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# ON THE UPPER AND LOWER CLASS FOR STATIONARY GAUSSIAN PROCESSES(1)

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1. **Introduction.** Let $X=\{x(t); 0\leqq t\leqq 1\}$ be a real stationary Gaussian process defined on a probability space $(\Omega, \mathscr{B}, P)$. As is well known, a stationary Gaussian process is uniquely determined by its mean value $m=E(x(t))$ and covariance function $\rho(h)=E((x(t+h)-m)(x(t)-m))$. Without loss of generality, we may assume that $m=0$ and $\rho(0)=1$. Then we have $\sigma^2(h)=2(1-\rho(h))$, where $\sigma^2(h)$ denotes $E((x(t+h)-x(t))^2)$, and accordingly all the information about $X$ is contained in $\sigma^2$.

Ju. K. Beljaev [1] has shown that for stationary Gaussian processes $X$ the following alternatives take place: either all sample functions are continuous, or all sample functions are unbounded in every interval of finite length (0-1 law). In the continuous case, he has further generalized G. A. Hunt's results [5] concerning Hölder continuity of sample functions. Before stating his result, we shall define the upper and lower classes for $X$. If there exists a positive number $\delta$ such that $0<|t-s|<\delta$ $(0\leqq t, s\leqq 1)$ implies $|f(t)-f(s)|\leqq g(|t-s|)$, then it is said that $f$ satisfies Lipschitz's condition relative to $g$.

Definition 1. Let $X=\{x(t); 0\leqq t\leqq 1\}$ be a stationary Gaussian process. Then a monotone nondecreasing continuous function $\varphi$ defined on $[a, \infty)$ with $a>0$ is called a function belonging to *the upper class* (*with respect to the uniform continuity of* $X$), if almost all sample functions $x(t, \omega)$ satisfy Lipschitz's condition relative to $g(h)=\sigma(h)\varphi(h^{-1})$, i.e. for almost all $\omega$ there exists a $\delta(\omega)>0$ such that $0<|t-s|<\delta(\omega)$ implies

$$|x(t, \omega)-x(s, \omega)| \leqq \sigma(t-s)\varphi(1/|t-s|).$$

A monotone nondecreasing continuous function $\varphi$ is called a function belonging to *the lower class* (*with respect to the uniform continuity of* $X$), if almost all sample functions $x(t, \omega)$ do not satisfy Lipschitz's condition relative to $g(h)=\sigma(h)\varphi(h^{-1})$, i.e. for almost all $\omega$ there exists a sequence $\{t_n(\omega); n=1, 2, 3, \ldots\}\subset[0, 1]$ such that

$$|x(t_{2n})-x(t_{2n-1})| > \sigma(|t_{2n}-t_{2n-1}|)\varphi(1/|t_{2n}-t_{2n-1}|), \qquad n \geqq 1,$$

and $|t_{2n}-t_{2n-1}|\to 0$ as $n\to\infty$.

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(1) The results of this paper were partially reported in [8] without proof.

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The collection of functions belonging to the upper class is denoted by $\mathscr{U}^u$ and the one for the lower class is denoted by $\mathscr{L}^u$(4).

Using these notations, Beljaev's result is stated as follows: Let $X$ be a continuous stationary Gaussian process with 0-mean and assume that there exist positive constants $\delta$, $C_1 < C_2$, and $\alpha$ $(0<\alpha<2)$ such that for any $h \in (0, \delta)$

$$C_1 h^\alpha/|\log h| \leqq \sigma^2(h) \leqq C_2 h^\alpha/|\log h|,$$

and further $\sigma^2(h)$ is concave in $(0, \delta)$. Then the function $\varphi(t)=c\{\sigma(t^{-1})t^{\alpha/2}\}^{-1}$ belongs to $\mathscr{U}^u$ or $\mathscr{L}^u$ according as $c>(2C_2)^{1/2}2^{\alpha+1/2}(2^{\alpha/2}-1)^{-1}$ or $c<(2C_1)^{1/2}$. As is easily seen, we have unfortunately

$$(2C_2)^{1/2}\frac{2^{\alpha+1/2}}{2^{\alpha/2}-1} > (2C_1)^{1/2}, \qquad 0<\alpha<2,\ 0<C_1<C_2.$$

On the other hand, the final form about the Hölder continuity of Wiener process is known which states: A monotone nondecreasing and continuous function $\varphi$ defined on $[a, \infty)$ belongs to $\mathscr{U}^u$ or $\mathscr{L}^u$ according as

$$\text{(1)} \qquad \int_a^\infty \varphi(t)^3 \exp\left[-\tfrac{1}{2}\varphi^2(t)\right] dt < \infty \quad \text{or} \quad = \infty,$$

(cf. Chung-Erdös-Sirao [3]). So we can see that

$$\varphi(t) = \{2\log t + 5\log_{(2)} t + 2\log_{(3)} t + \cdots + 2\log_{(n-1)} t + (2+\varepsilon)\log_{(n)} t\}^{1/2},$$
$$t \geqq e^{e^{\cdot^{\cdot^{\cdot^{e}}}}} \qquad (n-1 \text{ times})$$

belongs to $\mathscr{U}^u$ or $\mathscr{L}^u$ according as $\varepsilon>0$ or $\varepsilon\leqq 0$, where $\log_{(k)} t$ denotes the $k$-fold iterated logarithm, i.e.

$$\log_{(k)} t = \underbrace{\log\log\cdots\log}_{(k \text{ times})} t.$$

Our main purpose in this paper is to give a criterion like (1), under certain conditions on $\sigma^2$, which decide if $\varphi$ belongs to $\mathscr{U}^u$ ($\mathscr{L}^u$) or not(5).

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2. **Results.** Throughout this paper, $X=\{x(t); 0\leqq t\leqq 1\}$ is a real, continuous and stationary Gaussian process with zero mean, defined on the probability space $(\Omega, \mathscr{B}, P)$. We denote the correlation function of $X$ by $\rho$ and assume $\rho(0)=1$, and accordingly $\sigma^2(h)=2(1-\rho(h))$.

(4) The superscript $u$ expresses "*with respect to the uniform continuity*".

(5) $\mathscr{U}^u$ and $\mathscr{L}^u$ denote the collection of upper functions and lower functions defined for Wiener process, respectively.

Let $g$ be a positive continuous function defined on $(0, 1)$. For $a \geqq 0$, define $\mathscr{E}(a)$ by

$$\mathscr{E}(a) = \left\{\omega; \limsup_{h\downarrow 0} \left[\frac{x(t,\omega)-x(s,\omega)}{g(|t-s|)}; 0 \leqq s, t \leqq 1, 0 < |t-s| \leqq h\right] \leqq a\right\}.$$

Then we have

THEOREM 1. *If $\sigma(h)/g(h)$ tends to zero with $h$, then it follows that for any $a>0$, $P(\mathscr{E}(a))=0$ or 1.*

From this theorem, we can easily see the following

COROLLARY 1.1. *If $\sigma(h)/g(h)$ tends to zero with $h$, then for any $a \geqq 0$*

$$P\left(\limsup_{h\downarrow 0} \left[\frac{x(t)-x(s)}{g(|t-s|)}; 0 \leqq s, t \leqq 1, 0 < |t-s| \leqq h\right] = a\right) = 1 \quad or \quad 0.$$

REMARK 1. The above corollary suggests that under Beljaev's assumption mentioned in §1, there exists a constant $C_0$ between $C_1$ and $C_2$ such that $\varphi(t)= c\{t^{\alpha/2}\sigma(t^{-1})\}^{-1}$ belongs to $\mathscr{U}^u$ or $\mathscr{L}^u$ according as $c>C_0$ or $c<C_0$.

Now we consider the following condition (A):

(A.1) For suitably chosen constants $\alpha$, $\beta$, $C_3$, $C_4$ and $\delta$ such that $0<\alpha<2$, $-\infty<\beta<\infty$, $0<C_3<C_4<\infty$ and $0<\delta<1$, it holds that for any $h\in(0,\delta)$

$$C_3h^\alpha/|\log h|^\beta \leqq \sigma^2(h) \leqq C_4h^\alpha/|\log h|^\beta \text{ (6)},$$

(A.2) $\sigma^2(h)$ is concave or convex in $(0, \delta)$, where $\delta$ is a constant mentioned in (A.1).

REMARK 2. The condition (A.1) is a slight generalization of the corresponding one in Beljaev's case, because $\beta$ is arbitrary in our case.

REMARK 3. For the existence of processes satisfying the condition (A), we have the following sufficient condition.

*A sufficient condition*: Let $f$ be the spectral density function of correlation function $\rho$. If $f$ satisfies the following two conditions, then $X$ satisfies the condition (A).

(i) There exist positive constants $C_3$, $C_4$ and $K$ such that

$$C_3 \leqq f(x)x^{\alpha+1}(\log x)^\beta \leqq C_4, \qquad x \geqq K,$$

where $0<\alpha<2$, $-\infty<\beta<\infty$.

(ii) $g(x)=x^2f(x)$ is two times differentiable in $x$, and for some $0<\varepsilon<1$ either one of the following (a) or (b) holds.

(a) $x^{3-\varepsilon}g''(x)$ is bounded from below, and $\liminf_{x\to\infty} x^{3-\varepsilon}g''(x)>0$.

(b) $x^{3-\varepsilon}g''(x)$ is bounded from above, and $\limsup_{x\to\infty} x^{3-\varepsilon}g''(x)<0$.

The proof of this statement will be given in §7. It is also shown that, the cases (a) and (b) correspond to the convexity and concavity of $\sigma^2$, respectively.

(6) If a separable stationary Gaussian process $X$ satisfy the condition (A.1), then $x(t, \omega)$ is continuous in $t$ with probability 1 (cf. [4]).

Now let $f$ be a spectral density such that

$$f(x) = c/(x^{\alpha+1}(\log x)^{\beta}), \qquad x \geqq K,$$

where $c>0$, $K\geqq 1$, $0<\alpha<2$ and $-\infty<\beta<\infty$. If $f'(x)$ and $f''(x)$ are bounded on $[0, K]$, then the corresponding process $X$ satisfies the condition (A), and $\sigma^2(h)$ is convex or concave in a small interval $(0, \delta)$ according as $1<\alpha<2$ (or $\alpha=1$, $\beta>0$) or $0<\alpha<1$ (or $\alpha=1$, $\beta<0$). In fact, we have for $g(x)=x^2 f(x)$

$$\frac{1}{c}\, g''(x) = \frac{1}{x^{\alpha+1}(\log x)^{\beta}}\left\{\alpha(\alpha-1)+\frac{(2\alpha-1)\beta}{\log x}+\frac{\beta(\beta+1)}{(\log x)^2}\right\}.$$

Then we can see for $0<\varepsilon<2-\alpha$

$$\liminf_{x\to\infty} x^{3-\varepsilon}g''(x) = \infty, \qquad 1 < \alpha < 2 \quad \text{or} \quad \alpha = 1 \quad \text{and} \quad \beta > 0$$

and

$$\limsup_{x\to\infty} x^{3-\varepsilon}g''(x) = -\infty, \qquad 0 < \alpha < 1 \quad \text{or} \quad \alpha = 1, \quad \beta < 0,$$

which show the convexity and concavity of $\sigma^2$ respectively. In the case $\alpha=1$, $\beta=0$, $\sigma^2$ is still concave if other conditions hold. But if we replace constant $c$ by a positive and bounded function, then there may happen both cases.

Now we state:

THEOREM 2. *Let $\varphi$ be a positive, continuous and nondecreasing function defined on $[a, \infty)$ with $a>0$. If the process satisfies the condition* (A) *and it holds that*

$$\int_a^{\infty} \varphi(t)^{4/\alpha-1} \exp\left[-\tfrac{1}{2}\varphi^2(t)\right] dt < \infty, \tag{2}$$

*then the function $\varphi$ belongs to $\mathscr{U}^u$.*

Under the same assumption on $X$ as in Theorem 2, we have

COROLLARY 2.1. *For any $\varepsilon>0$,*

$$\{2\log t+(4/\alpha+1)\log_{(2)} t+2\log_{(3)} t+\cdots+2\log_{(n-1)} t+(2+\varepsilon)\log_{(n)} t\}^{1/2} \in \mathscr{U}^u.$$

COROLLARY 2.2.

$$P\left(\limsup_{h\downarrow 0}\left[\frac{x(t)-x(s)}{\sigma(t-s)\{2|\log|t-s||\}^{1/2}};\, 0 \leqq s, t \leqq 1, 0 < |t-s| \leqq h\right] \leqq 1\right) = 1.$$

THEOREM 3. *Let $\varphi$ be a positive, continuous and nondecreasing function defined on $[a, \infty)$ with $a>0$. If the process $X$ satisfies the condition* (A.1) *and $\sigma^2$ is concave in a small interval* $(0, \delta)$, *and further*

$$\int_a^{\infty} \varphi(t)^{4/\alpha-1} \exp\left[-\tfrac{1}{2}\varphi^2(t)\right] dt = \infty, \tag{3}$$

*then the function $\varphi$ belongs to $\mathscr{L}^u$.*

Under the same assumption on $X$ as in Theorem 3, it follows

COROLLARY 3.1. *For any* $\varepsilon \geqq 0$,

$$\{2 \log t+(4/\alpha+1) \log_{(2)} t+2 \log_{(3)} t+\cdots+2 \log_{(n-1)}t+(2-\varepsilon) \log_{(n)} t\}^{1/2} \in \mathscr{L}^u.$$

Combining Corollary 2.1 and 3.1, we have

COROLLARY 3.2. *Under the same assumption on X as in Theorem* 3, *it holds that*

$$P\left(\limsup_{h\downarrow 0} \left[\frac{x(t)-x(s)}{\sigma(t-s)\{2|\log|t-s||\}^{1/2}}; 0 \leqq s, t \leqq 1, 0 < |t-s| \leqq h\right] = 1\right) = 1.$$

REMARK 4. As was stated already, our main purpose is to prove Theorems 2 and 3. But we remark here that the condition (A) excludes all the cases for $\alpha=0$ which contains the critical case whether all sample functions are continuous or not (cf. X. Fernique [4]).

Next, we shall state the corresponding results concerning the local continuity of $X$.

DEFINITION 2. Let $\psi$ be a function defined on $[a, \infty)$ with $a>0$. If, for almost all $\omega$, there exists a positive $\delta(\omega)$ such that $0<h<\delta(\omega)$ implies $|x(h, \omega)-x(0, \omega)| <\sigma(h)\psi(1/h)$, then $\psi$ is called a function belonging to *the upper class with respect to the local continuity of X*. If, for almost all $\omega$, there does not exist any positive $\delta$ with the above stated property, then $\psi$ is called a function belonging to *the lower class with respect to the local continuity of X.*

The collections of functions belonging to the upper and lower classes with respect to the local continuity of $X$ are denoted by $\mathscr{U}$ and $\mathscr{L}$, respectively.

REMARK 5. Let $t$ be a number between 0 and 1. If $\psi$ belongs to $\mathscr{U}$, then for almost all $\omega$ there exists $\delta(t, \omega)>0$ such that

$$|x(t+h, \omega)-x(t, \omega)| < \sigma(h)\psi(1/h), \qquad 0 < h < \delta(t, \omega),$$

because $X$ is stationary. We may also consider that the above inequality holds for any $h\,(\neq 0)$ between $-\delta(t, \omega)$ and $\delta(t, \omega)$, because $Y=\{y(t, \omega)=x(1-t, \omega); 0\leqq t\leqq 1\}$ is stochastically equivalent to $X$.

Using the notations $\mathscr{U}$ and $\mathscr{L}$, we have

THEOREM 4. *Let $\psi$ be a positive, continuous nondecreasing function on $[a, \infty)$ with $a>0$, $X$ be a process satisfying the condition* (A.1), *and $\sigma(h)$ be monotone non-decreasing for small $h>0$. If*

$$\int_a^\infty \frac{1}{t}\, \psi(t)^{2/\alpha-1} \exp\left[-\tfrac{1}{2}\psi^2(t)\right] dt < \infty, \tag{4}$$

*then $\psi$ belongs to $\mathscr{U}$.*

COROLLARY 4.1. *Under the same assumption on X as in Theorem* 4,

$$\{2 \log_{(2)} t+(2/\alpha+1) \log_{(3)} t+2 \log_{(4)} t+\cdots+2 \log_{(n-1)} t+(2+\varepsilon) \log_{(n)} t\}^{1/2}$$

*belongs to $\mathscr{U}$ if $\varepsilon>0$.*

THEOREM 5. *Let $\psi$ be a positive, continuous and nondecreasing function defined on* $[a, \infty)$ *with* $a>0$. *If X satisfies the condition* (A.1) *and* $\sigma^2$ *is concave in a small interval* $(0, \delta)$, *and further*

$$\int_a^\infty \frac{1}{t}\psi(t)^{2/\alpha-1} \exp\left[-\tfrac{1}{2}\psi^2(t)\right] dt = \infty, \tag{5}$$

*then $\psi$ belongs to $\mathscr{L}$.*

COROLLARY 5.1. *Under the same assumption on X as in Theorem* 5,

$$\{2\log_{(2)} t+(2/\alpha+1)\log_{(3)} t+2\log_{(4)} t+\cdots+2\log_{(n-1)} t+(2-\varepsilon)\log_{(n)} t\}^{1/2}$$

*belongs to $\mathscr{L}$ if* $\varepsilon \geqq 0$.

Combining Corollary 4.1 with the above one, we have

COROLLARY 5.2. *Under the same assumption on X as in Theorem* 5, *it holds that*

$$P\left(\limsup_{h\downarrow 0} \frac{x(h)-x(0)}{\sigma(h)\{2\log_{(2)}(1/h)\}^{1/2}} = 1\right) = 1.$$

Let us return to the uniform continuity of $X$.

THEOREM 6. *Let* $0<\alpha<2$, $-\infty<\beta<\infty$, $C_4>0$, *and suppose that*

$$\sigma^2(h) \leqq C_4 h^\alpha/|\log h|^\beta.$$

*If $\sigma^2$ is concave or convex in a small interval* $(0, \delta)$ *according as* $0<\alpha<1$ (*or* $\alpha=1$, $\beta\leqq 0$) *or* $1<\alpha<2$ (*or* $\alpha=1$, $\beta>0$), *then for any* $\varepsilon>0$ *there exists an* $h_0(\omega)$, *with probability* 1, *such that* $0<|t-s|<h_0(\omega)$ *implies*

$$|x(t, \omega)-x(s, \omega)| \leqq \left\{(2+\varepsilon)C_4 \frac{|t-s|^\alpha}{|\log|t-s||^{\beta-1}}\right\}^{1/2}, \qquad 0 \leqq s, t \leqq 1.$$

The above result is an improvement of the first half of Theorem 7 in Beljaev [1].
Now let $F$ be the spectral function of $X$, i.e.

$$\rho(h) = \int_{-\infty}^\infty \exp[ihx]\, dF(x) = 2\int_0^\infty \cos hx\, dF(x)(^7). \tag{6}$$

Then we have

COROLLARY 6.1. *Let X be a process satisfying*

$$\int_0^\infty x^\alpha |\log x|^\beta\, dF(x) < \infty, \qquad 0 < \alpha < 2, \quad -\infty < \beta < \infty.$$

*If $\sigma^2$ is concave or convex in a small interval* $(0, \delta)$ *according as* $0<\alpha<1$ (*or* $\alpha=1$,

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(7) $F$ is a nondecreasing function of symmetric variation with $F(\infty)-F(-\infty)=1$.

$\beta \leqq 0$) *or* $1<\alpha<2$ (*or* $\alpha=1$, $\beta>0$), *then for any* $\varepsilon>0$ *there exists an* $h_0(\omega)$, *with probability* 1, *such that*

$$|x(t,\omega)-x(s,\omega)| \leqq \varepsilon\left\{\frac{|t-s|^{\alpha}}{|\log|t-s||^{\beta-1}}\right\}^{1/2}, \quad 0 \leqq s, t \leqq 1,\ 0 < |t-s| < h_0(\omega).$$

This corresponds to Theorem 4 in G. A. Hunt [5].

3. **Proof of Theorem 1.** Since $\rho(t-s)$ is continuous in $(t,s) \in [0,1]\times[0,1]$, we have by Mercer's theorem

$$\rho(t-s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\varphi_n(t)\varphi_n(s),$$

where $\lambda_n$, $\varphi_n$ $(n=1,2,3,\ldots)$ are the eigenvalues and orthonormal eigenfunctions of the integral equation

$$\varphi(t) = \lambda\int_0^1 \rho(t-s)\varphi(s)\,ds,$$

and $\lambda_n>0$ because $\rho$ is positive definite. We remark that

$$\begin{aligned} |\varphi_n(t+h)-\varphi_n(t)| &= \left|\lambda_n\int_0^1 \{\rho(t+h-s)-\rho(t-s)\}\varphi_n(s)\,ds\right| \\ &\leqq \lambda_n\left\{\int_0^1 |\rho(t+h-s)-\rho(t-s)|^2\,ds\right\}^{1/2} \\ &= \lambda_n\left\{\int_0^1 |E[(x(t+h)-x(t))x(s)]|^2\,ds\right\}^{1/2} \\ &\leqq \lambda_n\sigma(h). \end{aligned} \tag{7}$$

Now let $\{y_n; n=1,2,3,\ldots\}$ be a sequence of mutually independent standard Gaussian random variables. Then, for any $t\in[0,1]$, the series

$$y(t,\omega) = \sum_{n=1}^{\infty}\frac{1}{\sqrt{\lambda_n}}y_n(\omega)\varphi_n(t)$$

converges in the square mean (cf. M. Loève [6, p. 478]), and by the three series theorem, converges with probability 1. So the Gaussian system $Y=\{y(r); 0\leqq r\leqq 1\}$ where $r$ denotes rational number is equivalent to the system $X'=\{x(r); 0\leqq r\leqq 1\}$. Next we consider the events $\mathscr{E}'(a)$ and $\mathscr{E}''(a;j)$ defined by

$$\mathscr{E}'(a) = \left\{\omega; \limsup_{h\downarrow 0}\left[\frac{x(r)-x(r')}{g(|r-r'|)}; 0\leqq r, r'\leqq 1, 0<|r-r'|\leqq h\right]\leqq a\right\},$$

$$\mathscr{E}''(a;j) = \left\{\omega; \limsup_{h\downarrow 0}\left[\sum_{n=j}^{\infty}\frac{1}{\sqrt{\lambda_n}}\frac{\varphi_n(r)-\varphi_n(r')}{g(|r-r'|)}y_n; 0\leqq r, r'\leqq 1,\right.\right.$$
$$\left.\left. 0<|r-r'|\leqq h\right]\leqq a\right\},$$
$$j=1,2,3,\ldots,$$

where $r$ and $r'$ denote rational numbers. Since $X$ and $g$ are continuous, the equivalence of $X'$ and $Y$ implies

$$P(\mathscr{E}(a)) = P(\mathscr{E}'(a)) = P(\mathscr{E}''(a; 1)).$$

Moreover (7) and the assumption on $\sigma(h)/g(h)$ tells

$$\mathscr{E}''(a; 1) = \mathscr{E}''(a; j), \qquad j = 1, 2, 3, \ldots.$$

Therefore, using the fact that $\mathscr{E}''(a; j)$ is a tail event, i.e. $\mathscr{E}''(a; j)$ depends only on $y_n$ with $n \geqq j$, we can see by Kolmogorov's 0-1 law $P(\mathscr{E}''(a; 1)) = 0$ or 1, and accordingly $P(\mathscr{E}(a)) = 0$ or 1. Q.E.D.

4. **Proof of Theorem 2.** We proceed along the line of T. Sirao [7], but the proof is somewhat complicated. So we divide it into several lemmas.

LEMMA 1. *Theorems* 2 *and* 3 *are true if they hold under the assumption that for* $t > e^e \vee a$[8]

$$\{2 \log t\}^{1/2} \leqq \varphi(t) \leqq \{2 \log t + (7/\alpha) \log_{(2)} t\}^{1/2} \text{[9]}. \tag{8}$$

**Proof.** Cf. T. Sirao [7, Lemma 1]. (The proof given there does not need any change for the present case.)

By Lemma 1, we may assume in the following proofs of Theorems 2 and 3 that (8) holds for any $t \geqq a$ and accordingly $\varphi(t)$ tends to infinity with $t$. Moreover Theorems 2 and 3 treat the Hölder continuity of stationary processes. So it is enough to consider the behavior of $x(t)$ in the time interval $[0, \delta]$. (Divide $[0, 1]$ into $2([1/\delta]+1)$[10] subintervals of $[k\delta, (k+1)\delta]$, $[(k+1/2)\delta, (k+3/2)\delta]$, $k = 0, 1, 2, \ldots, [1/\delta]+1$, and consider a pair of $(t, s)$ such that $|t-s| < \delta/2$, if necessary.) Therefore, despite the unnaturalness of assuming (A.1) in (0, 1), we may assume that the condition (A), especially (A.2), holds in (0, 1) if we regard $[0, \delta]$ as $[0, 1]$ for the convenience of description. Further we may regard by the same reason that $\sigma^2$ is monotone nondecreasing in (0, 1).

Now we define the event $E(p; k, l)$ by

$$\begin{aligned} E(p; k, l) &= \{\omega;\, x((k+l)/2^p, \omega) - x(k/2^p, \omega) \geqq \sigma(l/2^p)\varphi(2^p/l)\}, \\ &\qquad 0 \leqq k \leqq 2^p, \quad 0 \leqq l \leqq p^{1/\alpha}, \quad p = 1, 2, 3, \ldots. \end{aligned} \tag{9}$$

Then we have

LEMMA 2. *For any* $c \in (0, 1)$, *we have*

$$\sum_{p=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=[cp^{1/\alpha}]+1}^{[p^{1/\alpha}]} P(E(p; k, l)) < \infty \quad \textit{or} \quad = \infty$$

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(8) $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

(9) The coefficient 7 on the second term has no special meaning except that the integral (2) for right hand side is finite.

(10) $[x]$ denotes the greatest integer which does not exceed $x$.

*according as*

$$\int_a^\infty \varphi(t)^{4/\alpha-1} \exp\left[-\tfrac{1}{2}\varphi^2(t)\right] dt < \infty \quad or \quad = \infty.$$

**Proof.** By the monotonicity of $\varphi$, we may assume without loss of generality that $a$ is sufficiently large so that $\varphi(t)>1$ and accordingly $\varphi(t)\exp[-\varphi^2(t)/2]$ is non-increasing in $t \geqq a$. Then for $(p, l)$ with $2^p/l \geqq a$, we have

$$\begin{aligned}(10)\qquad &\frac{1}{2(2\pi)^{1/2}}\frac{1}{\varphi(2^p/l)}\exp\left[-\tfrac{1}{2}\varphi^2(2^p/l)\right]\\ &< P(E(p;k,l)) = \frac{1}{(2\pi)^{1/2}}\int_{\varphi(2^p/l)}^\infty \exp[-x^2/2]\,dx\\ &< \frac{1}{(2\pi)^{1/2}}\frac{1}{\varphi(2^p/l)}\exp\left[-\tfrac{1}{2}\varphi^2(2^p/l)\right].\end{aligned}$$

We can further see by the monotonicity of $\varphi$ that

$$\begin{aligned}(11)\qquad &\frac{1}{\varphi((1/c)p^{-1/\alpha}2^p)}\exp\left[-\frac{1}{2}\varphi^2\left(\frac{1}{c}p^{-1/\alpha}2^p\right)\right]\\ &\leqq \frac{1}{\varphi(2^p/l)}\exp\left[-\tfrac{1}{2}\varphi^2(2^p/l)\right]\\ &\leqq \frac{1}{\varphi(p^{-1/\alpha}2^p)}\exp\left[-\tfrac{1}{2}\varphi^2(p^{-1/\alpha}2^p)\right], \qquad [cp^{1/\alpha}] \leqq l \leqq [p^{1/\alpha}].\end{aligned}$$

So if $p_0$ and $C_5$ are so large that

$$\varphi(p_0^{-1/\alpha}2^{p_0}) \wedge (p_0^{-1/\alpha}2^{p_0}) > a \quad \text{and} \quad C_5 \geqq \frac{1}{(2\pi)^{1/2}}\left\{1-\frac{1}{2(1-p_0^{-1})^{1/\alpha}}\right\}^{-1},$$

then it follows from (10), (11) and (8) that

$$\begin{aligned}(12)\qquad &\sum_{p=p_0}^\infty \sum_{k=1}^{2^p} \sum_{l=[cp^{1/\alpha}]+1}^{[p^{1/\alpha}]} P(E(p;k,l))\\ &\leqq \frac{1}{(2\pi)^{1/2}}\sum_{p=p_0}^\infty \frac{2^p p^{1/\alpha}}{\varphi(p^{-1/\alpha}2^p)}\exp\left[-\tfrac{1}{2}\varphi^2(p^{-1/\alpha}2^p)\right]\\ &\leqq C_5\sum_{p=p_0}^\infty \left\{\frac{2^p}{p^{1/\alpha}}-\frac{2^{p-1}}{(p-1)^{1/\alpha}}\right\}p^{2/\alpha}\frac{1}{\varphi(p^{-1/\alpha}2^p)}\exp\left[-\tfrac{1}{2}\varphi^2(p^{-1/\alpha}2^p)\right]\\ &\leqq C_5\int_a^\infty \varphi(t)^{4/\alpha-1}\exp\left[-\tfrac{1}{2}\varphi^2(t)\right]dt.\end{aligned}$$

Similarly we can see the existence of $C_6$ such that

$$\begin{aligned}(13)\qquad &\sum_{p=p_0}^\infty \sum_{k=1}^{2^p} \sum_{l=[cp^{1/\alpha}]+1}^{[p^{1/\alpha}]} P(E(p;k,l))\\ &\geqq \frac{(1-c)}{2(2\pi)^{1/2}}\sum_{p=p_0}^\infty \frac{2^p p^{1/\alpha}}{\varphi(c^{-1}p^{-1/\alpha}2^p)}\exp\left[-\tfrac{1}{2}\varphi^2(c^{-1}p^{-1/\alpha}2^p)\right]\\ &\geqq C_6\sum_{p=p_0}^\infty \left(\frac{2^{p+1}}{c(p+1)^{1/\alpha}}-\frac{2^p}{cp^{1/\alpha}}\right)\varphi(c^{-1}p^{-1/\alpha}2^p)^{4/\alpha-1}\exp\left[-\tfrac{1}{2}\varphi^2(c^{-1}p^{-1/\alpha}2^p)\right]\\ &\geqq C_6\int_{c^{-1}p_0^{-1/\alpha}2^{p_0}}^\infty \varphi(t)^{4/\alpha-1}\exp\left[-\tfrac{1}{2}\varphi^2(t)\right]dt.\end{aligned}$$

Now (12) and (13) prove Lemma 2. Q.E.D.

According to Lemma 2, we can see in the present case

$$\sum_{p=1}^{\infty}\sum_{k=1}^{2^p}\sum_{l=[p^{1/\alpha}/2]}^{[p^{1/\alpha}]} P(E(p;k,l)) < \infty. \tag{14}$$

This means by Borel-Cantelli's lemma that for almost all $\omega$ there exists $p_0(\omega)$ such that $p>p_0(\omega)$ implies

$$x((k+l)/2^p,\omega)-x(k/2^p,\omega) < \sigma(l/2^p)\varphi(2^p/l), \quad 0 \leqq k \leqq 2^p,\ [\tfrac{1}{2}p^{1/\alpha}] \leqq l \leqq [p^{1/\alpha}].$$

It is our purpose to show that the above inequality holds for any pair $(s,t)$ with $0<|t-s|<1/2^{p_0(\omega)}$. To show this, we consider the following events $F(q;m,n)$ for fixed $(p,k,l)$. Let $c$ be a large number which makes $e^c$ an integer and satisfies inequalities

$$\begin{aligned}(2/\pi)^{1/2}\exp\left[-\frac{C_7^2}{8}(e^{c/4}/2)^{2\alpha(q-1)}\right] &< C_7 e^{-3qc}, \qquad q \geqq 2,\\ C_7(e^{c/4}/2)^{\alpha(q-1)} &> 2C_8\end{aligned} \tag{15}$$

where $C_7$ and $C_8$ are positive constants to be defined later. Now let us set $b_1=k/2^p$, $b_2=(k+l)/2^p$, $h_q=\exp[-qc]/2^p$ and

$$\begin{aligned}F(q;m,n) = \Big\{\omega;\ & x(b_2-mh_q,\omega)-x(b_1+nh_q,\omega)\\ & \geqq \sigma(b_2-b_1-(m+n)h_q)\Big\{\varphi+\frac{2c}{\varphi}\sum_{i=0}^{q-1}\frac{1}{2^{\alpha i}}\Big\}\Big\},\\ & 0 \leqq m,n \leqq \exp[qc],\end{aligned}$$

where $\varphi$ denotes $\varphi(2^p/l)=\varphi(1/(b_2-b_1))$. Further set

$$S_q = \{m;\, 0 \leqq m \leqq \exp[qc]\}, \qquad F_q = \bigcup_{m,n\in S_q} F(q;m,n),$$

and denote the smallest integer $p'$ satisfying (16) by $p_1$

$$2^{p+1} \geqq p^{1/\alpha}, \qquad p\log 2 \geqq 2\log p/\alpha. \tag{16}$$

The following lemma plays an essential role, and we need several lemmas to prove it.

LEMMA 3. *Under the assumption of Theorem 2, there exists an absolute constant*(11) $C_9$ *such that for* $p \geqq p_1$

$$P(F_q) \leqq C_9 P(E(p;k,l)), \qquad q \geqq 1. \tag{17}$$

**Proof.** We consider a fixed triple $(p,k,l)$ with relation of $p\geqq p_1$. Denoting the

(11) An absolute constant means a constant independent of $(p,k,l)$ and $q$.

complementary event of $F$ by $F'$, we have

$$P(F_q) = P(F_{q-1})+P(F'_{q-1} \cap F_q) \leqq P(F_{q-1})+ \sum_{m,\, n\in S_q} P(F'_{q-1} \cap F(q; m, n)), \qquad q \geqq 2. \tag{18}$$

Let us denote the second term in the right-hand side of (18) by $P_1$ and estimate it. We denote the interval $(b_1+nh_q, b_2-mh_q)$ by $I(q; m, n)$. Next, for each pair $(m, n)$ of $m, n \in S_q$, we choose two elements $m_1$, $n_1$ of $S_{q-1}$ in such a way that

$$|mh_q-m_1h_{q-1}|, \qquad |nh_q-n_1h_{q-1}| \leqq h_{q-1}, \tag{19}$$

and, if $\rho$ is convex in $(0, \delta)$, $I(q; m, n)\subset I(q-1; m_1, n_1)$ and, if $\rho$ is concave in $(0, \delta)$,

$$I(q; m, n)-I(q-1; m_1, n_1) \neq \varnothing, \qquad I(q-1; m_1, n_1)-I(q; m, n) \neq \varnothing.$$

Then we have

$$P_1 = \sum_{m,\, n\in S_q} P(F'_{q-1} \cap F(q; m, n)) \leqq \sum_{m,\, n\in S_q} P(F(q-1; m_1, n_1)' \cap F(q; m, n)). \tag{20}$$

To estimate the right-hand side of (20), we use the following lemmas.

LEMMA 4. (i) *If the correlation function* $\rho$ *is convex in* $(0, 1)$, *then it holds that for any pair of nonoverlapping intervals* $(a, b)$, $(c, d)\subset(0, 1)$

$$E((x(b)-x(a))(x(d)-x(c))) \leqq 0, \tag{21}$$

*and for* $(c, d)\subset(a, b)\subset(0, 1)$

$$\sigma^2(d-a)+\sigma^2(b-c) \geqq \sigma^2(b-a)+\sigma^2(d-c). \tag{22}$$

(ii) *If the correlation function* $\rho$ *is concave in* $(0, 1)$, *then it holds that for any pair of nonoverlapping intervals* $(a, b)$, $(c, d)\subset(0, 1)$

$$E((x(b)-x(a))(x(d)-x(c))) \geqq 0, \tag{23}$$

*and for any pair of overlapping intervals* $(a, b)$, $(c, d)\subset(0, 1)$

$$\sigma^2(d-a)+\sigma^2(b-c) \geqq \sigma^2(b-a)+\sigma^2(d-c). \tag{24}$$

**Proof.** We prove only (i) because (ii) can be proved similarly. Let $0\leqq a\leqq b\leqq c \leqq d\leqq 1$. Then we have

$$E((x(b)-x(a))(x(d)-x(c))) = \{\rho(d-b)-\rho(d-a)\}-\{\rho(c-b)-\rho(c-a)\}.$$

Since $(d-b)-(d-a)=(c-b)-(c-a)$ and $\rho$ is convex in $(0, 1)$, the right-hand side of the above equality is nonpositive. So we have (21).

Next, let $0 \leqq a \leqq c \leqq d \leqq b \leqq 1$. Then we have

$$\begin{aligned}\{\sigma^2(d-a)+\sigma^2(b-c)\}-\{\sigma^2(b-a)+\sigma^2(d-c)\} \\ &= 2\{\rho(b-a)+\rho(d-c)\}-2\{\rho(d-a)+\rho(b-c)\} \\ &\geqq 0,\end{aligned}$$

because $d-c \leqq (d-a) \wedge (b-c) \leqq b-a$ and $(b-a)+(d-c)=(d-a)+(b-c)$. Q.E.D

LEMMA 5. *Let $(U, V)$ be a two dimensional Gaussian random variable with $E(U)=E(V)=0$ and $E(U^2)=E(V^2)=1$. Then, for a pair $(a, b)$ with $0<a<b$, the function $p(a, b; \rho)=P(U<a, V>b)$ is monotone decreasing in $\rho=E(UV)$.*

**Proof.** See Lemma 2 in T. Sirao [7].

Let us set, for any $m, n \in S_q$ and $m_1, n_1 \in S_{q-1}$ which are chosen by the way stated already, $a=b_1+nh_q$, $b=b_2-mh_q$, $c=b_1+n_1h_{q-1}$ and $d=b_2-m_1h_{q-1}$.

LEMMA 6. *Let $\rho$ be the correlation coefficient between $x(b)-x(a)$ and $x(d)-x(c)$. Then there exists a positive constant $C_{10}$ such that*

$$\rho \geqq 1-C_{10}(1/p) \exp[-(q-1)c\alpha/2]. \tag{25}$$

**Proof.** We can see from (22) and (24) that

$$\sigma^2(d-a)+\sigma^2(b-c) \geqq \sigma^2(b-a)+\sigma^2(d-c).$$

Hence we have

$$\begin{aligned}E((x(b)-x(a))(x(d)-x(c))) &= \tfrac{1}{2}\{\sigma^2(d-a)+\sigma^2(b-c)-\sigma^2(d-b)-\sigma^2(c-a)\} \\ &\geqq \tfrac{1}{2}\{\sigma^2(b-a)+\sigma^2(d-c)-\sigma^2(d-b)-\sigma^2(c-a)\}.\end{aligned}$$

So it follows from the monotonicity of $\sigma^2$ that

$$\begin{aligned}\rho &\geqq 1-\frac{1}{2}\frac{\sigma^2(d-b)+\sigma^2(c-a)}{\sigma(b-a)\sigma(d-c)} \\ &\geqq 1-\frac{\sigma^2(h_{q+1})}{\sigma^2(b_2-b_1-2h_q\exp[qc])}.\end{aligned}$$

Now put

$$M = \sup\left\{\frac{p}{(l-2)^\alpha \exp[(q-1)c\alpha/2]}\left|\frac{p\log 2-\log(l-2)}{p\log 2+(q-1)c}\right|^\beta; p \geqq p_1, q \geqq 2, \right.$$
$$\left. [\tfrac{1}{2}p^{1/\alpha}] \leqq l \leqq [p^{1/\alpha}], c \geqq 2\right\}.$$

Evidently, by (16), $M$ is finite. Then we have by (A.1)

$$\begin{aligned}\rho &\geqq 1-\frac{C_4}{C_3}\left\{\frac{l}{(l-2)\exp[(q-1)c]}\right\}^\alpha \left|\frac{p\log 2-\log(l-2)}{p\log 2+(q-1)c}\right|^\beta \\ &\geqq 1-\frac{C_4}{C_3}\frac{M}{p}\exp[-(q-1)c\alpha/2].\end{aligned}$$

So, putting $C_{10}=C_4M/C_3$, we have (25). Q.E.D.

Now let us go back to the estimation of $P_1$. Let $Y$ and $Z$ be mutually independent Gaussian random variables with $E(Y)=E(Z)=0$ and $E(Y^2)=E(Z^2)=1$. Then, using the notations in Lemma 6, we have

$$
\begin{aligned}
&P(F(q-1;m_1,n_1)'\cap F(q;m,n))\\
&\quad = P\left(x(b)-x(a) \geqq \sigma(b-a)\left(\varphi+\frac{2c}{\varphi}\sum_{i=0}^{q-1}2^{-\alpha i}\right),\right.\\
&\qquad\qquad \left. x(d)-x(c) < \sigma(d-c)\left(\varphi+\frac{2c}{\varphi}\sum_{i=0}^{q-2}2^{-\alpha i}\right)\right) \qquad (26)\\
&\quad = P\left(Y \geqq \varphi+\frac{2c}{\varphi}\sum_{i=0}^{q-1}2^{-\alpha i},\ (1-\rho^2)^{1/2}Z+\rho Y<\varphi+\frac{2c}{\varphi}\sum_{i=0}^{q-2}2^{-\alpha i}\right).
\end{aligned}
$$

Put $\rho_0=1-C_{10}\exp[-(q-1)c\alpha/2]/p$, and we can see from (25), (26) and Lemma 5 that

$$
\begin{aligned}
&P(F(q-1;m_1,n_1)'\cap F(q;m,n))\\
&\quad \leqq P\left(Y \geqq \varphi+\frac{2c}{\varphi}\sum_{i=0}^{q-1}2^{-\alpha i},\ (1-\rho_0^2)^{1/2}Z<(1-\rho_0)\left(\varphi+\frac{2c}{\varphi}\sum_{i=0}^{q-2}2^{-\alpha i}\right)\right.\\
&\qquad\qquad\qquad \left. -\rho_0\frac{2c}{\varphi}2^{-\alpha(q-1)}\right)\\
&\quad = P\left(Y \geqq \varphi+\frac{2c}{\varphi}\sum_{i=0}^{q-1}2^{-\alpha i}\right)P\left(Z < (1-\rho_0^2)^{-1/2}\right.\\
&\qquad\qquad \left.\cdot\left\{(1-\rho_0)\left(\varphi+\frac{2c}{\varphi}\sum_{i=0}^{q-2}2^{-\alpha i}-\rho_0\frac{2c}{\varphi}2^{-\alpha(q-1)}\right)\right\}\right)\\
&\quad = P_2 \text{ (say)}.
\end{aligned}
$$

Now we have by Lemma 1

$$\varphi = \varphi(2^p/l) \leqq \{2p\log 2+7\log\log p/\alpha\}^{1/2}.$$

Hence there exist absolute constants $c$, $C_8$ and $C_7$ such that they satisfy (15) and the following two inequalities

$$(1-\rho_0^2)^{-1/2}(1-\rho_0)\left(\varphi+\frac{2c}{\varphi}\sum_{i=0}^{q-2}2^{-\alpha i}\right) \leqq C_8 < \infty,$$

$$(1-\rho_0^2)^{-1/2}\rho_0\frac{2c}{\varphi} \geqq C_7\exp[\tfrac{1}{4}\alpha(q-1)c].$$

So we can see by (15) and (10)

$$
\begin{aligned}
P_2 &\leqq P(Y\geqq\varphi)P(Z < C_8-C_7(e^{c/4}/2)^{\alpha(q-1)})\\
&\leqq P(Y\geqq\varphi)P\left(Z > \frac{C_7}{2}(e^{c/4}/2)^{\alpha(q-1)}\right)\\
&\leqq P(Y\geqq\varphi)\left(\frac{1}{2\pi}\right)^{1/2}\frac{2}{C_7}(e^{c/4}/2)^{-\alpha(q-1)}\exp\left[-\frac{C_7^2}{8}(e^{c/4}/2)^{2\alpha(q-1)}\right] \qquad (27)\\
&< e^{-3qc}P(E(p;k,l)).
\end{aligned}
$$

Then it follows from (27) that

$$P(F(q-1; m_1, n_1)' \cap F(q, m, n)) \leqq e^{-3qc}P(E(p; k, l)), \qquad m, n \in S_q,\ q \geqq 2,$$

and accordingly

$$P_1 \leqq e^{-qc}P(E(p; k, l)). \tag{28}$$

Since

$$P(F_1) \leqq \sum_{m,\, n \in S_1} P(F(1; m, n)) = e^{2c}P(E(p; k, l)),$$

we get from (18) and (28)

$$P(F_q) \leqq \left(e^{2c} + \sum_{t=2}^{\infty} e^{-tc}\right) P(E(p; k, l)), \qquad q \geqq 2,$$

which proves Lemma 3. Q.E.D.

Now we are in a position to prove Theorem 2. Set

$$G(p; k, l) = \bigcup_{n=1}^{\infty} \bigcap_{q=n}^{\infty} F_q$$

and

$$H(p; k, l) = \left\{\omega; \sup \frac{x(t)-x(s)}{\sigma(t-s)} \geqq \varphi(2^p/l) + \frac{2c}{\varphi(2^p/l)} \sum_{t=0}^{\infty} 2^{-\alpha t}\right\},$$

where $s$ and $t$ runs over the intervals $[k2^{-p}, (k+1)2^{-p}]$ and $[(k+l-1)2^{-p}, (k+l)2^{-p}]$ respectively. Then it follows from the continuity of $X$ and Lemma 3 that

$$\begin{aligned} P(H(p; k, l)) &\leqq P(G(p; k, l)) \\ &\leqq \liminf_{q\to\infty} P(F_q) \\ &\leqq C_9 P(E(p; k, l)). \end{aligned}$$

Therefore we have by Lemma 2

$$\sum_{p=p_1}^{\infty} \sum_{k=0}^{2^p} \sum_{l=[p^{1/\alpha}/2]+1}^{[p^{1/\alpha}]} P(H(p; k, l)) < \infty. \tag{29}$$

According to Borel-Cantelli's lemma, (29) shows that for almost all $\omega$ there exists $p_2(\omega)$ such that

$$\omega \notin H(p; k, l) \quad \text{for } p \geqq p_2(\omega), \qquad 0 \leqq k \leqq 2^p, \quad [\tfrac{1}{2}p^{1/\alpha}] < l \leqq [p^{1/\alpha}].$$

Now, for any pair $(s, t)$ satisfying $0 \leqq s < t \leqq 1$ and $0 < t-s < p_2(\omega)^{1/\alpha}/2^{-p_2(\omega)}$, choose $p$, $k$ and $l$ such that

$$\frac{(p+1)^{1/\alpha}}{2^{p+1}} \leqq t-s < \frac{p^{1/\alpha}}{2^p}, \qquad \frac{k}{2^p} \leqq s < \frac{k+1}{2^p} < \frac{k+l-1}{2^p} \leqq t < \frac{k+l}{2^p}.$$

Then it holds that $p_2(\omega) \leqq p$, $0 \leqq k < 2^p$ and $p^{1/\alpha}/2 \leqq l \leqq p^{1/\alpha}$. Accordingly the fact that $\omega \notin H(p; k, l)$ implies

$$x(t, \omega)-x(s, \omega) < \sigma(t-s)\left\{\varphi(2^p/l)+\frac{2c}{\varphi(2^p/l)}\sum_{i=0}^{\infty} 2^{-\alpha i}\right\}$$
$$\leqq \sigma(t-s)\left\{\varphi(1/(t-s))+\frac{2c}{\varphi(1/(t-s))}\sum_{i=0}^{\infty} 2^{-\alpha i}\right\}.$$

Then, taking into consideration the symmetry of a Gaussian process, we can see that $\varphi+2c'/\varphi$ belongs to $\mathscr{U}^u$, where $c'=c\sum_{i=0}^{\infty} 1/2^{\alpha i}$. Since this result is obtained from the assumption of convergence of (2), the same result also should hold for $\tilde{\varphi}(t)=\varphi(t)-3c'/\varphi(t)$, because $\tilde{\varphi}$ is nondecreasing, continuous on $[a, \infty)$ and further the integral (2) for $\tilde{\varphi}$ is finite. Moreover $\tilde{\varphi}(t)+2c'/\tilde{\varphi}(t)<\varphi(t)$ for large $t$, as is easily seen. Hence $\varphi$ should belong to $\mathscr{U}^u$. Q.E.D.

5. **Proof of Theorem 3.** We shall use the same notations as in §4.

According to Lemma 2, the divergence of integral (3) implies

$$\sum_{p=1}^{\infty}\sum_{k=0}^{2^p}\sum_{l=[p^{1/\alpha}/3]}^{[p^{1/\alpha}]} P(E(p; k, l)) = \infty. \tag{30}$$

By the definition of $\mathscr{L}^u$, the function $\varphi$ belongs to $\mathscr{L}^u$ if $E(p; k, l)$ occurs "*infinitely often*" for almost all $\omega$. To prove this is the case, we apply the Chung-Erdös lemma[12].

In the sequel, we often denote $E(p; k, l)$ by $E_n$, where the subscript $n$ is given in the following way. Let $E_n=E(p; k, l)$ and $E_m=E(p'; k', l')$. Then $n$ stands before $m$ if and only if either one of the following hold: (i) $p<p'$, (ii) $p=p'$ and $l'<l$, (iii) $p=p'$, $l=l'$ and $k<k'$. Hence $n<m$ implies $l'/2^{p'} \leqq l/2^p$, where $(p, l)$ and $(p', l')$ correspond to $E_n$ and $E_m$ respectively.

Now, by the Chung-Erdös lemma and (30), it suffices to prove the following Lemma 7 and Lemma 8.

LEMMA 7. *For every pair of* $(n, h)$ *with* $n \geqq h$, *there exist* $c(h)>0$ *and* $H(n, h)>n$ *such that for any* $m \geqq H(n, h)$

$$P(E_m/E_h' \cap E_{h+1}' \cap \cdots \cap E_n') \geqq c(h)P(E_m)\text{[13]}. \tag{31}$$

LEMMA 8. *There exist two absolute constants* $K_1$ *and* $K_2$ *with the following property: to each* $E_j$ *there corresponds a set of events* $\{E_{j_i}; i=1, 2, 3, \ldots, s\} \subset \{E_n; n=1, 2, 3, \ldots\}$ *such that*

$$\sum_{i=1}^{s} P(E_j \cap E_{j_i}) \leqq K_1 P(E_j), \tag{32}$$

(12) See [2].

(13) $P(E/F)$ denotes the conditional probability of $E$ under the condition $F$.

*and that for all* $E_k \neq E_{j_i}$ $(i=1, 2, 3, \ldots, s)$ *provided* $k>j$

$$P(E_j \cap E_k) \leqq K_2 P(E_j)P(E_k). \tag{33}$$

Before we prove Lemma 7, we shall state a remark and Lemma 9. For each $E_n = E(p; k, l)$, we define $U_n = U(p; k, l)$ by

$$U_n = U(p; k, l) = x((k+l)/2^p) - x(k/2^p),$$

where $U(p; k, l)$ is used when we want to emphasize that $U_n$ corresponds to $E(p; k, l)$. Then we have for every $n$

$$\lim_{m\to\infty} \rho(U_n, U_m) = 0, \tag{34}$$

where $\rho(U, V)$ denotes the correlation coefficient between $U$ and $V$. In fact, (34) is proved in the following way. Let $U_n = x(b) - x(a)$ and $U_m = x(d) - x(c)$. Since we may regard $\sigma^2$ as a monotone and concave function as was remarked already, it holds by (21)

$$\begin{aligned} \rho(U_n, U_m) &\leqq \frac{\sigma^2(d-c)}{\sigma(b-a)\sigma(d-c)} \\ &= \frac{\sigma(d-c)}{\sigma(b-a)} \to 0 \quad \text{as } m \to \infty, \end{aligned}$$

because $d-c$ tends to 0 as $m \to \infty$.

LEMMA 9. *Let* $\{X_1, X_2, \ldots, X_k, Y_m; m=1, 2, 3, \ldots\}$ *be a sequence of standard Gaussian random variables, and assume that* $\rho_m = \max\{|\rho_{i,m}|; 0 \leqq i \leqq k\} \to 0$ *as* $m \to \infty$, *where* $\rho_{i,m}$ *denotes* $E(X_i Y_m)$. *Then, for any sequence of Borel sets* $B_m \subset [\rho_m^{-c}, \rho_m^{c}]$ *provided* $0 \leqq c < 1$ *and for any bounded Borel sets* $A_i$ $(i=1, 2, \ldots, k)$, *it holds*

$$P(Y_m \in B_m / X_i \in A_i,\ i = 1, 2, \ldots, k)/P(Y_m \in B_m) \to 1 \quad \text{as } m \to \infty.$$

**Proof.** See Lemma 4 in T. Sirao [7].

Now we shall prove Lemma 7.

**Proof of Lemma 7.** Let us set for $U_n = U(p; k, l)$ and $c > 0$

$$\begin{aligned} F_n(c) &= \{\omega; \varphi(2^p/l) \leqq U_n/\sigma(l/2^p) \leqq \varphi(2^p/l) + c\}, \\ E_n(c) &= \{\omega; U_n + c \geqq 0\}. \end{aligned}$$

Then we have by (10)

$$\begin{aligned} P(F_n(c)) &\geqq \frac{1}{2(2\pi)^{1/2}} \frac{1}{\varphi_n} \exp[-\tfrac{1}{2}\varphi_n^2] - \left(\frac{2}{2\pi}\right)^{1/2} e^{-c\varphi_n} \frac{1}{\varphi_n} \exp[-\tfrac{1}{2}\varphi_n^2] \\ &\geqq \tfrac{1}{2} P(E_n)(1 - 2e^{-c\varphi_n}), \end{aligned}$$

where $\varphi_n$ denotes $\varphi(2^p/l)$. So, for a given pair $(n, h)$, we can take $c > 0$ such that

$$P(F_m(c)) \geqq \tfrac{1}{3} P(E_m), \qquad m \geqq n, \tag{35}$$

and

$$P\left(\bigcap_{i=h}^{n}(E_i' \cap E_i(c))\right) \geqq \tfrac{1}{2}P\left(\bigcap_{i=h}^{n} E_i'\right).$$

Then it holds that

$$\begin{aligned} &P(E_m/E_h' \cap E_{h+1}' \cap \cdots \cap E_n') \\ (36)\qquad &= P(E_m \cap E_h' \cap E_{h+1}' \cap \cdots \cap E_n')/P(E_h' \cap E_{h+1}' \cap \cdots \cap E_n') \\ &\geqq \tfrac{1}{2}P\left(F_m(c)\Big/\bigcap_{i=h}^{n}(E_i' \cap E_i(c))\right). \end{aligned}$$

On the other hand, if we put

$$X_i = U_{h+i}, \qquad A_i = [-c, \varphi(2^{p_i}/l_i)], \qquad i = 0, 1, 2, \ldots, n-h,$$

and

$$Y_m = U_m, \qquad B_m = [\varphi(2^{p_m}/l_m), \varphi(2^{p_m}/l_m)+c], \qquad m = n+1, n+2, \ldots,$$

then, applying Lemma 9, we have for large $m$

$$P\left(F_m(c)\Big/\bigcap_{i=h}^{n}(E_i' \cap E_i(c))\right) > \tfrac{1}{2}P(F_m(c)).$$

Therefore (35) and (36) show that there exists an $H(n, h) > n$ such that

$$P(E_m/E_h' \cap E_{h+1}' \cap \cdots \cap E_n') \geqq P(E_m)/12, \qquad m \geqq H(n, h),$$

which proves Lemma 9 for $C(h) = 1/12$. Q.E.D.

The proof of Lemma 8 is complicated and we need some lemmas for it.

LEMMA 10. *Let $(U, V)$ be a two dimensional Gaussian random variable with $E(U)=E(V)=0$ and $E(U^2)=E(V^2)=1$. Then for any $a, b>0$, there exist positive constants $K$ and $d$ such that* (i)

$$(37)\qquad P(U > a, V > a) \leqq K \exp[-d(1-\rho^2)a^2]P(U > a),$$

*where $\rho$ denotes $\rho(U, V)$, and* (ii) *if $\rho < 1/ab$, then*

$$(38)\qquad P(U > a, V > b) \leqq KP(U > a)P(V > b).$$

**Proof.** See Lemmas 3 and 4 of Chung-Erdös-Sirao [3].

Now, for each $E_j = E(p; k, l)$, let $\boldsymbol{E}_j$ be the collection of $E_n = E(p'; k', l')$ such that $n > j$ and

$$(39)\qquad \rho(U_j, U_n) \geqq \{\varphi(2^p/l)\varphi(2^{p'}/l')\}^{-1}.$$

Then we have

LEMMA 11. *For each $E_j = E(p; k, l)$, $\boldsymbol{E}_j$ is a finite set. More precisely, there exists an absolute constant $C_{11}$ such that for $E_n = E(p'; k', l') \in \boldsymbol{E}_j$*

$$(40)\qquad p' < p + C_{11}\log p.$$

**Proof.** Let us set $a=k/2^p$, $b=(k+l)/2^p$, $c=k'/2^{p'}$ and $d=(k'+l')/2^{p'}$. Since $\rho$ is convex, we can see from Lemma 4 that $\rho(U_j, U_n)\leqq 0$ if the intervals $(a, b)$ and $(c, d)$ are disjoint. So each $E_n\in \boldsymbol{E}_j$ should satisfy either one of the following:

($\alpha$) $a\leqq c\leqq b\leqq d$,

($\beta$) $a\leqq c\leqq d\leqq b$,

($\gamma$) $c\leqq a\leqq d\leqq b$.

In the case ($\alpha$), using (21) again, we have

$$E((x(b)-x(a))(x(d)-x(c))) \leqq \sigma^2(b-c).$$

Hence the monotonicity of $\sigma^2$ implies

$$\rho(U_j, U_n) \leqq \frac{\sigma^2(b-c)}{\sigma(b-a)\sigma(d-c)} \leqq \frac{\sigma(d-c)}{\sigma(b-a)}.$$

Similar computation in the cases ($\beta$) and ($\gamma$) shows that in all cases ($\alpha$), ($\beta$), ($\gamma$), we have

$$\rho(U_j, U_n) \leqq \sigma(d-c)/\sigma(b-a), \qquad E_n\in \boldsymbol{E}_j. \tag{41}$$

Combining this with (39), we have

$$\{\varphi(2^p/l)\varphi(2^{p'}/l')\}^{-1} \leqq \sigma(l'/2^{p'})/\sigma(l/2^p).$$

Therefore we can see from (8) and (A.1) that there exists an absolute constant $L>0$ such that

$$\frac{(p'/p)^{1-\beta}}{2^{(p'-p)\alpha}} > L\left(\frac{1}{pp'}\right)^{1/2}$$

or

$$p'+\frac{2\beta-3}{2\alpha\log 2}\log p' < p+\frac{1}{2\alpha\log 2}\{(2\beta-1)\log p-\log L\},$$

which implies the existence of $C_{11}$ satisfying (40). Q.E.D.

Next, let us put

$$\begin{aligned} &\boldsymbol{E}_j = \{E_{j_i}; i = 1, 2, \dots, s\}, \\ &E_j = E(p; k, l), \qquad E_{j_i} = E(p_i; k_i, l_i), \\ &a = k/2^p, \qquad b = (k+l)/2^p, \qquad a_i = k_i/2^{p_i}, \qquad b_i = (k_i+l_i)/2^{p_i}, \\ &\varphi = \varphi(1/(b-a)), \qquad \varphi_i = \varphi(1/(b_i-a_i)). \end{aligned} \tag{42}$$

Then it holds that $b_i-a_i\leqq b-a$, $\varphi\leqq\varphi_i$, $i=1, 2, \dots, s$, because $j_i>j$.

Now it follows from Lemma 10, (i) that

$$\begin{aligned} P(E_j\cap E_{j_i}) &\leqq P(U_j \geqq \sigma(b-a)\varphi,\ U_{j_i} \geqq \sigma(b_i-a_i)\varphi) \\ &\leqq K\exp[-d(1-\rho_i^2)\varphi^2]P(E_j), \end{aligned} \tag{43}$$

where $\rho_i$ denotes $\rho(U_j, U_{j_i})$. Hence, for the estimation of the left-hand side of (32), it suffices to do

$$\sum_{i=1}^{s} \exp[-d(1-\rho_i^2)\varphi^2].$$

To do this, we divide the above summation into two parts denoted by $\sum^{(1)}$ and $\sum^{(2)}$ as follows:

$$\sum_{i=1}^{s} \exp[-d(1-\rho_i^2)\varphi^2] = \sum^{(1)} \exp[-d(1-\rho_i^2)\varphi^2]+\sum^{(2)} \exp[-d(1-\rho_i^2)\varphi^2], \tag{44}$$

where $\sum^{(1)}$ expresses the summation over all $i$'s such that

$$\rho_i \geqq (1-p^{-1/2})^{1/2}, \tag{45}$$

and $\sum^{(2)}$ expresses the summation of remainder.

LEMMA 12. *There exists an absolute constant $C_{12}$ such that*

$$\sum^{(2)} \exp[-d(1-\rho_i^2)\varphi^2] \leqq C_{12}. \tag{46}$$

**Proof.** It suffices to prove the boundedness of $\sum^{(2)}$ for large $p$. So we may assume that

$$p \log 2-(1/\alpha)\log p > p/2.$$

Since we have for all $i$ considered in $\sum^{(2)}$ $1-\rho_i^2 > p^{-1/2}$, it follows from (8) that

$$(1-\rho_i^2)\varphi^2 > p^{-1/2}\{p\log 2-(1/\alpha)\log p\} > \tfrac{1}{2}p^{1/2}. \tag{47}$$

Then Lemma 4, (i) shows

$$(a, b) \cap (a_i, b_i) \neq \varnothing \tag{48}$$ [14].

Now let $\#(p')$ be the number of $i$ which is considered in $\sum^{(2)}$ and satisfies the relation $p_i=p'$. Using (48) and Lemma 11, $\#(p')$ is estimated as follows.

$$\begin{aligned} \#(p') &< (b-a)2^{p'}(p')^{1/\alpha} \\ &\leqq (pp')^{1/\alpha}2^{p'-p} \\ &< p^{C_{11}+2/\alpha}(1+C_{11}\log p/p)^{1/\alpha}. \end{aligned}$$

Combine this with (40) and (47), and we can see the existence of an absolute constant $C_{12}$ such that

$$\begin{aligned} \sum^{(2)} \exp[-d(1-\rho_i^2)\varphi^2] &< p^{C_{11}+2/\alpha}(1+C_{11}\log p/p)^{1/\alpha}C_{11}\log p \exp[-\tfrac{1}{2}dp^{1/2}] \\ &\leqq C_{12}. \end{aligned}$$ Q.E.D.

LEMMA 13. *There exists an absolute constant $C_{13}$ such that*

$$\sum^{(1)} \exp[-d(1-\rho_i^2)\varphi^2] \leqq C_{13}. \tag{49}$$

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[14] $\varnothing$ expresses the empty set.

**Proof.** The proof is divided into several steps.

Using the notations given in (42), we first note that there exist three cases:

($\alpha$) $a \leqq a_i \leqq b \leqq b_i$,

($\beta$) $a \leqq a_i \leqq b_i \leqq b$,

($\gamma$) $a_i \leqq a \leqq b_i \leqq b$.

1°. There exists an absolute constant $C_{14}$ such that for all $p_i$'s considered in $\sum^{(1)}$

$$(50) \qquad p \leqq p_i \leqq p + C_{14}.$$

In fact, as was shown in (41) already, $\rho_i = \rho(U_j, U_{j_i})$ satisfies $\rho_i \leqq \sigma(b_i - a_i)/\sigma(b-a)$. And accordingly we can see by (45) and (A.1) that for large $p$

$$1 - p^{-1/2} \leqq \rho_i^2 < 2\frac{C_4}{C_3}\left(\frac{3}{2}\right)^{\alpha} 2^{-(p_i-p)\alpha}\left(\frac{p_i}{p}\right)^{1-\beta}$$

Then Lemma 11 shows the existence of $C_{15}$ such that

$$1 - p^{-1/2} \leqq C_{15} 2^{-(p_i-p)\alpha},$$

which implies (50).

2°. In the case ($\alpha$), there exists an absolute constant $C_{16}$ such that

$$(51) \qquad \rho_i \leqq 1 - C_{16}(k_i - k2^{p_i-p})^{\alpha}/p.$$

To prove this, we first remark that (41) and (45) imply

$$\lim_{p\to\infty}\frac{\sigma(b-a_i)}{\sigma(b-a)} = \lim_{p\to\infty}\frac{\sigma(b_i-a_i)}{\sigma(b-a)} = 1.$$

Then we have

$$(52) \qquad \lim_{p\to\infty}\frac{\sigma(a_i-a)}{\sigma(b-a)} = \lim_{p\to\infty}\frac{\sigma(a_i-a)}{\sigma(b-a_i)} = 0,$$

because we have by Lemma 4, (i)

$$(53) \qquad \begin{aligned}\rho_i &\leqq \frac{E((x(b)-x(a))(x(b)-x(a_i)))}{\sigma(b-a)\sigma(b_i-a_i)} \\ &= \frac{\sigma^2(b-a)+\sigma^2(b-a_i)-\sigma^2(a_i-a)}{2\sigma(b-a)\sigma(b_i-a_i)}\end{aligned}$$

and $\rho_i \to 1$ as $p \to \infty$. Moreover we can see from (50) that for any $\varepsilon > 0$ there exists $p(\varepsilon)$ such that

$$(54) \quad 1 \leqq \left|\frac{\log(a_i-a)}{\log(b-a)}\right| = \left|\frac{p_i \log 2 - \log(k_i - k2^{p_i-p})}{p\log 2 - \log l}\right| < 1+\varepsilon, \qquad p \geqq p(\varepsilon),$$

because $a_i > a$ in the case ($\alpha$).

Now, if we remark that for large $p$

$$\begin{aligned}\sigma^2(a_i-a)-\{\sigma(b-a)-\sigma(b-a_i)\}^2 &\geqq \sigma^2(a_i-a)-\{(\sigma^2(a_i-a)+\sigma^2(b-a_i))^{1/2}-\sigma(b-a_i)\}^2 \text{ [15]}\\ &\geqq \sigma^2(a_i-a)-\sigma^2(b-a_i)\{(1+\sigma^2(a_i-a)/\sigma^2(b-a_i))^{1/2}-1\}^2\\ &\geqq \sigma^2(a_i-a)/2, \qquad \text{(by (52))},\end{aligned}$$

it follows from (53), (54), and (A.1) that for large $p$

$$\begin{aligned}\rho_i &\leqq \frac{2\sigma(b-a)\sigma(b-a_i)+\{\sigma(b-a)-\sigma(b-a_i)\}^2-\sigma^2(a_i-a)}{2\sigma(b-a)\sigma(b_i-a_i)}\\ &\leqq 1-\frac{1}{4}\frac{\sigma^2(a_i-a)}{\sigma^2(b-a)}\\ &\leqq 1-\frac{1}{4}(1+\varepsilon)^{-|\beta|}\frac{C_3}{C_4}(a_i-a)^\alpha(b-a)^{-\alpha}\\ &\leqq 1-\frac{1}{4}(1+\varepsilon)^{-|\beta|}\frac{C_3}{C_4}(k_i-k2^{p_i-p})^\alpha l^{-\alpha}2^{-(p_i-p)\alpha}\\ &\leqq 1-\frac{1}{4}(1+\varepsilon)^{-|\beta|}\frac{C_3}{C_4}(k_i-2k^{p_i-p})^\alpha p^{-1}2^{-(p_i-p)\alpha}.\end{aligned}$$

Then (50) implies (51).

3°. In the case ($\gamma$), we have similarly as in 2°

(55) $$\rho_i \leqq 1-C_{17}\{(k+l)2^{p_i-p}-(k_i+l_i)\}^\alpha/p,$$

where $C_{17}$ is an absolute constant.

4°. In the case ($\beta$), we have by Lemma 4, (i) $\rho_i \leqq \rho(x(b)-x(a_i),\ x(b_i)-x(a_i))$. So, if we put $k2^{p_i-p}=k_i$ in (55), it follows from (55) that

(56) $$\rho_i \leqq 1-C_{17}(l2^{p_i-p}-l_i)^\alpha/p.$$

5°. Let us divide the summation $\sum^{(1)}$ into three parts as follows:

$$\sum^{(1)} = \sum^{(\alpha)}+\sum^{(\beta)}+\sum^{(\gamma)},$$

where $\sum^{(\alpha)}$, $\sum^{(\beta)}$ and $\sum^{(\gamma)}$ denote the summations over all $i$'s corresponding to the cases ($\alpha$), ($\beta$), ($\gamma$) respectively. Then we can see from (8), (51), (55) and (56) that there exists an absolute constant $d_1>0$ such that

(57) $$\begin{aligned}d(1-\rho_i^2)\varphi^2 &\geqq d_1(k_i-2^{p_i-p}k)^\alpha, \quad \text{for the case } (\alpha),\\ &\geqq d_1(l2^{p_i-p}-l_i)^\alpha, \quad \text{for the case } (\beta),\\ &\geqq d_1((k+l)2^{p_i-p}-(k_i+l_i))^\alpha, \quad \text{for the case } (\gamma).\end{aligned}$$

[15] $\sigma^2$ is concave.

Next, for fixed $p'$ ($\geqq p$) and $k'$, we consider the numbers $\#(p', k'; \alpha)$ of $i$ which corresponds to the case ($\alpha$) and satisfies the relations $p_i = p'$ and $k_i = k'$. Then we have $\#(p', k'; \alpha) \leqq k' - k2^{p'-p}$.

Further, for given $p'$, $l'$, $h'$, let us set

$\#(p', l'; \beta)$ = the number of $i$ which corresponds to the case ($\gamma$) and satisfies the conditions $p_i = p'$ and $l_i = l'$.

$\#(p', h'; \gamma)$ = the number of $i$ which corresponds to the case ($\gamma$) and satisfies the conditions $p_i = p'$ and $k_i + l_i = h'$.

Then it holds that $\#(p', l'; \beta) \leqq l2^{p'-p} - l'$, $\#(p', h'; \gamma) \leqq (k+l)2^{p'-p} - h'$.

Now we have by (50) and (57)

$$\begin{aligned}
&\sum\nolimits^{(1)} \exp[-d(1-\rho_i^2)\varphi^2] \\
&\leqq \sum_{p'=p}^{p+C_{14}} \Bigg\{ \sum_{k'=k2^{p'-p}}^{(k+l)2^{p'-p}} (k' - k2^{p'-p}) \exp[-d_1(k' - 2k^{p'-p})^\alpha] \\
&\qquad + \sum_{l'=0}^{l2^{p'-p}} (l2^{p'-p} - l') \exp[-d_1(l2^{p'-p} - l')^\alpha] \\
&\qquad + \sum_{h'=k2^{p'-p}}^{(k+l)2^{p'-p}} ((k+l)2^{p'-p} - h') \exp[-d_1((k+l)2^{p'-p} - h')^\alpha] \Bigg\} \\
&< 3C_{14} \sum_{k=0}^{\infty} k \exp[-d_1 k^\alpha] \\
&< \infty,
\end{aligned}$$

which proves (49). Q.E.D.

Now we shall prove Lemma 8.

**Proof of Lemma 8.** For any $E_j$, let us take $\boldsymbol{E}_j$ (the collection of $E_n$ satisfying (39)) as the set $\{E_{j_i}; i = 1, 2, \ldots, s\}$ in Lemma 8. Since (39) and Lemma 10 imply (33), it suffices to prove the validity of (32).

According to (43), we have

$$\sum_{i=1}^{s} P(E_j \cap E_{j_i}) \leqq K \sum_{i=1}^{s} \exp[-d(1-\rho_i^2)\varphi^2] P(E_j).$$

Then we get from (44), Lemmas 12 and 13

$$\sum_{i=1}^{s} P(E_j \cap E_{j_i}) \leqq K(C_{12} + C_{13}) P(E_j).$$

This shows that (32) holds for $K_1 = K(C_{12} + C_{13})$.

Thus we have completed the proof of Theorem 3. Q.E.D.

6. **Proof of Theorem 4.** The proof of Theorem 4 proceeds parallel with the one of Theorem 2.

LEMMA 14. *Theorems 4 and 5 hold if they do under the following condition*:

(58) $$\{2 \log_{(2)} t\}^{1/2} \leqq \psi(t) \leqq \{3 \log_{(2)} t\}^{1/2}, \qquad t \geqq e^e \vee a.$$

The proof is analogous to the one of Lemma 6 in T. Sirao [7], and we omit it. In the following discussion, we shall always assume (58). Let us define $E(p; k)$ by

$$E(p; k) = \{\omega;\, x(k/2^p)-x(0) \geqq \sigma(k/2^p)\psi(2^p/k)\},$$
$$p = 1, 2, 3, \ldots,\quad k = 1, 2, 3, \ldots, 2^p.$$

Then we have the following one corresponding to Lemma 2.

LEMMA 15. *For any* $c \in (0, 1)$,

$$\sum_{p=1}^{\infty} \sum_{k=[c(\log p)^{1/\alpha}]}^{[(\log p)^{1/\alpha}]} P(E(p; k))$$

*converges or diverges according as*

$$\int_a^{\infty} \frac{1}{t}\, \psi(t)^{2/\alpha-1} \exp\left[-\tfrac{1}{2}\psi^2(t)\right] dt$$

*converges or diverges.*

**Proof.** By the similar way as in the proof of Lemma 2, we can see that there exist $p_0$ and $p_1$ such that

$$\begin{aligned}
\sum_{p=p_0}^{\infty} \sum_{k=[c(\log p)^{1/\alpha}]+1}^{[(\log p)^{1/\alpha}]} & P(E(p; k)) \\
&\leqq \frac{2}{(2\pi)^{1/2}} \sum_{p=p_0}^{\infty} \left\{\frac{2^p}{(\log p)^{1/\alpha}} - \frac{2^{p-1}}{(\log (p-1))^{1/\alpha}}\right\} \frac{(\log p)^{2/\alpha}}{2^p} \\
&\qquad \cdot \frac{1}{\psi(2^p(\log p)^{-1/\alpha})} \exp\left[-\tfrac{1}{2}\psi^2(2^p(\log p)^{-1/\alpha})\right] \\
&\leqq \frac{2}{(2\pi)^{1/2}} \sum_{p=p_0}^{\infty} \int_{2^{p-1}(\log (p-1))^{-1/\alpha}}^{2^p(\log p)^{-1/\alpha}} \frac{(\log p)^{1/\alpha}}{2^p} \{\psi(2^p(\log p)^{-1/\alpha})\}^{2/\alpha-1} \\
&\qquad \cdot \exp\left[-\tfrac{1}{2}\psi^2(2^p(\log p)^{-1/\alpha})\right] dt \\
&\leqq \frac{2}{(2\pi)^{1/2}} \int_a^{\infty} \frac{1}{t}\, \psi(t)^{2/\alpha-1} \exp\left[-\tfrac{1}{2}\psi^2(t)\right] dt,
\end{aligned}$$

and similarly

$$\sum_{p=p_1}^{\infty} \sum_{k=[c(\log p)^{1/\alpha}]}^{[(\log p)^{1/\alpha}]} P(E(p; k)) \geqq \frac{1}{2(2\pi)^{1/2}} \int_{a/c}^{\infty} \frac{1}{t}\, \psi(t)^{2/\alpha-1} \exp\left[-\psi^2(t)\right] dt.$$

These two inequalities prove the lemma. Q.E.D.

According to Lemma 15, it holds that under the assumption (4)

$$\sum_{p=1}^{\infty} \sum_{k=[(\log p)^{1/\alpha}/3]}^{[(\log p)^{1/\alpha}]} P(E(p; k)) < \infty. \tag{59}$$

Now, for a given pair of $(p, k)$, we set

$$b = k2^{-p},\qquad h_q = 2^{-p} \exp\left[-qc\right],\qquad \psi \equiv \psi(1/b),\qquad q = 1, 2, 3, \ldots,$$

where $c$ is a large number which makes $e^c$ an integer and satisfies the following inequalities where absolute constants $C_{18}$ and $C_{19}$ will be chosen later.

$$(60)\qquad \left(\frac{2}{\pi}\right)^{1/2}\frac{1}{C_{18}}\exp\left[-\frac{C_{18}^2}{8}\left(\frac{e}{4}\right)^{\alpha(q-1)c}\right] < e^{-2qc}, \qquad q \geqq 2,$$

$$C_{18}(e^{c/4}/2)^{\alpha(q-1)} > 2C_{19}.$$

Then the set $S_q$ and the events $F(q;m)$, $F(q)$ are defined by $S_q=\{m; 0\leqq m\leqq e^{qc}\}$, $q\geqq 1$,

$$F(q;m) = \left\{\omega; x(b-mh_q)-x(0) > \sigma(b-mh_q)\left\{\psi+\frac{2c}{\psi}\sum_{i=0}^{q-1}2^{-\alpha i}\right\}\right\},$$

$$q \geqq 1,\ m\in S_q,$$

and $F_q=\bigcup_{m\in S_q}F(q;m)$.

LEMMA 16. *There exists an absolute constant $C_{20}$ such that*

$$P\left(\bigcup_{q=1}^{\infty}F_q\right) \leqq C_{20}P(E(p;k)).$$

**Proof.** For any $m\in S_q$, $q\geqq 2$, we choose $m_1\in S_{q-1}$ such that $|mh_q-m_1h_{q-1}|\leqq h_{q-1}$. And we set $U_m=x(b-mh_q)-x(0)$, $V_{m_1}=x(b-m_1h_{q-1})-x(0)$. Then we have by Schwartz's inequality and (A.1)

$$\rho(U_m, V_{m_1}) = \frac{\sigma^2(b-mh_q)+\sigma^2(b-m_1h_{q-1})-\sigma^2(mh_q-m_1h_{q-1})}{2\sigma(b-mh_q)\sigma(b-m_1h_{q-1})}$$

$$\geqq 1-\frac{1}{2}\frac{\sigma^2(mh_q-m_1h_{q-1})}{\sigma(b-mh_q)\sigma(b-m_1h_{q-1})}.$$

Moreover we can see, by the same way as in the proof of Lemma 6, that there exists an absolute constant $M$ such that

$$\rho(U_m, V_{m_1}) \geqq 1-M\frac{1}{\log p}\exp\left[-\frac{(q-1)c}{2}\alpha\right] = \rho_0 \text{ (say)}.$$

Now put

$$C_{19} = \sup\left\{(1-\rho_0^2)^{-1/2}(1-\rho_0)\left(\psi+\frac{2c}{\psi}\sum_{i=0}^{q-2}2^{-\alpha i}\right); p \geqq e^M \vee e, q \geqq 2\right\},$$

and choose $C_{18}$ which satisfies (60) and the following

$$(1-\rho_0^2)^{-1/2}\rho_0\frac{2c}{\varphi} \geqq C_{18}\exp\left[\tfrac{1}{4}\alpha(q-1)c\right], \qquad p \geqq e^M, q \geqq 2.$$

(Evidently we can choose such triple $(c, C_{18}, C_{19})$ if we take sufficiently large $c$.) Then we have by the procedure as was used in Lemma 6

$$P(F(q-1;m_1)'\cap F(q;m)) \leqq e^{-2qc}P(E(p;k)),$$

and accordingly

$$P(F_q) \leqq P(F_{q-1}) + \sum_{m \in S_q} P(F'_{q-1} \cap F(q; m)) \tag{61}$$
$$\leqq P(F_{q-1}) + e^{-qc} P(E(p; k)), \qquad q \geqq 2.$$

Since $P(F_1) \leqq e^c P(E(p; k))$, (61) proves the lemma. Q.E.D.

Next, let $H(p; k)$ be the collection of $\omega$ such that there exists $t$ satisfying $(k-1)/2^p \leqq t < k/2^p$ and

$$x(t, \omega) - x(0, \omega) \geqq \sigma(t)\left\{\psi(2^p/k) + 2c(\psi(2^p/k))^{-1} \sum_{i=0}^{\infty} 2^{-\alpha i}\right\}.$$

Then we can see by Lemma 16 and the continuity of $X$

$$P(H(p; k)) \leqq \liminf_{q \to \infty} P(F_q) \leqq C_{20} P(E(p; k)),$$

and accordingly

$$\sum_{p=p_0}^{\infty} \sum_{k=[(\log p)^{1/\alpha}/3]}^{[(\log p)^{1/\alpha}]} P(H(p; k)) < \infty,$$

where $p_0 = [e^M \vee e] + 1$. Now the procedure used in the proof of Theorem 2 implies Theorem 4. Q.E.D.

7. **Proof of Theorem 5.** We have by Lemma 15

$$\sum_{p=p_0}^{\infty} \sum_{k=[2(\log p)^{1/\alpha}/3]}^{[(\log p)^{1/\alpha}]} P(E(p; k)) = \infty,$$

where $p_0$ denotes a sufficiently large integer so that for $p \geqq p_0$, we can do all the computations in the sequel which are available for large $p$([16]).

As in §4, we denote $E(p; k)$ by $E_n$, where the subscript $n$ is given as follows: If $E_n = E(p; k)$ and $E_m = E(p'; k')$, then $n < m$ if and only if either one of the following holds:

(i) $p < p'$,

(ii) $p = p'$ and $k' < k$.

So $n < m$ implies $k'/2^{p'} \leqq k/2^p$.

Now it suffices to show that Lemmas 7 and 8 hold for our sequence

$$\{E_j; j = 1, 2, 3, \ldots\}.$$

For $E_n = E(p; k)$ and $E_m = E(p'; k')$, put $b = k/2^p$, $b' = k'/2^{p'}$, $U_n = x(b) - x(0)$, $U_m = x(b') - x(0)$.

Then we have by (34) $\lim_{m \to \infty} \rho(U_n, U_m) = 0$, or more precisely

$$\rho(U_n, U_m) \leqq \sigma(b')/\sigma(b) \tag{62}$$

([16]) This assumption does not take any loss of generality.

as was obtained in §3. Therefore the proof of Lemma 7 is valid for the present case if we replace there $\varphi$ and $l/2^p$ by $\psi$ and $k/2^p$ respectively, i.e. Lemma 7 holds for our sequence $\{E_j; j=1, 2, 3, \ldots\}$.

Next, we shall consider Lemma 8. For each $E_j=E(p;k)$, let $\boldsymbol{E}_j$ be the collection of events $E_n=E(p';k')$ such that $n>j$ and

$$\rho(U_j, U_n) \geqq \{\psi(2^p/k)\psi(2^{p'}/k')\}^{-1}. \tag{63}$$

Then we have

LEMMA 17. *For each $E_j=E(p;k)$, $\boldsymbol{E}_j$ is a finite set. More precisely, there exists an absolute constant $C_{21}$ such that for $E_n=E(p';k')\in \boldsymbol{E}_j$*

$$p' < p+C_{21}\log\log p.$$

**Proof.** Since we have for $E_n \in \boldsymbol{E}_j$

$$\{\psi(a)\psi(b)\}^{-1} \leqq \rho(U_j, U_n) \leqq \sigma(b')/\sigma(b)$$

where $b=k/2^p$ and $b'=k'/2^{p'}$, it follows by (A.1) that there exists an absolute constant $L>0$ such that

$$\frac{1}{2^{(p'-p)\alpha}}\left(\frac{\log p'}{\log p}\right)^{\alpha}\left(\frac{p}{p'}\right)^{\beta} \geqq \frac{L}{\log p\,\log p'}$$

which proves Lemma 17. Q.E.D.

Now let

$$\boldsymbol{E}_j = \{E_{j_i}; i = 1, 2, \ldots, s\}, \qquad E_j = E(p;k),\ E_{j_i} = E(p_i;k_i),$$
$$b = k/2^p,\quad b_i = k_i/2^{p_i}, \qquad \psi = \psi(1/b), \quad \text{and} \quad \rho_i = \rho(U_j, U_{j_i}).$$

According to Lemma 10, the validity of Lemma 8 for our sequence

$$\{E_j; j = 1, 2, 3, \ldots\}$$

is obtained from the boundedness of

$$\sum_{i=1}^{s} \exp\,[-d(1-\rho_i^2)\psi^2], \tag{64}$$

where $d$ denotes an absolute positive constant. To show the boundedness of the above series, we divide it into two parts as follows:

$$\sum_{i=1}^{s} \exp\,[-d(1-\rho_i^2)\psi^2] = \sum\nolimits^{(1)} \exp\,[-d(1-\rho_i^2)\psi^2]+\sum\nolimits^{(2)} \exp\,[-d(1-\rho_i^2)\psi^2], \tag{65}$$

where $\sum^{(1)}$ expresses the summation over all $i$'s such that $\rho_i \geqq (1-(\log p)^{-1})^{1/2}$ and $\sum^{(2)}$ does the summation of remainder. Then we have

LEMMA 18. *There exists an absolute constant $C_{22}$ such that*

$$\sum\nolimits^{(2)} \exp\,[-d(1-\rho_i^2)\psi^2] < C_{22}.$$

**Proof.** Since it follows from the definition of $\sum^{(2)}$ that for any $i$ considered in $\sum^{(2)}$, $1-\rho_i^2 > (\log p)^{-1/2}$, we have by Lemma 14

$$(1-\rho_i^2)\psi \geqq (\log p)^{1/2} \log p \geqq (\log p)^{1/2}.$$

On the other hand, the cardinal number of collection of $k'$ satisfying $k'/2^{p'} \leqq k/2^p$ for previously given $p'$ does not exceed $2^{p'-p}(\log p)^{1/\alpha}$. Moreover we can see by Lemma 17

$$2^{p'-p}(\log p)^{1/\alpha} < (\log p)^{C_{21}+1/\alpha}.$$

Then we have

$$\sum^{(2)} \exp\,[-d(1-\rho_i^2)\psi^2] \leqq (\log p)^{C_{21}+1/\alpha} \exp\,[-d(\log p)^{1/2}],$$

which proves Lemma 18. Q.E.D.

LEMMA 19. *There exists an absolute constant $C_{23}$ such that*

$$\sum^{(1)} \exp\,[-d(1-\rho_i^2)\psi^2] \leqq C_{23}.$$

**Proof.** 1°. For any $i$ considered in $\sum^{(1)}$, it holds by (62) and (A.1) that

$$1-(\log p)^{-1/2} \leqq \rho_i^2 \leqq \frac{\sigma^2(b_i)}{\sigma^2(b)} \leqq 2\frac{C_4}{C_3} 2^{(p'-p)\alpha} \frac{\log p}{\log p'} \left(\frac{p}{p'}\right)^{\beta}.$$

So we can see that there exists an absolute constant $C_{25} > 0$ such that

$$p \leqq p' < p + C_{25}. \tag{66}$$

2°. Considering the special case of ($\beta$) in the proof of Lemma 13, where $a = a_i = 0$ and $b_i < b$, we can see from (56) that for a properly chosen absolute constant $C_{25}$ $\rho_i \leqq 1 - C_{25}((b-b_i)/b)^{\alpha}$ because the term $(l2^{p_i-p}-l_i)^{\alpha}/p$ in (56) comes from $\sigma^2(b-b_i)/\sigma^2(b-a)$ and now $a=0$. Therefore we have by (A.1)

$$\rho_i \leqq 1 - C_{25}(k2^{p'-p}-k')^{\alpha}/\log p$$

or

$$1-\rho_i^2 > C_{25}(k2^{p'-p}-k')^{\alpha}/\log p. \tag{67}$$

3°. We have by (66), (67) and Lemma 14

$$\sum^{(1)} \exp\,[-d(1-\rho_i^2)\psi^2] \leqq \sum^{(1)} \exp\,[-dC_{25}(k2^{p'-p}-k')^{\alpha}]$$
$$\leqq C_{24} \sum_{k=1}^{\infty} \exp\,[-dC_{25}k^{\alpha}] < \infty,$$

which proves Lemma 19. Q.E.D.

Now we can get from (65), Lemmas 18 and 19 the boundedness of series in (64), as was to be proved. Q.E.D.

8. **Proof of Theorem 6 and Corollary 6.1.** At first we shall prove the following lemma which was stated in §2 as a sufficient condition to make $X$ a process satisfying the condition (A).

LEMMA 20. *Let $f$ be the spectral density function of stationary Gaussian process $X$. If $f$ satisfies the following conditions, then $X$ satisfies the condition* (A).

(i) *There exist positive constants $C_3$, $C_4$ and $K$ such that*

$$C_3 \leqq f(x)x^{\alpha+1}(\log x)^{\beta} \leqq C_4, \qquad x \geqq K, \tag{68}$$

*where $0<\alpha<2$, $-\infty<\beta<\infty$.*

(ii) *$g(x)=x^2f(x)$ is two times differentiable in $x$, and for some $0<\varepsilon<1$, either one of the following* (a) *or* (b) *holds.*

(a) *$x^{3-\varepsilon}g''(x)$ is bounded from below, and* $\liminf_{x\to\infty} x^{3-\varepsilon}g''(x)>0$.

(b) *$x^{3-\varepsilon}g''(x)$ is bounded from above, and* $\limsup_{x\to\infty} x^{3-\varepsilon}g''(x)<0$.

**Proof.** For the process $X$ with a spectral density $f$ satisfying (68) and $\rho(0)=1$, we have

$$\begin{aligned}\sigma^2(h) &= 2(1-\rho(h)) = 4\int_0^\infty \sin^2(hx/2)f(x)\,dx\\ &= 4\int_0^K \sin^2(hx/2)f(x)\,dx + 4\int_K^\infty \sin^2(hx/2)f(x)\,dx\\ &= 4(I_1(h)+I_2(h)) \qquad \text{(say)}.\end{aligned}$$

Then we have

$$\lim_{h\to 0} I_1(h)/h^2 = \int_0^K x^2 f(x)\,dx/4 \tag{69}$$

and

$$C_3\int_K^\infty \sin^2(hx/2)f_1(x)\,dx \leqq I_2(h) \leqq C_4\int_K^\infty \sin^2(hx/2)f_1(x)\,dx, \tag{70}$$

where $f_1(x)=x^{-(\alpha+1)}(\log x)^{-\beta}$. Taking a computation on the integral in (70), we have

$$\int_K^\infty \sin^2(hx/2)f_1(x)\,dx = h^\alpha(\log 1/h)^{-\beta}\int_{hK}^\infty \sin^2(x/2)x^{-(\alpha+1)}|1-(\log x/\log h)|^{-\beta}\,dx,$$

and accordingly

$$C_3K_1 \leqq \lim_{h\to 0} I_2(h)/h^\alpha(\log 1/h)^{-\beta} \leqq C_4K_1, \tag{71}$$

where $K_1=\int_0^\infty \sin^2(x/2)x^{-(\alpha+1)}\,dx$.

Now (69) and (71) implies (A.1).

Next, we shall prove the convexity of $\sigma^2$ if (a) and (ii) hold. Let $h>0$ and consider the quantity

$$\Delta_h^{(2)}\sigma^2 = \sigma^2(h_1+2h)+\sigma^2(h_1)-2\sigma^2(h_1+h).$$

Then we have

$$\begin{aligned}\Delta_h^{(2)}\sigma^2 &= 4\int_0^\infty \{\sin^2((h_1+2h)x/2)+\sin^2(h_1x/2)-2\sin^2((h_1+h)x/2)\}f(x)\,dx\\ &= 4\int_0^\infty \frac{\sin^2(x/2)}{x}\left\{\frac{x}{h_1+2h}f\left(\frac{x}{h_1+2h}\right)+\frac{x}{h}f\left(\frac{x}{h_1}\right)-2\frac{x}{h_1+h}f\left(\frac{x}{h_1+h}\right)\right\}dx.\end{aligned} \tag{72}$$

Putting $\hat{f}(h; x) = (x/h)f(x/h)$, $g(x) = x^2 f(x)$, we get

$$\frac{\partial^2 \hat{f}}{\partial h^2} = xh^{-3}\left\{2f\left(\frac{x}{h}\right) + 4\frac{x}{h}f'\left(\frac{x}{h}\right) + \left(\frac{x}{h}\right)^2 f''\left(\frac{x}{h}\right)\right\} = xh^{-3}g''(x/h).$$

So it follows from (72) that

$$\begin{aligned} \Delta_h^{(2)}\sigma^2 &= 4\int_0^\infty x^{-1}\sin^2(x/2)\Delta_h^{(2)}\hat{f}(h_1; x)\,dx \\ &= 4\int_0^\infty \frac{\sin^2(x/2)}{x}\frac{h^2 x}{(h_1+\theta h)^3} g''\left(\frac{x}{h_1+\theta h}\right)dx, \qquad 0 < \theta < 2. \end{aligned}$$

Then by (a) and Fatou's lemma

$$\begin{aligned} &\liminf_{h, h_1 \to 0} \frac{(\Delta_h^{(2)}\sigma^2)(h_1+2h)^\varepsilon}{4h^2} \\ &\geqq \int_0^\infty x^{-(3-\varepsilon)}\sin^2(x/2)\liminf_{h, h_1\to 0}(x/(h_1+\theta h))^{3-\varepsilon} g''(x/(x_1+\theta h))\,dx > 0, \end{aligned}$$

which proves the convexity of $\sigma^2$ in a small interval $(0, \delta)$.

If (b) holds, then we can prove the concavity of $\sigma^2$ in a similar way. Q.E.D.

Using Lemma 20, we can prove Theorem 6 as follows.

**Proof of Theorem 6.** According to the preceding lemma, there exists a stationary Gaussian process $Y = \{y(t); 0 \leqq t \leqq 1\}$ with 0-mean which is independent of $X$ and satisfies the condition (A) for a given pair $(\alpha, \beta)$ with $0 < \alpha < 2$, $-\infty < \beta < \infty$([17]). For an arbitrary $\varepsilon' > 0$, let us consider a stationary Gaussian process

$$Z = \{z(t); 0 \leqq t \leqq 1\}$$

defined by

$$z(t) = (1+\varepsilon'^2)^{-1/2}\{x + \varepsilon' y(t)\}, \qquad 0 \leqq t \leqq 1.$$

Then we have $E(z(t)) = 0$, $\rho_z(0) = E(z(t)^2) = 1$ and for small $h > 0$ and a properly chosen constant $C_{26} > 0$

$$C_{26}\frac{h^\alpha}{|\log h|^\beta} \leqq \sigma_z^2(h) \leqq \frac{(C_4 + \varepsilon'^2 C_{26})h^\alpha}{(1+\varepsilon'^2)|\log h|^\beta}$$

where

$$\sigma_z^2(h) = E((z(t+h) - z(t))^2).$$

Since the concavity or convexity of $\sigma_z^2$ is obtained from those of $\sigma^2$ and $\sigma_y^2$, where $\sigma_y^2(h) = E((y(t+h) - y(t))^2)$, the process $Z$ satisfies the condition (A). Then it follows from Corollary 3.2 that for an arbitrary $\varepsilon'' > 0$ and for almost all $\omega$ there exists an

([17]) See Remark 3 in §2.

$h_0(\omega)$ such that $0<h<h_0(\omega)$ implies

$$\sup\{|z(t+h,\omega)-z(t,\omega)|;0\leqq t\leqq 1-h\}<\sigma_z(h)\{(2+\varepsilon'')\log(1/h)\}^{1/2} \leqq\left\{\frac{1+\varepsilon'^2C_{26}/C_4}{1+\varepsilon'^2}\right\}^{1/2}\left\{\frac{(2+\varepsilon'')C_4h^\alpha}{(\log(1/h))^{\beta-1}}\right\}^{1/2}$$

and

$$\sup\{|y(t+h)-y(t)|;0\leqq t\leqq 1-h\}<\left\{(2+\varepsilon'')C_4\frac{h^\alpha}{(\log(1/h))^{\beta-1}}\right\}^{1/2}.$$

So we have for $h<h_0(\omega)$

$$\begin{aligned}\sup\{|x(t+h)-x(t)|;0\leqq t\leqq 1-h\}&\\ \leqq(1+\varepsilon'^2)^{1/2}\sup\{|z(t+h)-z(t)|;0\leqq t\leqq 1-h\}&\\ +\varepsilon'\sup\{|y(t+h)-y(t)|;0\leqq t\leqq 1-h\}&\\ \leqq\left\{C_4\frac{h^\alpha}{(\log(1/h))^{\beta-1}}\right\}^{1/2}\{(1+\varepsilon'^2C_{26}/C_4)^{1/2}+\varepsilon'\}(2+\varepsilon'')^{1/2}.&\end{aligned}\tag{73}$$

For any $\varepsilon>0$, if we take $\varepsilon'$ and $\varepsilon''$ such that

$$\{(1+\varepsilon'^2C_{26}/C_4)^{1/2}+\varepsilon'\}^2(2+\varepsilon'')<(2+\varepsilon),$$

then (73) proves Theorem 6. Q.E.D.

As an application of Theorem 6, we can prove Corollary 6.1.

**Proof of Corollary 6.1.** According to Theorem 6, it suffices to show that the relation

$$\int_0^\infty x^\alpha|\log(1+x)|^\beta\,dF(x)<\infty\tag{74}$$

implies

$$\sigma^2(h)=o(h^\alpha/(\log(1/h))^\beta)\text{[18]}.\tag{75}$$

To show this we first consider the function

$$g(h;x)=\frac{(\log(1/h))^\beta}{h^\alpha}(1-\cos hx).$$

Let $\delta\ (<1)$ be a positive number such that $x^{-\alpha}|\log x|^\beta$ is monotone decreasing in $x\in(0,\delta)$ and $x^{2-\alpha}|\log x|^\beta$ is monotone increasing in $x\in(0,\delta)$. Then, for $0<h<\delta$, it follows that for $x\geqq 1/h$

$$g(h,x)\leqq 2h^{-\alpha}|\log h|^\beta\leqq 2x^\alpha(\log x)^\beta,$$

and for $\delta<x<1/h$

$$\begin{aligned}g(h;x)&\leqq\frac{(\log(1/h))^\beta}{h^\alpha}\frac{h^2x^2}{2}=h^{2-\alpha}|\log h|^\beta x^2/2\\ &\leqq x^\alpha|\log x|^\beta/2.\end{aligned}$$

[18] $f(x)=o(g(x))$ expresses $\lim_{x\to 0}f(x)/g(x)=0$.

So we have $g(h; x) \leqq 2x^\alpha |\log x|^\beta$, $\delta < x < \infty$.

Now (74) shows that $x^\alpha |\log x|^\beta$ is integrable with respect to the measure $dF$. So we have

$$\lim_{h \to 0} \frac{\sigma^2(h)(\log(1/h))^\beta}{h^\alpha} = \lim_{h \to 0} 2 \int_0^\infty g(h; x)\, dF(x) = 0,$$

as was to be proved. Q.E.D.

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