lemmas echo those of §3.

**Lemma 1.** If \( l_1, l_2 \) bound disjoint orientable surfaces \( S_1, S_2 \), respectively, then \( l_2 \) bounds an orientable surface \( S_2 \) such that \( S_2 \subseteq Int T, S_2 \) is in general position relative to \( D_r \), and at most one component of \( l_2 — D_r \) is contained in a component of \( S_2 — D_r \), intersecting both sides of \( D_r \).

**Proof.** By the existence of \( S_1 \) we may suppose \( S_2 \subseteq Int T \) and \( S_2 \) is in general position relative to \( D_r \). Let \( l(i), i = 1, 2, 3 \) and 4, be the component (open arc) of \( l_2 — D_r \) going through the handle \( a_i \) of \( V \) and let \( C(i) \) be the component of \( S_2 — D_r \) containing \( l(i) \). Suppose \( C(1) \) and \( C(2) \) intersect both sides of \( D_r \). If every component of \( S_2 — D_r \), which intersects \( D_r^+ \) also intersects \( D_r^- \), then we could, by going around the components of \( S_2 — D_r \), find a loop \( l \) in \( S_2 \) such that \( Lk(l, Bd D_r) > 0 \) (counting +1 each time we pass through \( D_r \) going from \( D_r^- \) to \( D_r^+ \)). But from this it follows that \( S_1 \cap S_2 \neq \emptyset \), contradiction. Hence there is some component \( X \) of \( S_2 — D_r \), intersecting only \( D_r^+ \). Then \( X \) separates \( E^3 = Int T — D_r \), into exactly two components, one of which contains \( C(1) \cup C(2) \); call this component \( Ext X \), the other \( Int X \). Replace \( X \) by the punctured disk (or disks) \( Cl(Int X) \cap D_r \). (In this process we will also have to cut off all other parts of \( S_1 \cup S_2 \) in \( Int X \).) In any case by repeating this process a finite number of times, it follows that at least one of the resulting \( C(1), C(2) \) intersects only \( D_r^+ \).

By a similar reasoning relative to the components \( C(3), C(4) \) we may suppose at least one of them intersects only \( D_r^- \). By general position of \( S_2 \) and \( D_r \), either \( C(1) = C(2) \) or \( C(3) = C(4) \). Hence let \( S_2 = S_2' \), and \( S_2 \) satisfies the conclusion of this lemma.

Let \( S \) be an orientable surface in \( T — (D_r \cup V) \) such that \( Bd S = Bd D_0 \), \( Int S \subseteq Int (T — (D_r \cup V)) \) and \( S \) is in general position relative to \( \cup l a_i \). If \( l \) is a loop in \( S \), let \( Lk(l, a_i) \) be the linking number of \( l \) with respect to the handle \( a_i \) only and \( Lk(l, a_i \cup a_j) \) the linking number of \( l \) with respect to the handle \( a_i \) and \( a_j \) only.

We introduce now another set which will be useful in investigating the linking properties of \( S \) with respect to \( \cup l a_i \). Let \( T_A = T \cap xy \)-plane, \( T_A \) is an annulus in \( T \) which intersects each \( a_i \) in two arcs (see Figure 2). Adjust \( S \) slightly in \( T — V \) so that it is in general position relative to both \( T_A \) and \( \cup l a_i \). Note that the
components of \((S \cap \text{Int } T_A) - \bigcup_{i=1}^k a_i\) are scc's or open arcs whose closures are either arcs or scc's. Let \(\Gamma = \{\gamma : \gamma \text{ is a component of } (S \cap T_A) - \bigcup_{i=1}^k a_i\}\). We now alter \(S\) so that \(S \cap (T_A \cup \bigcup_{i=1}^k a_i)\) is in a certain sense simpler. By adjusting \(S\) close to \(\bigcup_{i=1}^k a_i\), we may suppose that if \(l\) is a scc of \(S \cap \bigcup_{i=1}^k a_i\) that bounds a disk in \(\bigcup_{i=1}^k a_i\), then \(l \cap T_A = \emptyset\) and if \(l\) does not bound a disk in \(\bigcup_{i=1}^k a_i\), then \(l \cap T_A\) consists of exactly two points. Suppose \(\gamma \in \Gamma\) and \(\text{Cl } \gamma\) is an arc starting and ending on the same side of some \(a_i\). We may then adjust \(S\) near \(T_A\) by pushing \(\gamma\) down through \(a_i\), changing two (or more) scc's of \(S \cap a_i\) to a scc \(l\) which bounds a disk on \(a_i\) and adjust \(S\) close to \(a_i\) so that \(l \cap T_A = \emptyset\). Since this process reduces the number of scc's of \(S \cap \bigcup_{i=1}^k a_i\) which separate an \(a_i\), it is finite. Now remove those scc's of \(S \cap \bigcup_{i=1}^k a_i\) which bound a disk in \(\bigcup_{i=1}^k a_i\) by cutting \(S\) off on \(\bigcup_{i=1}^k a_i\), let \(S(r)\) be the component of the resulting surface which contains \(\text{Bd } S\). Let \(G(r) = \{\gamma : \gamma \text{ is a component of } (S(r) \cap T_A) - \bigcup_{i=1}^k a_i\}\) and \(\Delta(r) = \{\delta : \delta \text{ is a component of } (S(r) - \bigcup_{i=1}^k a_i)\}\). Note that if \(\text{Cl } \gamma\) is an arc intersecting both sides of an \(a_i\), then we obtain a spiral in \(S(r) \cap T_A\) going around \(a_i\); eventually it reverses itself and here we obtain an arc intersecting only one side of \(a_i\), contradicting the form of \(S(r)\). Therefore if \(\gamma \in G(r)\) and \(\text{Cl } \gamma\) is an arc, then its endpoints lie in different components of \(\text{Bd } T \cup \bigcup_{i=1}^k a_i\).

To say \(\gamma \in G(r)\) ends on \(a_i^+\) means \(\gamma\) approaches \(a_i\) from its positive side (denoted by \(\gamma \cap a_i^+ \neq \emptyset\)). Similarly we may define \(\gamma \cap a_i^- \neq \emptyset\) and, for \(\delta \in \Delta(r)\), \(\delta \cap a_i^+ \neq \emptyset\) and \(\delta \cap a_i^- \neq \emptyset\). We say two distinct elements \(\gamma_1, \gamma_2\) of \(G(r)\) abut on \(a_i\) if they approach a point of \(T_A \cap \bigcup_{i=1}^k a_i\) from opposite sides of \(a_i\) and \(\delta_1, \delta_2 \in \Delta(r)\) abut on \(a_i\) if they approach a scc of \(S \cap a_i\) from opposite sides of \(a_i\). If \(\gamma\) ends on \(a_i\), then by the form of \(S(r)\), it follows that \(S \cap a_i\) contains a scc \(l\) such that \(l \cap T_A\) is two points \(b_1, b_2\) where \(\gamma\) approaches \(b_1\). Let \(\text{Op } \gamma(a_i^+)\) (or \(\text{Op } \gamma(a_i^-)\)) be the arc of \(G(r)\) which approaches \(b_2\) from the positive side of \(a_i\), \(a_i^+\) (or the negative side of \(a_i\), \(a_i^-\)). It helps to draw in Figures 2, 4 and 5 the various arcs \(\gamma\) which arise in the next lemmas.

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**Figure 4**
Lemma 2. Suppose $\delta \in \Delta(r)$, $\delta \cap a_3^- \neq \emptyset$, \(\text{Bd} \ S(r) \cup \delta\) and $\delta$ contains no $\gamma \in \Gamma(r)$ so that $\text{Cl} \ \gamma$ is a scc. Then $\delta \cap a_3^- \neq \emptyset$.

Proof. Suppose $\delta \cap a_3^- = \emptyset$. Since $\text{Bd} \ S(r) \cup \delta$ and $\delta$ contains no $\gamma$ so that $\text{Cl} \ \gamma$ is a scc, there is an arc $\gamma_1$ of $\Gamma(r)$ in $\delta$ going from $a_2^-$ to $a_1^-$. Let $\gamma_2 = \text{Op} \ \gamma_1(a_3^-)$, $\gamma_2$ must end on $a_2^-$. Let $\gamma_3 = \text{Op} \ \gamma_2(a_2^-)$, $\gamma_3 \neq \gamma_1$ and $\gamma_3$ must end on $a_1^-$. Let $\gamma_5 = \text{Op} \ \gamma_4(a_1^-)$. Continuing in this manner, it follows that $\Gamma(r)$ must be an infinite set, contradiction. Hence $\gamma_1$ must end on $a_3^-$ and $\delta \cap a_3^- \neq \emptyset$.

Also we have the following

Lemma 2'. Suppose $\delta \in \Delta(r)$, $\delta \cap a_3^- \neq \emptyset$, \(\text{Bd} \ S(r) \cup \delta\) and $\delta$ contains no $\gamma \in \Gamma(r)$ so that $\text{Cl} \ \gamma$ is a scc. Then $\delta \cap a_3^- \neq \emptyset$.

Lemma 3. Suppose $\delta \in \Delta(r)$, $\delta \cap a_3^+ \neq \emptyset$, \(\text{Bd} \ S(r) \cup \delta\) and $\delta$ contains no $\gamma \in \Gamma(r)$ so that $\text{Cl} \ \gamma$ is a scc. Then $\delta \cap a_3^+ \neq \emptyset$.

Proof. Suppose $\delta \cap a_3^+ = \emptyset$. Since $\text{Bd} \ S(r) \cup \delta$ and $\delta$ contains no $\gamma$ so that $\text{Cl} \ \gamma$ is a scc, there is an arc $\gamma_1$ of $\Gamma(r)$ in $\delta$ going from $a_3^+$ to $a_2^-$. Let $\gamma_2 = \text{Op} \ \gamma_1(a_3^+)$, by the proof of Lemma 2, $\gamma_2$ must end on $a_3^-$. Let $\gamma_3 = \text{Op} \ \gamma_2(a_3^-)$, $\gamma_3 \neq \gamma_1$, and $\gamma_3$ must end on $a_2^-$. Let $\gamma_5 = \text{Op} \ \gamma_3(a_2^-)$. Continuing in this manner, it follows that $\Gamma(r)$ must be an infinite set, contradiction. Hence $\gamma_1$ must end on $a_3^+$ and $\delta \cap a_3^+ \neq \emptyset$.

Also we have the following

Lemma 3'. Suppose $\delta \in \Delta(r)$, $\delta \cap a_3^+ \neq \emptyset$, \(\text{Bd} \ S(r) \cup \delta\) and $\delta$ contains no $\gamma \in \Gamma(r)$ so that $\text{Cl} \ \gamma$ is a scc. Then $\delta \cap a_3^+ \neq \emptyset$.

The next lemma follows easily by examining the various arcs of $\Gamma(r)$ in $\delta$.

Lemma 4. Suppose $\delta \in \Delta(r)$, $\delta \cap a_3^- \neq \emptyset$, \(\text{Bd} \ S(r) \cup \delta\) and $\delta$ contains no $\gamma \in \Gamma(r)$ so that $\text{Cl} \ \gamma$ is a scc. Then $\delta \cap a_3^- \neq \emptyset$.

Also we have the following.
Lemma 4'. Suppose $\delta \in \Delta(r)$, $\delta \cap a_i^+ \neq \emptyset$, $\text{Bd } S(r) \cap \delta$ and $\delta$ contains no $\gamma \in \Gamma(r)$ so that $\text{Cl } \gamma$ is a scs. Then $\delta \cap a_i^- \neq \emptyset$.

Lemma 5. There is a loop $l$ in $S$ so that either (1) $\text{Lk } (l, a_1 \cup a_2) \neq 0$ or (2) $\text{Lk } (l, a_3 \cup a_4) \neq 0$.

Proof. We first note that if there is a loop $l'$ in $S(r)$ satisfying (1) or (2) of this lemma, then there is a loop $l$ in $S$ satisfying (1) or (2). For, while retaining property (1) or (2), we may push $l'$ off the disks of $S(r)$ obtained by cutting $S$ off on $\bigcup_{i=1}^k a_i$. Pushing a $\text{Cl } \gamma$ down through some $a_i$ does not alter the linking properties of $l'$ with the various $a_i$. Hence reversing the steps needed to arrive at $S(r)$ from $S$, it follows that $l'$ gives rise to an $l$ in $S$ satisfying condition (1) or (2).

If there is an element $\gamma$ of $\Gamma(r)$ so that $\text{Cl } \gamma$ is a scs intersecting either $a_1$, $a_2$, $a_3$ or $a_4$, then $l' = \text{Cl } \gamma$ satisfies condition (1) or (2). Suppose then no such scs exists.

Let $\delta_0$ be the element of $\Delta(r)$ containing $\text{Bd } S(r)$ and suppose $\delta_0 \cap a_i^+ \neq \emptyset$.

By Lemma 2, $\delta_0$ abuts a $\delta_1$ on $a_2$ so that $\delta_1 \cap a_1^+ \neq \emptyset$ (assuming $\delta_0 \neq \delta_1$). By Lemma 3, $\delta_1$ abuts a $\delta_2$ on $a_3$ so that $\delta_2 \cap a_2^+ \neq \emptyset$ (again assuming $\delta_0 \neq \delta_2$). By Lemma 4, $\delta_2$ abuts a $\delta_3$ on $a_4$ so that $\delta_3 \cap a_3^+ \neq \emptyset$. Continuing this reasoning we obtain a sequence $\delta_0, \delta_1, \ldots, \delta_\ell$ which eventually repeats an element of $\Delta(r)$. Let $m$ be the first integer so that $n < m$, $\delta_n = \delta_m$ and $\delta_i \neq \delta_j$ if $i \neq j$ and $n < i, j < m$. If $m-n=1$, then $\text{Cl } (\bigcup_{n=1}^m \delta_i)$ contains a loop $l'$ so that $\text{Lk } (l', a_i) = \pm 1$ for some $i=1, 2$ or $3$ and $\text{Lk } (l', a_j) = 0$ for $j \neq i$. Hence $l'$ satisfies either (1) or (2). If $m-n \geq 2$, then it follows that $\text{Cl } (\bigcup_{n=1}^m \delta_i)$ contains a loop $l'$ so that $\text{Lk } (l', a_1 \cup a_2) \neq 0$.

If $\delta_0 \cap a_3^+ \neq \emptyset$ or $\delta_0 \cap a_4^+ \neq \emptyset$, then we may start at either of these places and, repeating the same pattern as before, obtain a sequence $\delta_0, \ldots, \delta_\ell$ so that $\text{Cl } (\bigcup_{n=1}^m \delta_i)$ contains a loop $l'$ satisfying (1) or (2).

If $\delta_0 \cap a_3^+ \neq \emptyset$, then making use of Lemmas 2', 3' and 4' it follows that $S(r)$ contains a loop $l'$ satisfying (1) or (2). Since $\delta_0$ must intersect one of $a_2^+, a_3^+, a_4^+$, or $a_5^+$, Lemma 5 follows.

Let $V_i = V$ minus its $i$th handle, $i=1, 2, 3$ and $4$, see Figures 4 and 5 for $V_1$ and $V_2$, respectively. Let $S_i$ be a surface in $T - (D_r \cup V_i)$ such that $\text{Bd } S_1 = \text{Bd } D_i$, $\text{Int } S_1 \subseteq \text{Int } (T - (D_r \cup V_i))$ and $S_1$ is in general position relative to $((\bigcup_{j=1}^i a_j) - a_i)$. As before, from $S_i$ we may obtain a surface $S_i(r)$ so that for each arc $\gamma$ of $\Gamma_i(r) = \{\gamma : \gamma$ is a component of $(S_i \cap T_r) - ((\bigcup_{j=1}^i a_j) - a_i)\}$ such that $\text{Cl } \gamma$ is not a scs we have that its endpoints lie in different components of $\text{Bd } T - ((\bigcup_{j=1}^i a_j) - a_i)$.

Lemma 6. There is a loop $l$ in $S_1$ so that either (1) $\text{Lk } (l, a_2) \neq 0$ or (2) $\text{Lk } (l, a_3 \cup a_4) \neq 0$.

Proof. As in Lemma 5, we note that if $S_1(r)$ contains a loop satisfying (1) or (2) then so does $S_1$. If there is an element $\gamma$ of $\Gamma_1(r)$ so that $\text{Cl } \gamma$ is a scs, then $l' = \text{Cl } \gamma$ satisfies (1) or (2) of this lemma. Suppose then no such scs exists. Note that $S_1(r) \cap a_3 = \emptyset$ and $S_1(r) \cap a_4 = \emptyset$. There is an element $\gamma_1$ of $\Gamma_1(r)$ going from $a_2^+$ to $a_3^+$. Let $\gamma_2 = \text{Op } \gamma_1(a_2^+)$. If $\gamma_2$ ends on $\text{Bd } T$, then the arc $\gamma_2$ abuts on $a_2$ would
start and end on $a_2^*$, contradiction. Hence $y_2$ must end on $a_2^*$. Let $y_3 = \text{Op}_2(a_2^*)$.
Continuing in this manner, it follows that $\Gamma_1(r)$ must be infinite, contradiction.

The proof of the next lemma is (geometrically) analogous to the proof of Lemma 6.

**Lemma 7.** There is a loop $l$ in $S_4$ so that either (1) $\text{Lk}(l, a_3) \neq 0$ or (2) $\text{Lk}(l, a_1 \cup a_3) \neq 0$.

**Lemma 8.** There is a loop $l$ in $S_2$ so that either (1) $\text{Lk}(l, a_3) \neq 0$ or (2) $\text{Lk}(l, a_1 \cup a_3) \neq 0$.

**Proof.** Again note that if $S_2(r)$ contains a loop satisfying (1) or (2), then so does $S_2$, hence assume no $\text{Cl} \gamma$ is a scc. Since $S_2 \cap a_3 \neq \varnothing$, there are arcs $y', y''$ in $\Gamma_2(r)$ so that both $y'$, $y''$ start on $a_3^*$, $y'$ ends on $a_3^*$ and $y''$ ends on $a_3^*$ and $y'$, $y''$ end on the innermost scc of $S_2(r) \cap a_3$ (see Figure 5). Let $y'_0$, $y''_0$ be the arcs $y'$, $y''$ abut on $a_3$, $a_4$, respectively. Since not both $y'_0$, $y''_0$ can end on $\text{Bd} T$, it follows that $y'_0 = y''_0$ and the scc $l = \text{Cl}(y' \cup y'' \cup y'_0)$ plus an arc in $S_2(r) \cap a_4$ satisfies (2) of this lemma.

We also have the following

**Lemma 9.** There is a loop $l$ in $S_3$ so that either (1) $\text{Lk}(l, a_4) \neq 0$ or (2) $\text{Lk}(l, a_1 \cup a_3) \neq 0$.

**Theorem.** $L$ is $1$-linked.

**Proof.** Suppose $l_1$, $l_2$ bound disjoint orientable surfaces $S_1$, $S_2$, respectively. Let $l(i)$, $C(i)$, $i = 1, 2, 3, 4$ be as given in the proof of Lemma 1. By Lemma 1, $l_2$ bounds an orientable surface $S_2$ such that $S_2 \subset \text{Int} T$, $S_2$ is in general position relative to $D_r$ and at most one component $C(i)$ of $S_2 - D_r$ intersects both sides of $D_r$. Suppose this $i = 1$.

Let $U$ be a closed regular neighborhood of $C(3) \cup C(4) \cup D_r$ chosen so $\text{Bd} U$ contains a surface $S$ such that $\text{Int} S \subset \text{Int} (T - (D_r \cup l_2))$ and $\text{Bd} S = \text{Bd} D_1$. Further, $U$ may be chosen so that $S \cap C(i) = \varnothing$ for $i = 2, 3$ and 4, since these $C(i)'$s intersect just one side of $D_r$. Since $S \cap l_2 = \varnothing$, we may push $S$ off $V$ (we may need to use a cut and paste argument on $S \cup \bigcup_{i=2}^4 C(i)$) to get $S$ off the handle of $V$ intersecting $a_3$ since $l_2$ goes through this handle twice).

There are disjoint arcs $x_1$, $x_2$ in $\text{Cl}(C(3) \cup C(4)) \cap D_r$ so that either (1) $l' = l(3) \cup l(4) \cup x_1 \cup x_2$ is a scc or (2) $l(3) \cup x_1$ and $l(4) \cup x_2$ are scc's. In Case (2) let $D$ be a disk in $\text{Int} D_r$ so that $x_1 \cup x_2 \subset \text{Bd} D$ and the arcs $x_1$, $x_2$ forming $\text{Bd} D - \text{Int} (x_1 \cup x_2)$ form a scc $l'' = l(3) \cup l(4) \cup x'_1 \cup x'_2$ which admits an orientation compatible with the orientation on $l(3)$ and $l(4)$ induced by the orientation of $l_2$. Let $d_n$, $n = 1, 2, \ldots, m$ be the disks in $D_r$ bounded by the scc's of $\text{Cl}(C(3) \cup C(4)) \cap D_r$. Let $C' = \text{Cl}(C(3) \cup C(4)) \cup \bigcup_{n=1}^m d_n$ and $C'' = C' \cup D_r$. It then follows that $l' \sim 0$ in $C'$ and $l'' \sim 0$ in $C''$ (using integer coefficients). Since $S \cap C' = S \cap C'' = \varnothing$, by Alexander Duality for each loop $l$ in $S$, $\text{Lk}(l, a_3 \cup a_4) = 0$. 

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If both $C(1)$, $C(2)$ intersect just one side of $D_n$, then by a similar argument as used above, it follows that for each loop $l$ in $S$, $\text{Lk}(l, a_1 \cup a_2) = 0$. But this is impossible by Lemma 5. If $C(1)$ intersects both sides of $D_n$, then, since $C(2)$ intersects only one side of $D_n$, $C(1) \cap C(2) = \emptyset$ and it follows that for each loop $l$ in $S$, $\text{Lk}(l, a_2) = 0$, but this contradicts Lemma 6. The cases $i=2$, 3 and 4 are similar. Hence the surfaces $S_1$, $S_2'$ could not have existed and $L$ is 1-linked.

5. **Concluding remarks.** Since our example $L = l_1 \cup l_2$ is 1-linked it follows that if $l_1$ bounds the orientable surface $S_1$ in $S^3 - l_2$, then $S_1$ contains a loop $l$ such that $l \sim 0$ in $H_1(S^3 - l_2)$. In particular we have the following

**Theorem.** If $K\equiv k_1 \cup k_2$ is a link of two components, then $K$ is a boundary link if and only if $k_1$ bounds an orientable surface $S_1$ in $S^3 - k_2$ such that the inclusion of $H_1(S_1)$ in $H_1(S^3 - k_2)$ is trivial.

**Proof.** The only if part follows immediately from the definition of boundary link. If the surface $S_1$ exists then $k_2$ bounds an orientable surface $S_2$ in $S^3 - k_1$. Put $S_2$ in general position with respect to the homology generators $H$ (figure eights) of $S_1$. Since the inclusion of $H_1(S_1)$ in $H_1(S^3 - k_2)$ is trivial, we may add handles to $S_2$ minus a small regular neighborhood of $H$ to form an orientable surface $S_2'$ such that $Bd S_2' = k_2$ and $S_2' \subseteq S^3 - (k_1 \cup H)$. Since $S_1 - H$ is a disk minus a finite number of points we may put $S_2'$ in general position relative to $S_1 - H$ and cut $S_2'$ off on $S_1$. We then obtain an orientable surface $S_2''$ such that $Bd S_2'' = k_2$ and $S_1 \cap S_2'' = \emptyset$. Hence $K$ is a boundary link.

This theorem and our example $L$ motivate the following

**Question.** Does there exist a link $K\equiv k_1 \cup \cdots \cup k_n$, $2 < n$, such that each $k_i$ bounds an orientable surface $S_i$ in $S^3 - (K - k_i)$, and the inclusion of $H_1(S_i)$ in $H_1(S^3 - (K - k_i))$ is trivial but $K$ is 1-linked?

Such a link $K$ of the question would also be an example of a 1-linked link whose longitudes all lie in the second commutator subgroup.

**References**


University of Iowa,
Iowa City, Iowa