TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, January 1970

LIE ISOMORPHISMS OF FACTORS[1]

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1. **Introduction.** By a von Neumann algebra M we mean a weakly closed, selfadjoint algebra of operators (containing the identity I) on a complex Hilbert space H. In this paper we consider a mapping ϕ between von Neumann algebras M and N which is one-one, onto, $*$-linear, and which preserves Lie brackets of operators (that is $\phi[A, B]=[\phi(A), \phi(B)]$ where $[X, Y]=XY-YX$). Such mappings will be called Lie $*$-isomorphisms. We prove, as our main result, that if M is a factor (=center trivial von Neumann algebra), then $\phi=\tilde{\theta}+\lambda$ where $\tilde{\theta}$ is either a $*$-isomorphism, or the negative of a $*$-anti-isomorphism and λ is a $*$-linear map from M into the center of N which annihilates brackets of operators in M. This was proved by L. Hua [6] in the case that M $(=N)$ is a factor of type I_n $(n>2)$. Subsequently Hua's result was generalized in an algebraic sense by W. S. Martindale [8], [9], to the case where M and N are simple rings with M containing two nonzero idempotents whose sum is the identity. The algebraic techniques of these papers, however, are not sufficient in our setting since von Neumann factors are not, in general, simple.

In all that follows $\phi: M \to N$ is a Lie $*$-isomorphism between the von Neumann algebras M and N.

2. **Preliminary results.**

LEMMA 1. *Let $X, Y \in M$. Then $XY=YX$ iff $\phi(X)\phi(Y)=\phi(Y)\phi(X)$.*

Proof. $XY=YX$ iff $[X, Y]=0$ iff $\phi[X, Y]=[\phi(X), \phi(Y)]=0$ iff $\phi(X)\phi(Y)=\phi(Y)\phi(X)$.

A von Neumann subalgebra $M_0 \subseteq M$ is called *normal* in M if $(M_0' \cap M)' \cap M=M_0$ where M_0' is the commutant of M_0.

As an easy consequence of this definition we have

THEOREM 1. *If M_0 is a normal von Neumann subalgebra of M, then $\phi(M_0)$ is a normal von Neumann subalgebra of N having the same linear dimension.*

COROLLARY. *If M is a factor, then so is N.*

Received by the editors March 13, 1969.

[1] This paper represents part of the author's doctoral dissertation written under the direction of Professor Henry Dye at the University of California, Los Angeles. Research was partially supported by NSF grant GP-6727.

Proof. ϕ sends the center of M, namely $[\alpha I \mid \alpha \in \mathbf{C}]$ onto the center of N. Thus the center of N is 1-dimensional.

LEMMA 2. *If M is a factor, and M_0 a finite-dimensional von Neumann subalgebra of M, then M_0 is normal in M.*

Proof. See [7, Theorems 1 and 4].

The following lemma is a variant of a result of Hua [6].

LEMMA 3. *Let M be a factor and P a noncentral projection (=selfadjoint idempotent) in M. Then $\phi(P)$ can be expressed uniquely in one of two forms:*

(i) $\phi(P)=\theta(P)+\lambda(P)I$,

(ii) $\phi(P)=-\theta'(P)+\lambda'(P)I$,

where $\theta(P)$, $\theta'(P)$ are noncentral projections in N, and $\lambda(P)$, $\lambda'(P)$ are scalars.

Proof. The von Neumann algebra generated by P is 2-dimensional and so by Lemma 2 and Theorem 1, $\phi(P)=\alpha Q+\beta I$ where Q is a projection in N and $\alpha, \beta \in \mathbf{C}$. Since $[[[X, P]P]P]=[X, P]$ for all $X\in M$, the same relation must hold for P replaced by $\phi(P)$ and all $X\in N$. Thus $\alpha^3[[[X, Q]Q]Q]=\alpha[X, Q]$ for all $X\in N$. Choosing an X such that $[X, Q]\neq 0$ we see that $\alpha=\pm 1$.

For the other part, suppose $Q+\mu I=Q'+\mu' I$. Then Q commutes with Q' and $(\mu-\mu')^2 I=(Q-Q')^2=Q+Q'-QQ'$. This is possible only if $Q=Q'$.

3. **The decomposition $\phi=\theta+\lambda$.** We now assume that M is a factor not of type I_n. The notation of Lemma 3 is retained.

LEMMA 4. *$\theta(I-P)=I-\theta(P)$, $\theta'(I-P)=I-\theta'(P)$.*

Proof. We prove the lemma for θ. $\theta(P)+\theta(I-P)$ is central in N; so, therefore, is $\theta(P)\theta(I-P)$. $\theta(P)\theta(I-P)$, being the product of commuting projections, is thus a central projection. If $\theta(P)\theta(I-P)=I$, then $\theta(P)=\theta(I-P)=I$ which is impossible. Hence $\theta(P)\theta(I-P)=0$ and $\theta(P)+\theta(I-P)=I$.

DEFINITION. Projections P and Q are called *coorthogonal*, written $\text{co}_\perp$, if

$$(I-P)(I-Q) = 0.$$

LEMMA 5. *If P and Q are orthogonal projections in M then either $\theta(P)\perp\theta(Q)$, or $\theta(P)\ \text{co}_\perp\ \theta(Q)$.*

Proof. $PQ=0$ implies $[[[[X, P], Q]P]Q]+[[X, P], Q]=0$ for all $X\in M$. Hence the same relation is satisfied with P replaced by $\theta(P)$, Q replaced by $\theta(Q)$, and all $X\in N$. Writing the latter relation out and multiplying on the left by $\theta(P)\theta(Q)$ we get $\theta(P)\theta(Q)X\theta(I-P)\theta(I-Q)=0$ for all $X\in N$. For notation, let $R=\theta(P)\theta(Q)$ and $R'=\theta(I-P)\theta(I-Q)$. Both R and R' are projections and so, since we are in a factor, they are comparable. Assume $R\precsim R'$. Let $V\in N$ be such that $VV^*=R$, $V^*V\leqq R'$. Then $R=RVR'V^*R=0$. Since $RXR'=0$ for all $X\in N$ implies $R'XR=0$ for all $X\in N$, we could use the same reasoning if $R'\precsim R$.

Definition. A *projection ortho-isomorphism* between von Neumann algebras M and N is a one-one mapping θ of the set of projections in M on that in N such that if P and Q are projections in M with $PQ=0$, then $\theta(P)\theta(Q)=0$.

Lemma 6. *Let $\mathscr{A}$ be an abelian von Neumann algebra contained in M of dimension $\geqq 3$, and let $\mathscr{A}_P=$ the set of projections in $\mathscr{A}$. Then either θ or θ' is an ortho-isomorphism on $\mathscr{A}_P$ and these possibilities are mutually exclusive.*

Proof. First suppose that $P_1, \ldots, P_n$ $(n\geqq 3)$ are mutually orthogonal projections in M. We claim that either the $\theta(P_1), \ldots, \theta(P_n)$ are mutually orthogonal or the $\theta'(P_1), \ldots, \theta'(P_n)$ are mutually orthogonal. For, suppose $\theta(P_1) \perp \theta(P_2)$. If $\theta(P_3)$ is coorthogonal to $\theta(P_1)$ then $I-\theta(P_1)\leqq\theta(P_3)$ and so $\theta(P_2)=\theta(P_2)\theta(P_3)$. Hence we have $\theta(P_2)\leqq\theta(P_3)$. But either $\theta(P_2) \perp \theta(P_3)$ or $1-\theta(P_2)\leqq\theta(P_3)$ a contradiction. Instead of P_3 we could have used any P_i $(i>2)$ and so $\theta(P_i) \perp \theta(P_1)$ for all i $(i\neq 1)$. Applying the above argument to $\theta(P_1)$, $\theta(P_i)$, $\theta(P_j)$ for $i, j>1$, $i\neq j$ we see that all the $\theta(P_i)$ are orthogonal. If $\theta(P_1)$ is coorthogonal to $\theta(P_2)$, then $\theta'(P_1)$ is orthogonal to $\theta'(P_2)$ and we repeat the argument. We note also that if the $\theta(P_1), \ldots, \theta(P_n)$ are orthogonal then $\theta(\sum P_i)=\sum\theta(P_i)$ (where the index i runs over any subset of $\{1, \ldots, n\}$) by the uniqueness of the representation of $\phi(P)$, P a projection. A similar comment holds for θ'.

Now choose three mutually orthogonal projections P_1, P_2, P_3 in $\mathscr{A}_P$ whose sum is I. Take any other two orthogonal projections Q, R in $\mathscr{A}_P$. These five projections generate an atomic subalgebra (with finitely many atoms) M_0 of M. Suppose the $\theta(P_i)$ $(i=1, 2, 3)$ are mutually orthogonal. We claim that the atoms of M_0 have orthogonal images. For, if not, the images of atoms under θ' are then orthogonal and thus, by the additivity of θ' discussed above, the $\theta'(P_i)$ are mutually orthogonal. This is impossible since the $\theta(P_i)$ are already mutually orthogonal. Applying the additivity of θ again we see that $\theta(R) \perp \theta(Q)$.

Remark. In the above lemma, if θ is an ortho-isomorphism on $\mathscr{A}_p$, then both θ and λ are additive on mutually orthogonal projections from $\mathscr{A}_p$. A similar statement holds for θ' and λ' if θ' is an ortho-isomorphism.

Lemma 7. *Let M be any von Neumann algebra, P and Q noncentral orthogonal projections in M, μ a scalar, and $X\in M$ be such that*

(i) $X^*X-XX^*=P-Q+\mu I$,

(ii) $XP-PX=X=QX-XQ$.

Then $P\sim Q(M)$ and $\mu=0$.

Proof. (ii) implies that $(P+Q)X=XP-X+XQ+X=X(P+Q)$. Moreover, since both (i) and (ii) hold if X is replaced by $X(P+Q)$, we may assume that $P+Q=I$ by considering the algebra $(P+Q)M(P+Q)$ if necessary. Relation (ii) implies that if R is a projection and $R\leqq P$ then $RX=0$. Hence the projection on the complement of the null space of X (called the *initial projection* of X) is contained in P, and the projection on the closure of the range of X (called the *terminal* *projection* of X) is contained in Q. We further note that if $X=0$ then (i) implies that $Q=P+\mu I$ which is impossible.

Suppose that the initial projection is not P. Then there exists a projection R such that $0\neq R\leqq P$ and $XR=0$. Multiplying (i) on both sides by R we have $0=(1+\mu)R$ and so $\mu=-1$. Since $X\neq 0$ there exists a projection $S\leqq P$ such that $XS=X$. Again by (i), $X^*X=(1+\mu)S=0$ which implies $X=0$. Hence the initial projection of X is P. By similar means we could show that the terminal projection (which is inside Q) is not smaller than Q.

If $X=VH$ is the polar decomposition of X, then $H=(X^*X)^{1/2}$ and, by the above, $V^*V=P$, $VV^*=Q$. Relation (i) implies that

$$X^*X = P(X^*X-XX^*)P = (1+\mu)P, \tag{1}$$

$$-XX^* = Q(X^*X-XX^*)Q = (-1+\mu)Q. \tag{2}$$

But

$$XX^* = VHHV^* = VX^*XV^* = V(1+\mu)PV^* = (1+\mu)Q. \tag{3}$$

Adding (2) and (3) we see that $\mu=0$.

COROLLARY. *Suppose that $P_1,\ldots,P_n$ are mutually orthogonal, equivalent projections in M. If the $\theta(P_i)$ $(i=1,\ldots,n)$ are mutually orthogonal in N then they are equivalent (N) and $\lambda(P_i)=\lambda(P_j)$ for all $i\neq j$. If the $\theta'(P_i)$ are mutually orthogonal a similar statement holds for the $\theta'(P_i)$ and $\lambda'(P_i)$ $(i=1,\ldots,n)$.*

Proof. If $P_1\sim P_2$ then there exists a $V\in M$ such that $V^*V-VV^*=P_1-P_2$, $VP_1-P_1V=VP_1=V=P_2V-VP_2$. Applying ϕ and letting $X=\phi(V)$ we have

$$X^*X-XX^* = \theta(P_1)-\theta(P_2)+(\lambda(P_1)-\lambda(P_2))I,$$
$$X\theta(P_1)-\theta(P_1)X = X = \theta(P_2)X-X\theta(P_2).$$

If $\theta(P_1)$ is orthogonal to $\theta(P_2)$ then by Lemma 7, $\theta(P_1)\sim\theta(P_2)$ and

$$\mu = \lambda(P_1)-\lambda(P_2) = 0.$$

If $\theta'(P_1)$ is orthogonal to $\theta'(P_2)$ then

$$\begin{aligned} X^*X-XX^* &= -(I-\theta(P_1))+(I-\theta(P_2))+(\lambda(P_1)-\lambda(P_2))I \\ &= \theta'(P_2)-\theta'(P_1)+(\lambda'(P_2)-\lambda'(P_1)) \end{aligned}$$

and the roles of P_1 and P_2 in Lemma 7 are reversed.

THEOREM 2. *Let $\phi\colon M\to N$ be a Lie isomorphism of the infinite factor M onto the von Neumann algebra N. Then N is an infinite factor, and if P is a projection in M, either $\phi(P)=\theta(P)$ where θ is an ortho-isomorphism, or $\phi(P)=-\theta'(P)$ where θ' is an ortho-isomorphism.*

Proof. Choose mutually orthogonal, equivalent projections P_i $(i=1,2,3)$ of sum I in M and assume that $\theta(P_i)$ are orthogonal. Hence the $\theta(P_i)$ are $\sim$ and of sum I, and $\lambda(P_i)=\lambda(P_j)$ for $i\neq j$. Since $P_1+P_2\sim P_3$ we have $\theta(P_1)\sim\theta(P_3)\sim\theta(P_1)+\theta(P_2)$, and so N is infinite. Moreover, since $\lambda(P_1)=\lambda(P_3)=\lambda(P_1+P_2)=2\lambda(P_1)$ all the $\lambda(P_i)=0$. The relation $\phi(I)=\sum_{i=1}^{3}\phi(P_i)=\sum_{i=1}^{3}\theta(P_i)$ shows that $\phi(I)$ is a nonzero central projection and so $\phi(I)=I$. A similar computation in the θ' case shows that all the $\lambda'(P_i)=0$ and $\theta'(I)=-I$.

Suppose now that the $\theta(P_i)$ are mutually orthogonal and that Q, R are orthogonal projections in M. We shall show that $\theta(Q)\theta(R)=0$. The technique used here is similar to that of [5, Lemma 13]. Let $S=I-(Q+R)$.

Case (i). Suppose Q, R, S are all infinite. Write $Q=\sum_{i=1}^{3}Q^{(i)}$, $R=\sum_{i=1}^{3}R^{(i)}$, $S=\sum_{i=1}^{3}S^{(i)}$ where the $Q^{(i)}$ (resp. $R^{(i)}$, $S^{(i)}$) are mutually orthogonal, equivalent projections of sum Q (resp. R, S). Let $T^{(i)}=Q^{(i)}+R^{(i)}+S^{(i)}$ $(i=1,2,3)$. Then the $T^{(i)}$ are equivalent and of sum I.

If $\theta(Q)$, $\theta(R)$, $\theta(S)$ are $\mathrm{co}_\perp$ then the $\theta(Q^{(i)})$, $\theta(R^{(i)})$, and $\theta(S^{(i)})$ are $\mathrm{co}_\perp$, and so then are the $\theta(T^{(i)})$. Thus $\phi(I)=-I$ which is impossible. We conclude $\theta(Q)\cdot Q(R)=0$. If the $\theta'(P_i)$ are orthogonal a similar argument shows $\theta'(Q)\theta'(R)=0$.

Case (ii). Q, R infinite, S finite.

Choose orthogonal infinite projections T_i $(i=1,2,3,4)$ such that $Q=T_1+T_2$, $R=T_3+T_4$. Applying Case (i) to the T_i we see that $\theta(T_i)\theta(T_j)=0$ if $i\neq j$. Using the additivity of θ on the T_i we have $\theta(Q)\cdot\theta(R)=0$.

Case (iii). Q infinite, R finite, $I-Q$ infinite.

Choose $R'\perp R$ such that $I-Q=R+R'$. Since R' is infinite we have $\theta(R')\theta(Q)=0$. Hence $0=\theta(Q)\theta(I-Q)=\theta(Q)\theta(R+R')=\theta(Q)\theta(R)+\theta(Q)\theta(R')=\theta(Q)\theta(R)$.

Case (iv). Q infinite, R finite, $I-Q$ finite.

Choose $Q_1\perp Q_2$ where the Q_i $(i=1,2)$ are infinite such that $Q=Q_1+Q_2$. Then $\theta(Q_i)\cdot\theta(R)=0$ by Case (iii). Hence $\theta(Q)\theta(R)=0$ by the additivity of θ.

Case (v). Q, R finite.

Choose S, T infinite orthogonal projections such that $1-(S+T)=Q+R$. Then $0=\theta(Q+S)\theta(R+T)=\theta(Q)\cdot\theta(R)$.

Thus, if $Q\perp R$, $\theta(Q)\perp\theta(R)$ for all projections Q, R in M.

Finally, suppose we are in the case when θ conserves orthogonality and let R be an infinite projection. Write $R=\sum_{i=1}^{3}R_i$ where the R_i are mutually orthogonal and equivalent. The argument of the first part of the proof applies to show all the $\lambda(R_i)=0$, and so $\lambda(R)=0$. If P is finite, $I-P$ is infinite and we have

$$I=\phi(I)=\phi(P)+\phi(I-P)=I+(\lambda(P)+\lambda(I-P))I=I+\lambda(P)I$$

and so $\lambda(P)=0$. A similar argument holds if θ' conserves orthogonality, thus completing the proof.

We now consider the case when M is a finite factor. Notice that if M is finite then N is also. For, if N were infinite we could apply the above reasoning to ϕ^{-1} and conclude that M is infinite also. Moreover, if M is of type II_1 then so must N be of type II_1. For, if N were of type I_n, the operator I in N could be expressed as a sum of at most n mutually orthogonal projections.

We shall assume for the II_1-case that $\phi(I)=I$. (If $\phi(I)=\alpha I$, replace ϕ by the mapping $\phi'(A)=\phi(A)+(1-\alpha)\operatorname{Tr}_M(A)I$.)

THEOREM 3. *If $\phi: M\to M$ is a Lie isomorphism of the II_1-factor M onto the II_1-factor N, then either θ is an ortho-isomorphism, or θ' is an ortho-isomorphism.*

Proof. Let $\mathscr{A}$ be a MASA, i.e. a maximal abelian subalgebra, in M. Then $\operatorname{Tr}_M|\mathscr{A}_p$ has range [0, 1] and so we can choose orthogonal projections P_i $(i=1,2,3)$ in $\mathscr{A}$ such that $\operatorname{Tr}_M(P_i)=1/3$. If θ' is an ortho-isomorphism on $\mathscr{A}_p$ then the $\theta'(P_i)$ are orthogonal, equivalent, and of sum I. Hence $\operatorname{Tr}_N(\theta'(P_i))=1/3$, $\operatorname{Tr}_N(\theta(P_i))=2/3$. If θ is an ortho-isomorphism on $\mathscr{A}_p$ then $\operatorname{Tr}_N(\theta(P_i))=1/3$.

Assume θ is an ortho-isomorphism on $\mathscr{A}$ and let $P\in\mathscr{A}_p$. We claim that it is an ortho-isomorphism on any other MASA $\mathscr{B}$. In fact, let $Q\in\mathscr{B}_P$ with $\operatorname{Tr}_M(Q)=1/3$. Then $\operatorname{Tr}_M(P\vee Q)\leqq 2/3$ so that we may choose nonzero projections R, S with $P\vee Q$, R, S mutually orthogonal. Let $\mathscr{A}_1$ be any MASA containing P, R, S and $\mathscr{A}_2$ any MASA containing Q, R, S. If θ is not an ortho-isomorphism on $\mathscr{B}$, then θ is not an ortho-isomorphism on $\mathscr{A}_2$ since $\theta(Q)\in\mathscr{A}_2$ and $\operatorname{Tr}_N(\theta(Q))=2/3$. Since $\operatorname{Tr}_N(\theta(P))=1/3$, θ is an ortho-isomorphism on $\mathscr{A}_1$. Hence on $\mathscr{A}_1\cap\mathscr{A}_2$, an abelian algebra of dimension $\geqq 3$, θ and θ' are both ortho-isomorphisms which is impossible. Similarly for the case when θ' is an ortho-isomorphism.

LEMMA 8. *Let P be a noncentral projection in M. Then each $A\in M$ has a unique dissection $A=A_1+A_2$ where $PA_1=A_1P$ and $(I-P)A_2(I-P)=0=PA_2P$. One has $A_1=PAP+(I-P)A(I-P)$, $A_2=[[P,A],I-P]$.*

Proof. We need only show uniqueness. Suppose A_1, A_2 satisfy the hypotheses and $A_1+A_2=0$. Then $0=P(A_1+A_2)P=A_1P$ and

$$0 = (I-P)(A_1+A_2)(I-P) = A_1(I-P).$$

Thus $A_1=A_2=0$.

LEMMA 9. *Let $\mathscr{A}$ be a MASA in M. Then $\mathscr{A}$ contains a projection P such that $P\sim I-P(M)$.*

Proof. Suppose $\mathscr{A}$ is nonatomic. Let $\{R_\alpha\}_{\alpha\in\mathfrak{A}}$ be a (possibly void) maximal set of mutually orthogonal finite projections in $\mathscr{A}$, and let $Q=I-\sum_{\alpha\in\mathfrak{A}}R_\alpha$. Also $Q=\sum_{\beta\in\mathfrak{B}}Q_\beta$ where the Q_β are mutually orthogonal, countably decomposable projections in $\mathscr{A}$. For each $\beta\in\mathfrak{B}$ we can choose Q'_β such that $0\neq Q'_\beta\leqq Q_\beta$, $Q_\beta\neq Q'_\beta$, and $Q_\beta-Q'_\beta\sim Q'_\beta$. Now each algebra M_{R_α} $(=[R_\alpha AR_\alpha \mid A\in M])$ has a faithful numerical trace T_α, and $(\mathscr{A}_{R_\alpha}, T_\alpha)$ is a nonatomic, finite, measure algebra. Thus we can choose $R'_\alpha\leqq R_\alpha$ such that $T_\alpha(R'_\alpha)=T_\alpha(R_\alpha-R'_\alpha)$ and consequently $R'_\alpha\sim R_\alpha-R'_\alpha$. The desired projection is $P=\sum_{\alpha\in\mathfrak{A}}R'_\alpha+\sum_{\beta\in\mathfrak{B}}Q'_\beta$.

If $\mathscr{A}$ has an atom, then this atom is a minimal projection in M and so M is of type I_∞. Suppose that the collection of orthogonal atoms is finite with sum R. Then $I-R$ is infinite and contains no finite projection of M in $\mathscr{A}$. The above argument shows that we may write $R=R_1+R_2$ with the R_i mutually orthogonal and equivalent in $\mathscr{A}$. Then $P=R+R_1\sim R_2$ is the desired projection.

If the collection of atoms is infinite with least upper bound R, then R decomposes in the desired way. Since now $I-R$ is infinite or zero it also decomposes in the desired way.

LEMMA 10. *Let P be a projection such that $P\sim I-P(M)$, and V a partial isometry such that $V^*V=P$, $VV^*=I-P$. Let A be a selfadjoint operator with $0\leqq A\leqq I$, $A=AP=PA$ and set $R(A)=A+V(I-A)V^*+V(A(I-A))^{1/2}+(A(I-A))^{1/2}V^*$. Then $R=R(A)$ is a projection and if $W=i(V+V^*)$ then $R+WRW^*=I$.*

Proof. This is a straightforward computation utilizing the relations $V^2=0$, $AV=PV=0$, $VP=V$, $V^*V=P$ and $VV^*=I-P$.

LEMMA 11. *If ϕ is a Lie isomorphism of a factor M on a factor N then there exists a C^*-isomorphism $\tilde{\theta}$ of M on N such that either* (i) *$\phi(A)-\tilde{\theta}(A)$ is central for all $A\in M$, or* (ii) *$\phi(A)+\tilde{\theta}(A)$ is central for all A.*

Proof. On the set of projections M_P of M, either (1) $\phi=\theta+\lambda$ or (2) $\phi=-\theta+\lambda$ where θ is an ortho-isomorphism of M_P on N_P and λ is additive on orthogonal projections. Moreover, $\lambda(P)=0$ if there exists a dissection $I=P_1+\cdots+P_n$ with the P_i mutually orthogonal and $P_i\sim P$ (in this case we say P divides I). The latter statement holds true if M is infinite, and in the finite case when $\phi=\theta+\lambda$. If $\phi=-\theta+\lambda$ with $\lambda(\cdot)=2\,\mathrm{Tr}_M(\cdot)$ on rational projections, we replace ϕ by $\phi'(\cdot)=\phi(\cdot)-2\,\mathrm{Tr}_M(\cdot)$ and note that if the lemma is true for ϕ' it is true for ϕ.

Now, applying a theorem of Dye [5, Theorem 1], there exists a C^*-isomorphism $\tilde{\theta}$ of M on N which agrees with θ on M_P. It suffices to prove the lemma for self-adjoint operators A with $0\leqq A\leqq I$. By Lemma 9 choose a projection P such that $PA=AP$ and $P\sim I-P$ via V where $V^*V=P$, $VV^*=I-P$. We may additionally assume that $PA=AP=A$, since in general $A=AP+A(I-P)$ and the lemma would hold for each part.

Form the projection $R(A)$ of Lemma 10. Using the notation of Lemma 8 we have $R_1=R(A)_1=A+V(I-A)V^*$ and, since $A-A_1$ is a Lie triple product, $\phi(R(A)_1)=\phi(R(A))_1$. $\tilde{\theta}$ conserves Lie triple products and carriers so that again by Lemma 8, $\tilde{\theta}(R(A)_1)=\tilde{\theta}(R(A))_1$. Moreover, by Lemma 10, $R(A)$ divides I. There are thus two cases: (i) $\phi(R(A))=\tilde{\theta}(R(A))$ or (ii) $\phi(R(A))=-\tilde{\theta}(R(A))$.

Case (i). Since $AP=A$, A commutes with all projections $Q\leqq I-P$. Therefore, $\phi(A)\theta(I-P)=\mu\theta(I-P)$, and we can write (uniquely) $\phi(A)=B+\lambda I$ where $B\theta(P)=B$. Also if $W=\phi(V)$, then W is a partial isometry such that $W^*W=\theta(P)$, $WW^*=\theta(I-P)$. Now $\phi(VA)=\phi[V,A]=[W,B+\lambda I]=WB$ and $\phi(A-VAV^*)=\phi[V^*,VA]=[W^*,WB]=B-WBW^*$. Thus $\phi(R_1)=\phi(A-VAV^*)+\phi(I-P)=B-WBW^*+WW^*=B+W(I-B)W^*$. $\tilde{\theta}(R_1)=\tilde{\theta}(A)+\tilde{\theta}(V^*(I-A)V)$ and, as $\tilde{\theta}$ conserves carriers, $\tilde{\theta}(A)$ lives on $\theta(P)$, $\tilde{\theta}(V^*(I-A)V)$ lives on $\theta(I-P)$. Thus we must have $B=\tilde{\theta}(A)$, and so $\phi(A)-\tilde{\theta}(A)=\lambda I$.

Case (ii). In this case we have $\phi(R_1) = -\tilde{\theta}(R_1)$, $\phi(A) = B + \lambda I$ with B living on $\theta(P)$. However, in this case if $W = \phi(V)$, $WW^* = \theta(P)$, $W^*W = \theta(I-P)$. Calculating as before,

$$\begin{aligned} \phi(VA) &= \phi[V, A] = [W, B+\lambda I] = -BW, \\ \phi(A - VAV^*) &= [W^*, -BW] = B - W^*BW + \phi(I-P) \\ &= B - W^*BW - \phi(I-P) - \tilde{\theta}(R_1) \\ &= -\tilde{\theta}(A) - \tilde{\theta}(V(I-A)V^*). \end{aligned}$$

The part of $\phi(R_1)$ living on $\theta(P)$ is B, and the part of $-\tilde{\theta}(R_1)$ living on $\theta(P)$ is $-\tilde{\theta}(A)$. Thus $B = -\tilde{\theta}(A)$ and $\phi(A) + \tilde{\theta}(A) = \lambda I$.

THEOREM 4. *If $\phi: M \to N$ is a Lie isomorphism between the factors M and N, then ϕ has one of two forms:* (i) *$\phi = \tilde{\theta} + \lambda$ where $\tilde{\theta}$ is an isomorphism and λ is a $*$-linear functional which annihilates brackets or* (ii) *$\phi = -\tilde{\theta} + \lambda$ where $\tilde{\theta}$ is an anti-isomorphism and λ as before.*

Proof. In Case (i) of Lemma 11 we define $\lambda(A) = \phi(A) - \tilde{\theta}(A)$ and in Case (ii) $\lambda(A) = \phi(A) + \tilde{\theta}(A)$. In (i) $\tilde{\theta}$ must be an isomorphism. For, if it were an anti-isomorphism, $\lambda[A, B] = \phi[A, B] - \tilde{\theta}[A, B] = [\phi(A), \phi(B)] + [\tilde{\theta}(A), \tilde{\theta}(B)] = 2[\phi(A), \phi(B)]$. This implies all commutators are central which is impossible. In Case (ii), $\tilde{\theta}$ must be an anti-isomorphism by similar reasoning.

In both cases λ vanishes on brackets. For example in (ii),

$$\lambda[A, B] - \tilde{\theta}[A, B] = \phi[A, B] = [\phi(A), \phi(B)] = [\tilde{\theta}(A), \tilde{\theta}(B)] = -\tilde{\theta}[A, B].$$

COROLLARY. *If ϕ is as in Theorem* 4 *and M of type* III *on a separable Hilbert space, or of type I_∞, then ϕ is bounded.*

Proof. In these cases, by theorems of Brown and Pearcy [1], [2], the span of commutators is all of M. Hence, the λ of Theorem 4 is identically zero.

LEMMA 12. *Let M be a von Neumann algebra. If λ is a norm continuous linear functional on M which annihilates brackets and the center of M then $\lambda \equiv 0$.*

Proof. Given $A \in M$, the uniform closure of the convex hull of $[UAU^{-1} \mid U$ a unitary operator in $M]$ contains at least one central element [3, p. 272, Théorème 1]. Choose C in the center of M and elements $S_n = \sum \alpha_i^{(n)} U_i^{(n)} A (U_i^{(n)})^{-1}$ such that $S_n \to C$ uniformly. Then $\lambda(S_n) \to \lambda(C) = 0$. But for any unitary U, $0 = \lambda([U^{-1}, UA]) = \lambda(A) - \lambda(UAU^{-1})$ so that $\lambda(S_n) = \lambda(A)$ for each n. Therefore $\lambda(A) = 0$.

COROLLARY. *Let ϕ be a uniformly continuous Lie isomorphism between factors M and N. Then* (1) *if M is infinite, ϕ is either an isomorphism or the negative of an anti-isomorphism, and* (2) *if M is finite, ϕ is either an isomorphism or the negative of an anti-isomorphism* $+2\,\mathrm{Tr}_M(\cdot)$.

Proof. The λ in Theorem 4 is continuous in this case. Applying the above lemma and conserving the normalization of Lemma 11 we have the result.

Added in proof. The author has proved, as a generalization of the above, that if $\phi: M \to N$ is a Lie*-isomorphism between the von Neumann algebras M and N, M, N with no central summands of type I_1 or I_2, there exists a central projection C in M such that $\phi|_{M_C} = \sigma + \lambda$ where σ is a *-isomorphism and λ is a *-linear map from M into Z_N which annihilates brackets, and $\phi|_{M_{I-C}} = \sigma' + \lambda'$ where σ' is the negative of a *-anti-isomorphism and λ' has properties similar to those of λ.

As an application of this result it can be shown that a uniformly continuous group isomorphism between the unitary groups of two simple C^*-algebras induces a Lie *-isomorphism $\bar{\phi}$ between the ultra-weak closures of the universal representation algebras of the respective C^*-algebras, and that the above generalization applies to $\bar{\phi}$.

These results will appear as part of a later paper.

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