TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, February 1970

THE DECOMPOSITION OF 3-MANIFOLDS WITH SEVERAL BOUNDARY COMPONENTS

BY
JONATHAN L. GROSS

1. **Introduction.** Any 3-manifold considered here is triangulated, oriented, connected, and compact. Any map considered is piecewise linear.

It will be shown in this paper that the problem of classifying the 3-manifolds with several boundary components (several $\geqq 1$) reduces to the problem of classifying certain 3-manifolds with connected boundary (i.e. the Δ-prime 3-manifolds—see below).

If M and M' are two disjoint 3-manifolds with connected nonvacuous boundary, one forms their *disk sum* $M \Delta M'$ by pasting a 2-cell on bd (M) to a 2-cell on bd (M'). Up to homeomorphism, the operation of disk sum is well-defined, associative, and commutative. One says that a 3-manifold P with connected nonvacuous boundary is Δ-*prime* if P is not a 3-cell and whenever P is homeomorphic to a disk sum $M \Delta M'$, either M or M' is a 3-cell. The following theorem is proved by the author in [1]:

THEOREM 1. *Let M be a* 3-*manifold with connected, nonvacuous boundary. If M is not a* 3-*cell, then M is homeomorphic to a disk sum $P_1\Delta\cdots\Delta P_n$ of Δ-prime* 3-*manifolds. The summands P_i are uniquely determined up to order and homeomorphism.*

The disk sum is generalized in §2 to permit application to 3-manifolds with several boundary components and to allow pasting across more than one disk. This generalization consists of a family of operations, called multi-disk sums, and the 3-manifolds which are essentially indecomposable under all of them are called m-prime 3-manifolds. It is proved in §3 that every irreducible 3-manifold (see below) with nonvacuous boundary has an essentially unique multi-disk sum decomposition into m-prime 3-manifolds. The concern of §4 is to reduce the problem of classifying the 3-manifolds with boundary to the problem of classifying the m-prime 3-manifolds. It will be shown in §5 that the problem of classifying the m-prime 3-manifolds reduces to the problem of classifying the Δ-prime 3-manifolds, which are simply the m-prime 3-manifolds with connected boundary.

A 3-manifold M is said to be *irreducible* if every 2-sphere in M bounds a 3-cell. In what follows a 3-cell or a 3-sphere will sometimes be called a *trivial* 3-manifold.

Presented to the Society, April 5, 1969; received by the editors May 13, 1969.

2. **Multi-disk sums.** Let M and M' be disjoint 3-manifolds, both with nonvacuous boundary. Let $D_1, \ldots, D_n$ be disks on bd (M), no two of them on the same component of bd (M). Let $D'_1, \ldots, D'_n$ be disks on bd (M'), no two of them on the same component of bd (M'). The 3-manifold obtained from $M \cup M'$ by identifying D_i and D'_i under an orientation reversing homeomorphism, for $i=1, \ldots, n$, is called a *multi-disk sum* of the 3-manifolds M and M'. If for $i=1, \ldots, n$, E_i and E'_i are disks which lie on the same components of bd (M) and bd (M') as D_i and D'_i, respectively, then the multi-disk sum obtained by pasting E_i to E'_i, for $i=1, \ldots, n$, is homeomorphic to the multi-disk sum obtained by pasting D_i to D'_i, for $i=1, \ldots, n$. In general, if bd (M) has r components and bd (M') has s components and $1 \leqq j \leqq \min(r, s)$, then the number of distinct (up to homeomorphism) multi-disk sums of M and M' involving pasting across j disks may be as large as

$$j!\binom{r}{j}\binom{s}{j}.$$

A 3-manifold P with nonvacuous boundary is called *m-prime* if P is not a 3-cell, and whenever P is a multi-disk sum of two 3-manifolds, one of the summands is a 3-cell.

A fixed iteration of $n-1$ multi-disk sum operations on the 3-manifolds $M_1, \ldots, M_n$ which results in a 3-manifold M partitions the components of bd $(M_1 \cup \cdots \cup M_n)$. That is, two components of bd $(M_1 \cup \cdots \cup M_n)$ are in the same partition class if they are both connected summands (under the iteration) of the same component of bd (M). Two iterations of $n-1$ multi-disk sum operations on $M_1, \ldots, M_n$ are called *equivalent* if they induce the same partitions on bd $(M_1 \cup \cdots \cup M_n)$. The proof of the following lemma is omitted:

LEMMA 1. *Let the results of two iterations of $n-1$ multi-disk sum operations on the 3-manifolds $M_1, \ldots, M_n$ be the 3-manifolds M and M', respectively. If the two iterations are equivalent, then M and M' are homeomorphic.*

The following example indicates that a 3-manifold which is not irreducible and which has more than one boundary component may have more than one decomposition into M-prime multi-disk summands:

EXAMPLE. Let P be any m-prime 3-manifold with more than one boundary component. Let Q be homeomorphic to the m-prime 3-manifold which is obtained by removing the interior of a 3-cell from $S^1 \times S^2$, where S^n denotes the n-sphere. Let B be homeomorphic to the m-prime 3-manifold $S^2 \times [0, 1]$. And let P, Q, and B be mutually disjoint. Then any multi-disk sum of P and Q is homeomorphic to any multi-disk sum of P and B across two disks.

REMARK. One might wish to "generalize" the multi-disk sum by permitting more than one of the pasting disks to lie on the same component of the boundary of a summand. However, such a sum of two 3-manifolds would generate handles, so even an irreducible 3-manifold might have more than one factorization.

3. **The irreducible case.** It will be shown in this section that every irreducible 3-manifold with nonvacuous boundary has an essentially unique multi-disk sum decomposition into m-prime 3-manifolds.

An imbedding $f: M \to N$ of one manifold in another is called *proper* if

$$f(\mathrm{bd}\,(M)) \subset \mathrm{bd}\,(N) \quad \text{and} \quad f(\mathrm{int}\,(M)) \subset \mathrm{int}\,(N).$$

Let R be a 2-manifold of genus zero whose boundary has $n+1$ components, for some nonnegative integer n, i.e. R is a disk with n holes. If a 3-manifold M is homeomorphic to $R \times [0, 1]$, then M is called a *handlebody of genus n*. In the case that $n=0$, one says M is a *trivial handlebody*. In the case that $n=1$, one says that M is a *handle*.

LEMMA 2. *Let P be an irreducible m-prime* 3-*manifold and let $E_1, \ldots, E_n$ be a collection of disjoint disks, each properly imbedded in P.*

(a) *The closure in P of all but possibly one component of $P-(E_1 \cup \cdots \cup E_n)$ is a* 3-*cell.*

(b) *If the closure in P of every component of $P-(E_1 \cup \cdots \cup E_n)$ is a* 3-*cell, then P is a handle.*

Proof. If the closures of two or more components of $P-(E_1 \cup \cdots \cup E_n)$ were not 3-cells, then there would be a subcollection of the disks E_j, say (after renumbering) $E_1, \ldots, E_r$, such that $P-(E_1 \cup \cdots \cup E_r)$ has exactly two components, both of which are nontrivial and the intersection of whose closures is $E_1 \cup \cdots \cup E_r$. This would contradict the m-primeness of P, so part (a) must hold. Part (b) is true because one can show by an induction on the number of components of

$$P-(E_1 \cup \cdots \cup E_n)$$

that if every component of $P-(E_1 \cup \cdots \cup E_n)$ is a 3-cell, then P is a handlebody.

LEMMA 3. *Let P be an m-prime* 3-*manifold with boundary components $R_1, \ldots, R_q$, and let $E_1, \ldots, E_n$ be a collection of mutually disjoint disks, each properly imbedded in P, such that the closure Y of some component of $P-(E_1 \cup \cdots \cup E_n)$ is not a* 3-*cell. Let $C_1, \ldots, C_q$ be mutually disjoint* 3-*cells, each disjoint from P. One forms a* 3-*manifold P' from $Y \cup C_1 \cup \cdots \cup C_q$ by pasting a single disk on each component of* bd (Y) *to a disk on* bd $(C_1 \cup \cdots \cup C_q)$ *according to the following rule: if the component S of* bd (Y) *meets R_j, then paste a disk on S to a disk on* bd (C_j). *One chooses the disks on each* bd (C_j) *to be disjoint. Then there is a homeomorphism from P to P' which carries R_j onto the component of* bd (P') *that meets C_j, for $j=1, \ldots, q$.*

Proof. Lemma 3 is an extension of Lemma 6 of [1]. One obtains a proof of Lemma 3 by generalizing the steps in the proof of Lemma 6 of [1].

The *total genus* of bd (N), where N is any 3-manifold, is the sum of the genera of the components of bd (N).

THEOREM 2. *Let M be an irreducible nontrivial 3-manifold with several boundary components. Then M is homeomorphic to an iterated multi-disk sum of irreducible m-prime 3-manifolds $P_1, \ldots, P_n$. The summands P_i are unique up to order and homeomorphism and the iteration of the multi-disk sum operations is unique up to equivalence.*

Proof. Since M is irreducible and nontrivial, no component of bd (M) is a 2-sphere. Therefore, the total genus of bd (M) is positive. If M is not m-prime, then M is homeomorphic to a multi-disk sum of two nontrivial irreducible 3-manifolds, M_1 and M_2. One observes that the total genus of bd (M) equals the sum of the total genus of bd (M_1) and the total genus of bd (M_2), both positive numbers less than the total genus of bd (M). Therefore, the decomposing process terminates in finitely many steps (i.e. fewer than the total genus of bd (M)) and yields some irreducible m-prime 3-manifolds which have a multi-disk sum homeomorphic to M.

The proof of uniqueness to be given here will be obtained by generalizing the techniques of [1]. The notation used here reflects the notation used in [1]. One considers here a fixed collection of mutually disjoint m-prime 3-manifold $P_1, \ldots, P_n$ and a fixed iteration of $n-1$ multi-disk sum operations on them which results in a 3-manifold homeomorphic to M. The next paragraph is concerned with the construction of a 3-manifold M^* which is homeomorphic to M and in which it will be convenient to perform cutting and pasting operations.

Let $R_1, \ldots, R_q$ be the components of bd (M), and let $K_1, \ldots, K_q$ be 3-cells, mutually disjoint and each disjoint from each of the m-primes P_j. The 3-manifold M^* is obtained from the union of the 3-cells K_i and the m-primes P_j by making the following identifications, for $i=1, \ldots, q$ and $j=1, \ldots, n$: If a component S of bd (P_j) is a connected summand of R_i (under the fixed iteration of multi-disk sum operations which produces M from $P_1, \ldots, P_n$), then identify a disk $D'_{i,j}$ on S with a disk $D''_{i,j}$ on bd (K_i). One chooses the disks $D''_{i,j}$ on bd (K_i) to be mutually disjoint, and one observes that on any component of the boundary of any of the m-primes P_j, there is exactly one disk $D'_{i,j}$. It follows from Lemma 1 that M^* is homeomorphic to M.

Let M_1 and M_2 be nontrivial 3-manifolds which have M as a multi-disk sum over r disks (note: $1 \leqq r \leqq q$). Then there is a family of mutually disjoint disks $E_1, \ldots, E_r$, each properly imbedded in M^*, such that $M^* - (E_1 \cup \cdots \cup E_r)$ has two components, whose closures are homeomorphic to M_1 and M_2, and which will be denoted by M_1^* and M_2^* respectively.

Indexing system. One indexes the components $R_1^*, \ldots, R_q^*$ of bd (M^*) so that $R_1^*, \ldots, R_r^*$ are the ones which meet M_1^* and M_2^*, that $R_{r+1}^*, \ldots, R_t^*$ are the ones which lie in M_1^*, and that $R_{t+1}^*, \ldots, R_q^*$ are the ones which lie in M_2^*. One indexes the 3-cells $K_1, \ldots, K_q$ so that bd (K_i) meets R_i^*, for $i=1, \ldots, q$. And one indexes the disks $E_1, \ldots, E_r$ so that bd $(E_i) \subset R_i^*$, for $i=1, \ldots, r$.

It follows from an induction on the number n of m-prime summands P_j that the uniqueness of the m-primes P_j and the iteration of multi-disk sum operations can be proved by establishing the following statement:

A. The 3-manifolds M_1^* and M_2^* are homeomorphic to respective iterated multi-disk sums of two mutually exclusive and exhaustive nonvoid subsets of $\{P_1, \ldots, P_n\}$; furthermore, the iteration of multi-disk sum operations which produces M^* via M_1^* and M_2^* is equivalent to the original fixed iteration.

It may be shown by a general position argument that one need consider only the case in which for $i=1, \ldots, r$ and $j=1, \ldots, q$ and $k=1, \ldots, n$ the components of $E_i \cap D_{j,k}$ are simple arcs and simple loops, each a crossing of surfaces.

(3.1) If statement A holds for the case in which no component of any intersection $E_i \cap D_{j,k}$ is a loop, then statement A holds in general.

Proof of (3.1). It is not difficult to see that statement A holds when every intersection $E_i \cap D_{j,k}$ is empty, because the 3-manifolds P_j are m-prime. Now suppose that statement A holds whenever there are fewer than u components in the union of all intersections $E_i \cap D_{j,k}$ and consider the case in which there are exactly u components, at least one of which is a loop.

In particular, suppose that an intersection loop occurs on $D_{a,b}$. Then $D_{a,b}$ contains a loop k which is innermost among all intersection loops on $D_{a,b}$, say $k \subset E_c \cap D_{a,b}$. Since M^* is irreducible, one may isotopically deform E_c onto a disk obtainable by cutting E_c at the loop k and pasting in the subdisk of $D_{a,b}$ which is bounded by the loop k, and one may then isotopically move this disk so as to restore general position. These isotopies on E_c can be accomplished in the complement in M^* of the union of the other disks E_i, so the loop k is removable. By the induction hypothesis, statement (3.1) holds.

It will hereafter be assumed that every component of every intersection $E_i \cap D_{j,k}$ is an arc. This implies that if $E_i \cap D_{j,k}$ is nonvoid, then $i=j$. It also implies that each component of each intersection $E_i \cap P_j$ is a properly imbedded disk.

From now on, let E denote the union of the disks $E_1, \ldots, E_r$.

DEFINITION. A Δ-prime summand P_j of M^* is called a *special handle* if the closure of every component of $P_j - E$ is a 3-cell. (It follows from (b) of Lemma 2 that a special handle actually is a handle.)

DEFINITION. Let P_j be a Δ-prime summand of M^* such that P_j is not a special handle. The *essence* of P_j, hereafter denoted by Y_j, is the closure in P_j of the component of $P_j - E$ which is not a 3-cell. (It follows from (a) of Lemma 2 that Y_j is well defined.)

Indexing the m-primes. One now reindexes the m-primes $P_1, \ldots, P_n$ so that $P_1, \ldots, P_u$ are the ones whose essences lie in M_2^*, and that $P_{s+1}, \ldots, P_n$ are the special handles. One observes that while this induces a reindexing of the pasting disks $D_{i,j}$, it does not affect the indexing of the components R_i^* of bd (M^*), or of the 3-cells K_i, or of the disks E_i.

(3.2) Each component of cl $(M_1^* - (Y_1 \cup \cdots \cup Y_n))$ is a (possibly trivial) handlebody.

Proof of (3.2). A component of cl $(M_1^* - (Y_1 \cup \cdots \cup Y_n))$ is the union of 3-cells, each of which is either the closure of a component of some $K_i - E_i$ or the closure of a component of some $P_j - E$. Each component of the intersection of any two of these 3-cells is a disk, i.e. it is the closure of a component of some $D_{i,j} - E_i$. By Lemma 1 of [1], statement (3.2) holds.

By Lemma 3 of [1], M_1^* is the union of a family of disjoint 3-manifolds $Y_1', \ldots, Y_u'$ and a family of disjoint handlebodies, whose union will be called T', such that for $j = 1, \ldots, u$, $Y_j' \approx Y_j$ and the following condition holds:

(3.3) Y_j' and T' intersect only in their boundaries and $Y_j' \cap T'$ is the union of a family of disjoint disks, one lying on each component of bd (Y_j').

(3.4) The number of components of T' equals the number t of components of bd (M_1^*).

Proof of (3.4). Let w be the number of components of T'. Since T' is the union of disjoint handlebodies, the number w equals the number of components of bd (T'). By statement (3.3), each component of each bd (Y_j') meets some component of bd (T'), so every component of bd (M_1^*) meets bd (T'). Hence, the number t of components of bd (M_1^*) is not greater than the number w. For $j = 1, \ldots, u$, let y_j be the number of components of bd (Y_j') and let p_j be the number of components of $Y_j' \cap T'$. Then the number t is not less than the number $w + \Sigma y_j - \Sigma p_j$. By statement (3.3), $y_j = p_j$ for $j = 1, \ldots, u$, so $t \geqq w$. Therefore $t = w$.

More indexing. Let $T_1', \ldots, T_t'$ denote the components of T', indexed so that for $i = 1, \ldots, t$, bd (T_i') meets R_i^*.

For $j = 1, \ldots, u$ and $i = 1, \ldots, t$ one defines $F_{j,i}$ to be the empty set if $Y_j' \cap T_i'$ is empty, and otherwise to be a disk on bd (T_i') which contains $Y_j' \cap T_i'$ in its interior. One requires that the sets $F_{j,i}$ be mutually disjoint.

For $j = 1, \ldots, u$ and $i = 1, \ldots, t$ let $B_{j,i}$ be the empty set if $F_{j,i}$ is empty and, otherwise, a 3-cell in T_i' such that $B_{j,i} \cap$ bd $(T_i') = F_{j,i}$. One requires that the sets $B_{j,i}$ be mutually disjoint.

(3.5) For $j = 1, \ldots, u$ and $i = 1, \ldots, t$, there is a homeomorphism from the 3-manifold $Y_j' \cup B_{j,1} \cup \cdots \cup B_{j,t}$ to the m-prime 3-manifold P_j which carries the component (if any) of bd $(Y_j' \cup B_{j,1} \cup \cdots \cup B_{j,t})$ that meets bd (T_i') onto the component (if any) of bd (P_j) that meets bd (K_i).

Proof of (3.5). The homeomorphism established by an application of Lemma 3 works here.

One observes that the construction here has represented M_1^* as an iterated multi-disk sum of the 3-manifolds $Y_j' \cup B_{j,1} \cup \cdots \cup B_{j,t}$, for $j = 1, \ldots, u$, which are homeomorphic to $P_1, \ldots, P_u$ respectively (by statement (3.5)) and the handlebodies cl $(T_i' - (B_{1,i} \cup \cdots \cup B_{u,i}))$, for $i = 1, \ldots, t$, which are homeomorphic to $T_1', \ldots, T_t'$ respectively (see, for example, Zeeman [4, p. III-19]). From this and statement (3.5), one obtains the following:

(3.6) The 3-manifold M_1^* is homeomorphic to an iterated multi-disk sum of the m-primes $P_1, \ldots, P_u$ and some handlebodies (some of which, perhaps, are trivial), such that a component of bd (P_i) and a component of bd (P_j) are connected summands (under this iterated sum) of the same component of bd (M_1^*) if and only if they are connected summands (under the original fixed iteration which produces M) of the same component of bd (M).

Correspondingly, one obtains statement (3.7).

(3.7) The 3-manifold M_2^* is homeomorphic to an iterated multi-disk sum of the m-primes $P_{u+1}, \ldots, P_s$ and some handlebodies (some of which, perhaps, are trivial), such that a component of bd (P_i) and a component of bd (P_j) are connected summands (under this iterated sum) of the same component of bd (M_2^*) if and only if they are connected summands (under the original fixed iteration which produces M) of the same component of bd (M).

(3.8) The number of handles among $P_1, \ldots, P_n$ is uniquely determined.

Proof of (3.8). Let h be the number of handles (special and nonspecial) among $P_1, \ldots, P_n$. And let p be the total number of pasting disks in the original representation of M as a multi-disk sum of $P_1, \ldots, P_n$. By the Grushko-Neumann theorem (see Kuroš [2, p. 58]) the rank of a maximal free summand of $\Pi_1(M)$ is an invariant of M. It follows that the number $p+h$ (i.e. the number of pasting disks plus the number of handles in any multi-disk sum decomposition of M into m-primes) is an invariant of M. For $j=1, \ldots, n$ let b_j be the number of components of bd (P_j). Then $(\sum_{j=1}^n b_j)-p$ is the number of components of bd (M). Hence, the number $(p+h)+((\sum_{j=1}^n b_j)-p)=h+\sum_{j=1}^n b_j$ is an invariant of M. Another invariant of the 3-manifold M is the number $b=\sum \{b_j : P_j \text{ is not a handle}\}$, because of statements (3.6) and (3.7). Therefore, the number $(h+\sum_{j=1}^n b_j)-b=h+((\sum_{j=1}^n b_j)-b)$ is an invariant of M. But $(\sum_{j=1}^n)-b=\sum \{b_j : P_j \text{ is a handle}\}=h$. Consequently, the number $h+h$ is an invariant of M, and so is the number h.

The proof of Theorem 2 is now close to completion. Statements (3.6), (3.7), and (3.8) taken together say that the m-primes $P_1, \ldots, P_n$ are uniquely determined up to order and homeomorphism. What remains to be proved is that the iteration of multi-disk sum operations on $P_1, \ldots, P_n$ that produces M^* via the iterated multi-disk sums M_1^* and M_2^* and pasting across the disks $E_1, \ldots, E_r$ (to be called the *M^*-iteration* below) is equivalent to the original fixed iteration of multi-disk sum operations on $P_1, \ldots, P_n$ that produces M (to be called the M-iteration below).

(3.9) The M^*-iteration is equivalent to the M-iteration.

Proof of (3.9). One reindexes the components $R_1, \ldots, R_q$ of bd (M) so that they again correspond to the components $R_1^*, \ldots, R_q^*$, respectively, of bd (M^*) in the construction of M^*. It follows from statement (3.5) and the corresponding statement about M_2^* that if the m-prime multi-disk summands P_i and P_j are not handles, then a component of bd (P_i) and a component of bd (P_j) are connected summands of the component R_k^* of bd (M^*) under the M^*-iteration if and only if they are connected summands of the component R_k of bd (M) under the M-iteration. The number of the handles at R_k^* under the M^*-iteration equals the number of handles at R_k under the M-iteration because there is a homeomorphism from M^* to M

which carries R_k^* onto R_k. That is, the number of independent simple loops on R_k^* which do not separate R_k^* but are homotopically trivial in M^* equals the number of independent simple loops on R_k which do not separate R_k but are homotopically trivial in M.

The proof of Theorem 2 is now complete.

4. **Decomposition results.** The *connected sum* $M \# M'$ of two 3-manifolds is obtained by removing from each the interior of a 3-cell and then pasting the resulting boundary components together. Up to homeomorphism, the operation of connected sum is well defined, associative, and commutative. One says that a 3-manifold P is *#-prime* if P is not a 3-sphere, and whenever P is homeomorphic to a connected sum $M \# M'$, either M or M' is a 3-sphere.

Remark 1 of Milnor [3, p. 5], given here as Theorem 3, reduces the problem of classifying the 3-manifolds-with-boundary to the problem of classifying the #-primes.

THEOREM 3. *Let M be a 3-manifold which is not a 3-sphere. Then M is homeomorphic to a connected sum $Q_1 \# \cdots \# Q_n$ of #-prime 3-manifolds. The summands Q_i are uniquely determined up to order and homeomorphism.*

Lemmas 4, 5, and 6 reduce the problem of classifying the #-primes to the problem of classifying the *m*-primes, which are "almost finer" than the #-primes (see Lemma 9).

LEMMA 4. *The correspondence between the homeomorphism classes of #-prime 3-manifolds with vacuous boundary and the homeomorphism classes of Δ-prime 3-manifolds whose boundaries are 2-spheres which is given by removing from a representative #-prime 3-manifold the interior of a 3-cell is a bijection.*

Proof. This is obvious.

LEMMA 5. *Let the 3-manifold M be a multi-disk sum of the irreducible 3-manifolds M' and M''. Then M is irreducible.*

Proof. One considers a 2-sphere S imbedded in M in general position with respect to the pasting disks. One shows that S bounds a 3-cell in M by an induction on the number of components in the intersection of the 2-sphere S with the union of the pasting disks. The key idea in the induction step is cutting at an innermost intersection loop on one of the pasting disks, which is a standard technique (for example, see Lemma 9 of [1], of which the present Lemma 5 is a generalization).

LEMMA 6. *Let M be a #-prime 3-manifold with nonvacuous boundary. Then M is irreducible.*

Proof. Let S be a 2-sphere in the interior of M. The 2-sphere S must separate M because otherwise M would have as connected summands the 3-manifold $S^1 \times S^2$

and some 3-manifold with nonvacuous boundary. Since the 2-sphere S separates M and M is #-prime, the 2-sphere S must bound a 3-cell.

THEOREM 4. *A classification of the m-prime 3-manifolds yields a classification of the #-prime 3-manifolds.*

Proof. Lemma 4 is the restriction of this theorem to #-prime 3-manifolds with vacuous boundary. For every positive integer n and every unordered n-tuple of irreducible m-prime 3-manifolds, make an entry on a list of each distinct (up to equivalence) iterated multi-disk sum of the n-tuple. By Lemma 5, every entry in the list is a #-prime 3-manifold with nonvacuous boundary. By Lemma 6 and Theorem 2, every #-prime 3-manifold with nonvacuous boundary appears on the list exactly once.

It will soon be shown that every 3-manifold with nonvacuous boundary has a multi-disk sum decomposition into m-primes. Then, the extent to which such a decomposition is nonunique will be described.

NOTATION. For any 3-manifold V, (from now on) let V^* denote the 3-manifold (unique up to homeomorphism) which is obtained by removing from V the interior of a 3-cell. The resulting boundary component of V^* will be called the *new* boundary component of V^*.

LEMMA 7. *Let B be a 3-manifold with nonvacuous boundary and let M be any 3-manifold which is disjoint from B. Then any 3-manifold which is obtained by pasting a disk on the new boundary component of M^* to a disk on any component of* bd (B) *is homeomorphic to the connected sum $M \# B$.*

Proof. This is a trivial generalization of Lemma 8 of [1].

LEMMA 8. *Let B be a 3-manifold with nonvacuous boundary. Then B^* is homeomorphic to any multi-disk sum of B and the m-prime 3-manifold $S^2 \times [0, 1]$ across one disk.*

Proof. This is an easy corollary of Lemma 7, or it can be easily derived independently.

THEOREM 5. *A 3-manifold M with nonvacuous boundary has a multi-disk sum decomposition into m-primes.*

Proof. By Theorem 3, the 3-manifold M is homeomorphic to a connected sum $Q_1 \# \cdots \# Q_n$ of #-prime 3-manifolds. One indexes the #-primes Q_i so that $Q_1, \dots, Q_k$ are the ones with nonvacuous boundary. By Lemma 7, M is homeomorphic to a multi-disk sum of the 3-manifolds $Q_1, Q_2^*, \dots, Q_n^*$. Therefore, by Lemma 8, the 3-manifold M is homeomorphic to a multi-disk sum of the m-prime 3-manifolds $Q_{k+1}^*, \dots, Q_n^*$ and $k-1$ copies of the m-prime 3-manifold $S^2 \times [0, 1]$ and the #-prime 3-manifolds $Q_1, \dots, Q_k$. By Lemma 6 and Theorem 2, each of the 3-manifolds $Q_1, \dots, Q_k$ has a multi-disk sum decomposition into #-primes. Therefore, the 3-manifold M has a multi-disk sum decomposition into m-primes.

LEMMA 9. *An m-prime 3-manifold P falls into exactly one of these three cases*: (i) $P\approx S^2\times[0,1]$. (ii) *There exists a #-prime 3-manifold Q with vacuous boundary such that* $P\approx Q^*$. (iii) *P is #-prime.*

Proof. Let s be the number of components of bd (P) which are 2-spheres. Lemma 8 implies that if $s\geqq 2$, then P has a multi-disk summand which is homeomorphic to $S^2\times[0,1]$, which implies that $P\approx S^2\times[0,1]$. Lemma 8 implies that if $s=1$, then bd $(P)\approx S^2$, in which case Lemma 4 implies that case (ii) holds. Lemma 7 implies that if $s=0$, then P is #-prime.

Lemma 9 indicates that every m-prime except $S^2\times[0,1]$ is #-prime or almost #-prime, while Lemma 5 implies that a multi-disk sum of arbitrarily many irreducible m-primes is #-prime. Thus, a multi-disk sum decomposition into m-primes may be better for certain purposes than a connected sum decomposition into #-primes, but unlike a connected sum decomposition, it has the possible disadvantage of nonuniqueness. Fortunately, this disadvantage is small for most applications, as indicated by Theorem 6 below.

One says that a 3-manifold M has decomposition nonuniqueness of the *first type* if there are two collections of m-prime 3-manifolds which are not identical up to order and homeomorphism and which both have iterated multi-disk sums homeomorphic to M. One says that M has decomposition nonuniqueness of the *second type* if there are two inequivalent iterations of multi-disk sum operations on a collection of m-prime 3-manifolds, both of which yield a 3-manifold homeomorphic to M. An example of nonuniqueness of the first type is given in §2. The following is an example of nonuniqueness of the second type:

EXAMPLE. Let B be m-prime 3-manifold with two or more boundary components which is not homeomorphic to $S^2\times[0,1]$, and let P be either a #-prime 3-manifold with vacuous boundary or a 3-cell, which is disjoint in either case from B. Then the 3-manifolds obtained by identifying a disk on the boundary of the m-prime 3-manifold P^* with a disk on any component of bd (B) are mutually homeomorphic.

THEOREM 6. *Every instance of decomposition nonuniqueness of the first type is generated as in the example of* §2. *Every instance of decomposition nonuniqueness of the second type is generated as in the example immediately above.*

The proof of Theorem 6 is omitted. It is a straightforward (but moderately lengthy) application of methods and results already presented here.

5. Reduction of the classification problem.

THEOREM 7. *Let P be an m-prime* 3-*manifold with more than one boundary component, and let M be a* 3-*manifold obtained by identifying two disks on different components of* bd (P). *Then M is m-prime.*

Proof. Suppose that M is the multi-disk sum of the 3-manifolds M_1 and M_2 across a family of disks whose union is E. Let D denote the image in M of the pasting disks on bd (P). The irreducibility of M and a general position argument justify the assumption here that the components of $D \cap E$ are simple arcs, each a crossing of surfaces. By Lemma 2, there is exactly one component of P-preimage (E) whose closure, denoted here by Y, is not a 3-cell. One takes the 3-manifolds M_1 and M_2 to be indexed so that $Y \subset M_1$. Theorem 7 will be proved by showing that M_2 is a 3-cell.

(5.1) M_2 is a handlebody.

Proof of (5.1). The 3-manifold M_2 is the union of 3-cells, i.e. the closures of components of P-preimage (E). A component of the intersection of any two of these 3-cells is a disk, i.e. a component of $M_2 \cap E$. By Lemma 1 of [1], M_2 is a handlebody.

Corollary. *E is a single disk.*

Let p be the maximum number of disks which can be imbedded in P without separating P, and let y be the maximum number of disks which can be imbedded in Y without separating Y. Also, let n be the number of components of $E-D$.

(5.2) P-preimage (E) has $n-(p-y)+1$ components.

Proof of (5.2). The preimage of E is the union of a family of mutually disjoint disks $E_1, \ldots, E_n$, each properly imbedded in P and indexed so that $E_1, \ldots, E_r$ is a maximal nonseparating subfamily in P. Hence, P-preimage (E) has $n-r+1$ components. One considers the process of making n consecutive cuts in P with the jth cut at the disk E_j. It is evident that "cutting P open" along any nonseparating family of disks yields an m-prime 3-manifold, because P is m-prime. In particular, the 3-manifold obtained by cutting P open at $E_1, \ldots, E_r$ is m-prime. Furthermore, cutting at each of the remaining disks $E_{r+1}, \ldots, E_n$ either "removes" a 3-cell from this new m-prime 3-manifold or else it separates a previously removed 3-cell. Thus, the homeomorphism type of the closure of the component of P-preimage (E) which is not a 3-cell is determined by the cuts at $E_1, \ldots, E_r$. That is, the 3-manifold obtained by cutting P open at $E_1, \ldots, E_r$ is homeomorphic to Y. Therefore $r=p-y$, and $n-r+1=n-(p-y)+1$.

(5.3) bd (Y) has $p-y$ more components than bd (P).

Proof of (5.3). In the proof of (5.2), it is shown that a 3-manifold homeomorphic to Y can be obtained by cutting P open along $p-y$ disks. Since P does not have a handle as a multi-disk summand, the boundary of each of these disks separates bd (P). Hence bd (Y) has $p-y$ more components than bd (P).

Let h be the number of components of M_2-D.

(5.4) The number of components of $M_1 \cap D$ is at least $n-h+1$.

Proof of (5.4). The number of components of M_1-D plus the number of components of M_2-D is $n-(p-y)+1$. Hence, M_1-D has $n-(p-y)+1-h$ components. The closure of one of these is Y and the closure of each of the others is a

3-cell. To obtain a connected 3-manifold from Y and the $n-(p-y)+h$ 3-cells, one needs $n-(p-y)+h$ components of $M_1 \cap D$. Another $(p-y)+1$ are required to reduce the number of boundary components from the number of components of bd (Y) down to the number of components of bd (M_2), because bd (M_2) has as many components as bd (M), i.e. one less than bd (P).

(5.5) M_2 is a 3-cell.

Proof of (5.5). The closure of each of the h components of M_2-D is a 3-cell. One observes that M_2 is a 3-cell if and only if $M_2 \cap D$ has $h-1$ components. Since M_2 is connected, $M_2 \cap D$ has at least $h-1$ components. On the other hand, the number of components of $M_1 \cap D$ plus the number of components of $M_2 \cap D$ is n, i.e. the same as the number of components of $E-D$. By statement (5.4), $M_2 \cap D$ has at most $h-1$ components.

This completes the proof of Theorem 7.

THEOREM 8. *A classification of the* Δ*-prime* 3*-manifolds yields a classification of the m-prime* 3*-manifolds.*

Proof. For each distinct (up to homeomorphism) Δ-prime 3-manifold Q and for each distinct (up to homeomorphism of pairs) nonseparating finite family of mutually disjoint properly imbedded disks in Q, enter on a list the 3-manifold obtained by cutting Q open along that family. For a fixed Δ-prime Q, no two of the 3-manifolds obtained in this manner are homeomorphic. Since the 3-manifold obtained by joining together with "boundary connecting handles" all the boundary components of a given 3-manifold is unique (up to homeomorphism), no 3-manifold on the list results from more than one Δ-prime 3-manifold. Therefore, the list has no repetitions whatsoever. It is obvious that every entry on the list is m-prime. By Theorem 7, every m-prime 3-manifold appears on the list.

REFERENCES

1. J. L. Gross, *A unique decomposition theorem for* 3*-manifolds with connected boundary*, Trans. Amer. Math. Soc. **142** (1969), 191–199.

2. A. G. Kuroš, *The theory of groups*. Vol. II, GITTL, Moscow, 1953; English transl., Chelsea, New York, 1956. MR **15**, 501; MR **18**, 188.

3. J. Milnor, *A unique decomposition theorem for* 3*-manifolds*, Amer. J. Math. **84** (1962), 1–7. MR **25** #5518.

4. E. C. Zeeman, *Seminar on combinatorial topology*, Inst. Hautes Études Sci. (Publ. Math.) (1963).

PRINCETON UNIVERSITY,
PRINCETON, NEW JERSEY
COLUMBIA UNIVERSITY,
NEW YORK, NEW YORK