TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 148, March 1970

# QUASICONFORMAL MAPPINGS AND SCHWARZ'S LEMMA

BY
PETER J. KIERNAN

**Abstract.** In this paper, $K$ quasiconformal maps of Riemann surfaces are investigated. A theorem, which is similar to Schwarz's lemma, is proved for a certain class of $K$ quasiconformal maps. This result is then used to give elementary proofs of theorems concerning $K$ quasiconformal maps. These include Schottky's lemma, Liouville's theorem, and the big Picard theorem. Some of Huber's results on analytic self-mappings of Riemann surfaces are also generalized to the $K$ quasiconformal case. Finally, as an application of the Schwarz type theorem, a geometric proof of a special case of Moser's theorem is given.

1. **Introduction.** In this paper, $K$ quasiconformal maps of Riemann surfaces are investigated. A theorem, which is similar to Schwarz's lemma, is proved for a certain class of $K$ quasiconformal maps. This result is then used to give elementary proofs of known theorems concerning $K$ quasiconformal maps. These include Schottky's lemma, Liouville's theorem, and the big Picard theorem. Some of Huber's results on analytic self-mappings of Riemann surfaces are also generalized to the $K$ quasiconformal case. Finally, as an application of the Schwarz type theorem, a geometric proof of a special case of Moser's theorem is given.

I wish to express my deep gratitude to Professor Kobayashi for his many helpful suggestions as well as his encouragement and patience. I also thank Professor Wu for several helpful discussions.

2. **Definitions.** Let $D=\{z \in \boldsymbol{C} \mid |z|<1\}$. The Poincaré-Bergman metric on $D$ is given by $ds^2=(1/(1-|z|^2)^2)\,dz\,d\bar{z}$. If we let $d_D(x, y)$ denote the corresponding distance function, then it is easy to show that $d_D(0, y)=\log((1+|y|)/(1-|y|))$. If we let $H$ denote the upper half plane $=\{z \in \boldsymbol{C} \mid \operatorname{im} z>0\}$, then $H$ is conformally equivalent to $D$. The Poincaré-Bergman metric on $H$ is given by $ds^2=(1/4y^2)\,dz\,d\bar{z}$, where $y=\operatorname{im} z$.

In the literature, it is often required that quasiconformal maps be homeomorphisms. For the purposes of this paper, this will not be necessary, mainly because the work of Bers and Mori enables us to reduce problems to this case. Thus we shall use the following definitions.

Received by the editors August 13, 1969.

*AMS Subject Classifications.* Primary 3047, 5399.

*Key Words and Phrases.* Quasiconformal, Schwarz's lemma, Poincaré-Bergman metric, Kobayashi pseudo-distance, distance decreasing, Picard theorem, harmonic mappings.

DEFINITION. By a rectangle $R$ in $\boldsymbol{C}$, we mean a Jordan region together with a pair of disjoint closed arcs on the boundary (the $b$ arcs). If $\bar{R}$ is conformally equivalent to $\{z=x+iy \mid 0\leqq x\leqq a,\ 0\leqq y\leqq b\}$ with the $b$ arcs being mapped to $\{z=x+iy \mid x=0 \text{ or } x=a\}$, then the modulus of $R$ is $\operatorname{mod} R=a/b$.

DEFINITION (GEOMETRIC). Let $V$ and $W$ be domains in $\boldsymbol{C}$ and let $w\colon V\to W$ be a homeomorphism such that

$$\operatorname{mod} R \leqq K \operatorname{mod} w(R),$$

for all rectangles $R$ with $\bar{R}\subset V$. Then $w$ is said to be a $K$ quasiconformal homeomorphism. A map $f\colon V\to \boldsymbol{C}$ is called $K$ quasiconformal if $f=g\circ w$, where $w$ is a $K$ quasiconformal homeomorphism and $g$ is holomorphic.

DEFINITION (ANALYTIC). Let $\mu$ be a complex valued measurable function on a domain $V\subset \boldsymbol{C}$ with $|\mu|\leqq (K-1)/(K+1)$. Let $w$ be a continuous function with $L_2$ derivatives satisfying the Beltrami equations $w_{\bar{z}}=\mu w_z$ almost everywhere. Then $w$ is called a $K$ quasiconformal map. Let $w\colon M\to M'$ be a mapping of Riemann surfaces. We say $w$ is $K$ quasiconformal if it is $K$ quasiconformal in terms of local coordinates.

In [3], Bers used the work of Mori and Morrey to show that the analytic and geometric definitions are equivalent. For further equivalent definitions, see Gehring [5].

3. **Schwarz's lemma.** From the analytic definition, it can be seen that the 1 quasiconformal maps on $V$ are precisely the holomorphic functions. Thus, it is natural to ask to what extent does the theory of holomorphic functions generalize to $K$ quasiconformal functions.

We shall begin by examining the Schwarz-Pick lemma. In its invariant form, it says that any holomorphic map $f\colon D\to D$ is distance decreasing with respect to the Poincaré-Bergman metric on $D$, i.e., that $d_D(f(x),f(y))\leqq d_D(x,y)$, $\forall x, y\in D$.

In order to obtain an analog for $K$ quasiconformal maps, we need the following theorem, which was discovered by Morrey, Ahlfors and Lavrent'ev and, in its sharp form, by Mori. For a proof, see Ahlfors [2].

THEOREM 1. *Let $w\colon D\to D$ be a $K$ quasiconformal homeomorphism with $w(0)=0$. Then*

$$|w(x)-w(y)| < 16|x-y|^{1/K}, \quad \textit{for all } x, y\in D. \tag{1}$$

This enables us to obtain the following.

THEOREM 2. *Let $K\geqq 1$ and $0<\varepsilon\leqq (1/32)^K$ be fixed. There exists a constant $C_{K,\varepsilon}$, which depends only on $K$ and $\varepsilon$, such that for any $K$ quasiconformal map $f\colon D\to D$, we have*

$$\begin{aligned} d_D(f(x),f(y)) &\leqq C_{K,\varepsilon}[d_D(x,y)]^{1/K}, && \textit{if } d_D(x,y)\leqq \varepsilon;\\ d_D(f(x),f(y)) &\leqq C_{K,\varepsilon}d_D(x,y), && \textit{if } d_D(x,y)\geqq \varepsilon. \end{aligned} \tag{2}$$

**Proof.** By using the Riemann mapping theorem, we can assume that $f=g\circ w$ where $w\colon D\to D$ is a $K$ quasiconformal homeomorphism and $g$ is holomorphic.

Since $g$ is distance decreasing by the classical Schwarz-Pick lemma, it suffices to show the inequality for $w$. Thus we can assume that $f$ is a homeomorphism.

It suffices to show that (2) holds when $f(0)=0$, $y=0$ and $x \in [0, 1)$. This follows since

(i) the set of holomorphic automorphisms of $D$ acts transitively on $D$;

(ii) $d_D$ is invariant under a holomorphic automorphism;

(iii) if $w$ is a $K$ quasiconformal map and $g$ and $h$ are holomorphic, then $g \circ w \circ h$ is $K$ quasiconformal.

Under the assumptions above, (1) yields

$$|f(x)| \leqq 16x^{1/K} \leqq 16\left[\log \frac{1+x}{1-x}\right]^{1/K} = 16[d_D(x, 0)]^{1/K}.$$

Choose $c>0$ such that $\log((1+r)/(1-r)) \leqq cr$ for $r \leqq \frac{1}{2}$. Since $|f(x)| \leqq \frac{1}{2}$ for $|x| \leqq \varepsilon \leqq (1/32)^K$, we have

$$d_D(f(x), f(0)) = \log \frac{1+|f(x)|}{1-|f(x)|} \leqq c|f(x)| \leqq 16c[d_D(x, 0)]^{1/K}, \quad \text{for } d_D(x, 0) \leqq \varepsilon.$$

By the remarks above, we conclude that

$$d_D(f(x), f(y)) \leqq 16c[d_D(x, y)]^{1/K} \quad \text{if } d_D(x, y) \leqq \varepsilon. \tag{3}$$

Choose $C_{K,\varepsilon}$ such that $16cr^{1/K} \leqq C_{K,\varepsilon} r$ for $\frac{1}{2}\varepsilon \leqq r \leqq \varepsilon$. Then (3) implies

$$d_D(f(x), f(y)) \leqq C_{K,\varepsilon} d_D(x, y) \quad \text{if } \tfrac{1}{2}\varepsilon \leqq d_D(x, y) \leqq \varepsilon. \tag{4}$$

Now if $x \in [0, 1)$ and $d_D(x, 0) \geqq \varepsilon$, we can find an increasing sequence of real numbers $0=x_0, x_1, \ldots, x_m=x$ such that $\frac{1}{2}\varepsilon \leqq d_D(x_i, x_{i-1}) \leqq \varepsilon$ for $1 \leqq i \leqq m$. Then by (4)

$$\begin{aligned} d_D(f(x), f(0)) &\leqq \sum_{i=1}^{m} d_D(f(x_i), f(x_{i-1})) \leqq C_{K,\varepsilon} \sum_{i=1}^{m} d_D(x_i, x_{i-1}) \\ &= C_{K,\varepsilon} d_D(x, 0). \end{aligned} \tag{5}$$

Since $16c \leqq C_{K,\varepsilon}$, equations (3) and (5) yield (2). Q.E.D.

REMARKS. (1) Theorems 1 and 2 give bounds for $|f(x)-f(y)|$ in terms of $x$ and $y$. Theorem 2 is meaningful for all $x$ and $y$, while Theorem 1 is meaningful only if $|x-y|$ is small. However, Theorem 1 gives a better estimate of $|f(x)-f(y)|$ if $|x-y|$ is small and $|y|$ is close to 1.

(2) If $K=1$, then Theorem 2 almost reduces to Schwarz's lemma. That is, it says that any holomorphic map $f\colon D \to D$ is distance decreasing up to a constant factor $c$ which is independent of $f$. It does not tell you that $c=1$.

(3) A simple computation shows that the function $f(z)=|z|^{1/K-1}z$ is $K$ quasiconformal. Suppose there existed a constant $c$ such that $d_D(f(x), f(y)) \leqq c d_D(x, y)$ for all $x, y \in D$. Then we could find $c'$ and $\delta>0$ such that $|f(x)| \leqq c'|x|$ for all $|x|<\delta$. This would imply that $|f(x)|/|x|$ is bounded for all $0<|x|<\delta$. But

$$\lim_{|z|\to 0} \frac{|f(z)|}{|z|} = \lim_{|z|\to 0} |z|^{1/K-1} = \infty.$$

Thus no such $c$ exists and, in this sense, Theorem 2 gives the best possible result. In [7], Ikoma has obtained results concerning the value of the

$$\liminf_{|z|\to 0} \frac{|f(x)|}{|z|^{1/K}}.$$

(4) The proof shows that $C_{K,\varepsilon}$ can be calculated explicitly.

In [9], Kobayashi defined an invariant pseudo-distance $d_M$ on each complex manifold $M$. Using Proposition 2.6 of that paper, we give the following equivalent definition of $d_M$ for a Riemann surface $M$. If $M$ is not covered by $D$, then $d_M \equiv 0$. Otherwise, let $\pi\colon D \to M$ be a universal covering map and let $p, q \in M$. If $\tilde{p} \in \pi^{-1}(p)$ and $A = \pi^{-1}(q)$, then define $d_M(p, q) = \inf_{\tilde{q}\in A} d_D(\tilde{p}, \tilde{q})$.

It is clear that for a Riemann surface $M$, either $d_M \equiv 0$ or $d_M$ is an actual distance. Since $d_D$ and $d_H$ are equivalent to the Poincaré-Bergman distances of $D$ and $H$ respectively, our notation is consistent.

For each $K \geqq 1$, we define a new pseudo-distance $h_{M,K}$ on each Riemann surface $M$. For each $K \geqq 1$, let the $\varepsilon$ and $C_{K,\varepsilon} = C_K$ of Theorem 2 be fixed for the remainder of the paper. Define

$$\begin{aligned} h_{M,K}(x, y) &= C_K d_M(x, y) && \text{if } d_M(x, y) \geqq 1, \\ &= C_K[d_M(x, y)]^{1/K} && \text{if } d_M(x, y) \leqq 1 \end{aligned} \tag{6}$$

for each $x, y \in M$. Then $h_{M,K}$ is a pseudo-distance on $M$, such that $h_{M,K} \equiv 0$ if $d_M \equiv 0$ and $h_{M,K}$ is an actual distance otherwise.

Theorem 2 can now be stated in the following form.

THEOREM 2′. *Let $f\colon D \to D$ be a $K$ quasiconformal map. Then $f$ is distance decreasing with respect to the metrics $h_{D,K}$ and $d_D$ in the following sense*:

$$d_D(f(x), f(y)) \leqq h_{D,K}(x, y), \qquad \forall x, y \in D.$$

Before proceeding, it is necessary to prove

LIOUVILLE'S THEOREM. *Let $f\colon \boldsymbol{C} \to \boldsymbol{C}$ be a bounded, $K$ quasi-conformal map. Then $f$ is constant.*

**Proof.** We can assume $f(\boldsymbol{C}) \subset D$ and $f(0) = 0$. Define $g_n\colon D \to D$ by $g_n(z) = f(nz)$. Each $g_n$ is $K$ quasiconformal and if $z_0 \in \boldsymbol{C}$ with $|z_0| < m$, then $g_n(z_0/n) = f(z_0)$ for all $n > m$. Using Theorem 2′, we have

$$d_D(f(z_0), f(0)) = d_D(g_n(z_0/n), g_n(0)) \leqq h_{D,K}(z_0/n, 0).$$

Since the right hand side goes to zero as $n \to \infty$, we have $d_D(f(z_0), 0) = 0$. Thus $f \equiv 0$. Q.E.D.

THEOREM 3. *Let $f\colon M \to M'$ be a $K$ quasiconformal map of Riemann surfaces. Then $f$ is distance decreasing with respect to the pseudo-metrics $h_{M,K}$ and $d_{M'}$ in the sense that*:

$$d_{M'}(f(x), f(y)) \leqq h_{M,K}(x, y) \quad \textit{for every } x, y \in M. \tag{7}$$

**Proof.** There are three cases

*Case* 1. $d_{M'} \equiv 0$. There is nothing to prove.

*Case* 2. $d_M \equiv 0$ and $d_{M'} \not\equiv 0$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & D \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

where $\tilde{M} = \boldsymbol{C}$ or the Riemann sphere $P_1(\boldsymbol{C})$, $\pi$ and $\pi'$ are holomorphic covering projections and $\tilde{f}$ is $K$ quasiconformal. Since $\boldsymbol{C} \subset P_1(\boldsymbol{C})$, Liouville's theorem implies that $\tilde{f}$ is constant. Thus $f$ is constant and (7) is trivially satisfied.

*Case* 3. $d_M \not\equiv 0$ and $d_{M'} \not\equiv 0$.

This implies that we have the following commutative diagram:

$$\begin{array}{ccc} D & \xrightarrow{\tilde{f}} & D \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

where $\pi$, $\pi'$ and $\tilde{f}$ are as above. Let $p, q \in M$, $\tilde{p} \in \pi^{-1}(p)$ and $A = \pi^{-1}(q)$. Then using Theorem 2′, we have:

$$\begin{aligned} d_{M'}(f(p), f(q)) &\leqq \inf_{\tilde{q} \in A} d_D(\tilde{f}(\tilde{p}), \tilde{f}(\tilde{q})) \\ &\leqq \inf_{\tilde{q} \in A} h_{M,K}(\tilde{p}, \tilde{q}) = h_{M,K}(p, q). \end{aligned}$$

Q.E.D.

Following the terminology of [9], we say that a Riemann surface $M$ is *hyperbolic* if $d_M$ is a proper distance. This leads to a generalization of the little Picard theorem.

COROLLARY. *Let $M$ and $M'$ be Riemann surfaces, with $M'$ hyperbolic and $d_M \equiv 0$. If $f$: $M \to M'$ is $K$ quasiconformal, then $f$ is a constant map. In particular, any $K$ quasiconformal map $f$: $\boldsymbol{C} \to P_1(\boldsymbol{C}) - \{3$ points$\}$ is constant.*

**Proof.** $d_M \equiv 0$ implies $h_{M,K} \equiv 0$. By Theorem 3, we have for any $x, y \in M$ $d_{M'}(f(x), f(y)) \leqq h_{M,K}(x, y) = 0$. Thus $f(x) = f(y)$ for any $x, y \in M$ and $f$ is constant. Clearly $d_{\boldsymbol{C}} \equiv 0$. The fact that $P_1(\boldsymbol{C}) - \{3$ points$\}$ is covered by $D$ is well known and implies that $P_1(\boldsymbol{C}) - \{3$ points$\}$ is hyperbolic. The second statement of the corollary also follows from Liouville's theorem. Q.E.D.

4. **Applications.** We now use the previous results to give elementary proofs of known theorems for $K$ quasiconformal maps. Throughout this section, $M$ will denote a Riemann surface which is covered by the unit disc $D$, i.e., $M$ will be hyperbolic.

SCHOTTKY'S LEMMA. *Let $U$ and $V$ be subsets of $M$ with $U$ compact, $V$ open and $U \subset V$. Let $f$: $D \to M$ be a $K$ quasiconformal map. Then there exists a constant $r > 0$, depending only on $U$ and $V$ but independent of $f$, such that $f(0) \in U$ implies $f(z) \in V$ for $|z| < r$.*

**Proof.** Since $U$ is compact and $V$ open, there exists $s > 0$ such that $d_D(x, y) \leqq s$ and $x \in U$ imply that $y \in V$. Choose $r > 0$ such that $|z| < r$ (where $z \in D$) implies that $h_{D,K}(z, 0) < s$. By Theorem 3,

$$d_M(f(0), f(z)) \leqq h_{D,K}(z, 0) < s \quad \text{if } |z| < r$$

and the result follows. Q.E.D.

Before proving a generalization of the big Picard theorem, we need the following:

LEMMA. *Consider the punctured disk $D^* = \{z \in \boldsymbol{C} \mid 0 < |z| < 1\}$ and let $f$: $D^* \to D^*$ be a $K$ quasiconformal map. Then $f$ can be extended to a $K$ quasiconformal map $f$: $D \to D$. If $f$: $D^* \to A_r = \{z \in \boldsymbol{C} \mid r < |z| < 1\}$ is $K$ quasiconformal and $f$ is not homotopic to a constant function, then $r = 0$.*

**Proof.** We can assume that $f = g \circ w$, where $w$: $D^* \to W$ is a $K$ quasiconformal homeomorphism and $g$: $W \to D^*$ is holomorphic. The complement of $W$ in $P_1(\boldsymbol{C})$ is the union of two disjoint simply connected sets $A$ and $B$. Since $g$ is bounded, we have $W \neq \boldsymbol{C} - \{pt\}$. Thus we can assume $A$ contains more than one point. Using the Riemann mapping theorem, we obtain a conformal mapping $v$: $W \cup B \to D$ with $v^{-1}(0) \in B$. This implies that we can assume that $W \subset D^*$ and that $w$ is not homotopic to a constant map. Using elementary topology, it is easy to show that $f$ will have a removable singularity if $w$ can be extended to a $K$ quasiconformal homeomorphism of $D$ into $D$.

Let $\varphi$: $H \to D^*$ be defined by $\varphi(z) = e^{2\pi i z}$ and let $\tilde{w} = w \circ \varphi$. Define a hermitian metric on $D^*$ by

$$ds^2_{D^*} = \frac{dz\, d\bar{z}}{|z|^2(\log |z|^2)^2}.$$

Then $\varphi^* ds^2_{D^*} = ds^2_H$. Since $\varphi$ is a covering map, this means that the metric $ds^2_{D^*}$ determines the Kobayashi distance $d_{D^*}$ (see [11, Chapter VI]). Define $\gamma_n$: $[0, 1] \to H$ by $\gamma_n(t) = t + in$. A simple computation shows that the length of $\gamma_n$ with respect to the Poincaré-Bergman metric $d_H$ is $L(\gamma_n) = 1/n$. If $a_n$ is the diameter of the set $\{\gamma_n(t) \mid t \in [0, 1]\}$ with respect to $h_{H,K}$, it follows that $a_n \to 0$ as $n \to \infty$. Theorem 3 now implies that if $b_n$ equals the diameter of the set $\{\tilde{w}(\gamma_n(t)) \mid t \in [0, 1]\}$ with respect to $d_{D^*}$, then $b_n \to 0$ as $n \to \infty$. This says that the circles $\varphi \circ \gamma_n$ are mapped by $w$ to simple Jordan curves which converge to 0. This follows from the formula for $ds^2_{D^*}$, and the fact that $w$ is not homotopic to a constant. Therefore $w$ can be extended to all of $D$ by setting $w(0) = 0$.

To prove the second statement, let $f$: $D^* \to A_r$ be a $K$ quasiconformal map which is not homotopic to a constant. If $0 < r' < r$, then the first part of the lemma

implies $f$ can be extended to $\tilde{f}$: $D \to A_{r'}$. We have the diagram

$$\begin{tikzcd} D^* \arrow[r, "f"] \arrow[d, "i"'] & A_{r'} \\ D \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

where $i$ is the inclusion. This is a contradiction since $\tilde{f} \circ i$ is homotopically trivial. Thus $r=0$. Q.E.D.

We are now in a position to generalize Ohtsuka's theorem (see [12]). When $M=P_1(C)-\{3 \text{ points}\}$, this reduces to the big Picard theorem.

OHTSUKA'S THEOREM. *Let $M$ be a hyperbolic Riemann surface and assume that $M$ is embedded in an arbitrary Riemann surface $M'$ with $\overline{M}$ compact. Let $f$: $D^* \to M$ be a $K$ quasiconformal map. Then $f$ can be extended to a map $f$: $D \to \overline{M}$.*

**Proof.** There are two cases.

(1) $f_*(\pi_1(D^*))=0$. In this case $f$ can be lifted to a $K$ quasiconformal map $\tilde{f}$: $D^* \to D$. That is, we obtain the diagram:

$$\begin{tikzcd} & D \arrow[d, "p"] \\ D^* \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & M \end{tikzcd}$$

The lemma implies that $\tilde{f}$ can be extended and therefore so can $f$.

(2) If $f_*(\pi_1(D^*)) \neq 0$, there exists (see [12]) $r \geqq 0$ such that the following diagram is commutative

$$\begin{tikzcd} & A_r \arrow[d, "p"] \\ D^* \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & M \end{tikzcd}$$

Here $A_r=\{z \in \boldsymbol{C} \mid r<z<1\}$, $p$ is a holomorphic covering projection and $\tilde{f}$ is a $K$ quasi-conformal map. The lemma implies that $r=0$ and that $\tilde{f}$ can be extended to a map $\tilde{f}$: $D \to D$. Now since $p$: $A_0=D^* \to M$ is holomorphic, the original version of Ohtsuka's theorem [12] says that $p$ can be extended to a map $p$: $D \to \overline{M}$. Thus $f$ can be extended. Q.E.D.

5. **Moser's Theorem.** In this section, we prove a special case of a theorem due to Moser [13]. For convenience, we shall assume that all functions are sufficiently differentiable.

Let $\bar{D}_r=\{z=x_1+ix_2\in \boldsymbol{C} \mid |z|\leqq r\}$ and let $a_{ij}\colon D\to R$ be functions for $1\leqq i, j\leqq 2$ such that $A=(a_{ij}(z))$ is symmetric and positive definite. Moreover, the positive eigenvalues of $A$ are assumed to lie between two positive values, say $1/\lambda$ and $\lambda$ for some fixed constant $\lambda\geqq 1$.

THEOREM (MOSER). *Let $u\colon D\to R^+$ be a positive solution of the equation*

$$\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)=0. \tag{8}$$

*Then for each $0\leqq r<1$, there exists a constant $C_r$, depending only on $\lambda$ and independent of $u$, such that*

$$\max_{z\in \bar{D}_r} u(z)\leqq C_r \min_{z\in \bar{D}_r} u(z).$$

**Proof.** Let $u$ be a positive solution. Equation (8) implies that the 1 form

$$\varphi=\left(-a_{2j}\frac{\partial u}{\partial x_j}\right)dx_1+\left(a_{1j}\frac{\partial u}{\partial x_j}\right)dx_2$$

is closed. Therefore, there exists a function $v\colon D\to R$ such that $dv=\varphi$. Now consider the map $f\colon D\to H$ defined by $f(z)=v(z)+iu(z)$.

Computing, we have

$$\begin{aligned}\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2+\left(\frac{\partial v}{\partial x_1}\right)^2+\left(\frac{\partial v}{\partial x_2}\right)^2 &= \left\|\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2}\right)\right\|^2+\left\|A\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2}\right)\right\|^2\\ &\leqq (1+\lambda^2)\left\|\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2}\right)\right\|^2\\ &\leqq (1+\lambda^2)\lambda\left\langle A\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2}\right),\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2}\right)\right\rangle\\ &= (1+\lambda^2)\lambda J\end{aligned}$$

where $\langle\ ,\ \rangle$ is the usual inner product in $R^2$ and $J$ is the Jacobian of $f$. This implies that $f$ is $K$ quasiconformal for some $K$ depending only on $\lambda$ (see [17, p. 223]).

Let $C_r'$ denote the diameter of $\bar{D}_r$ with respect to $h_{D,K}$. Using Theorem 3, we have:

$$d_H(f(z_1),f(z_2))\leqq C_r' \quad \text{for } z_1, z_2\in \bar{D}_r.$$

Since $d_H$ is the distance function corresponding to the metric $ds^2=(1/4u^2)(du^2+dv^2)$, we have

$$\left|\log\frac{u(z_1)}{u(z_2)}\right|\leqq d_H(f(z_1),f(z_2))\leqq C_r' \quad \text{for } z_1, z_2\in \bar{D}_r.$$

Let $u$ assume its maximum in $\bar{D}_r$ at $z_1$ and its minimum in $\bar{D}_r$ at $z_2$. Then $\log(u(z_1)/u(z_2))\leqq C_r'$ and setting $C_r=e^{C_r'}$, we obtain $u(z_1)\leqq C_r u(z_2)$. Q.E.D.

6. **$K$ quasiconformal self-mappings.** In this section, we generalize some of Huber's results on analytic self-mappings of Riemann surfaces to $K$ quasiconformal maps. Huber's results can be found in [6] or [12]. In this section, we define $B_r=\{z\in \boldsymbol{C} \mid 0<r<|z|<1/r\}$.

THEOREM 4. *Let $f$: $B_r \to B_r$ be a $K$ quasiconformal map. Then $|\deg f| \leqq K$ and if $|\deg f| = K$, then $f$ is given up to a rotation or inversion by $f(z) = |z| e^{iK \arg z}$.*

**Proof.** Let $f = g \circ w$ where $w$: $B_r \to B_s$ is a $K$ quasiconformal homeomorphism and $g$: $B_s \to B_r$ is analytic with $\deg g = \deg f$. We can assume $|\deg g| = m > 0$. Since $g$: $B_s \to B_r$ is holomorphic, a theorem of Huber (see [15, p. 207]) implies that $1 > s \geqq r^{1/m}$. Let $s = r^{1/Q}$ where $Q \geqq m$. In [14], Mori shows that any $Q$ quasiconformal homeomorphism $h$: $B_r \to B_s$ is defined up to a rotation or an inversion by $h(z) = |z|^{1/Q} e^{i \arg z}$. If $Q > K$, then $w$: $B_r \to B_s$ could not be of this form. Thus $Q \leqq K$ and the first statement is proved.

Now if $Q = K$, then up to a rotation or inversion $w(z) = |z|^{1/K} e^{i \arg z}$. By the theorem of Huber referred to above, $g(z) = z^K$ up to a rotation or an inversion. Thus up to a rotation or an inversion $f(z) = |z| e^{iK \arg z}$. Q.E.D.

COROLLARY. *Let $f$: $B_r \to B_t$ be a $K$ quasiconformal map. If $t > r^{1/K}$, then $f$ is homotopic to a constant map.*

**Proof.** Let $f = g \circ w$ where $w$: $B_r \to B_s$ is a homeomorphism and $g$: $B_s \to B_t$ is holomorphic. If $|\deg g| > 0$, then, as noted in the previous proof, $s > r^{1/K}$. Let $s = r^{1/Q}$ where $Q > K$. Now $w$ is $Q$ quasiconformal, but it is not of the form $w(z) = |z|^{1/Q} e^{i \arg z}$ (up to a rotation or inversion). This is impossible. Therefore $\deg g = 0$. Q.E.D.

The following theorem is a partial generalization of a theorem due to Huber. The proof is similar to the proof of Theorem 3 in [12], which can be referred to for some of the topological details.

THEOREM 5. *Let $M$ be a hyperbolic Riemann surface and let $\gamma$ be a closed curve in $M$ which is not homotopically trivial and which is not a point cycle. Let $f$: $M \to M$ be a $K$ quasiconformal map with $f(\gamma) \sim \gamma^q$. Then $|q| \leqq K$. If $q = -K$, then $f$ has a fixed point.*

**Proof.** Following essentially the proof of Theorem 3 in [12], we obtain a commutative diagram:

$$\begin{tikzcd} B_r \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & B_r \arrow[d, "p"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

where $p$ is a holomorphic covering projection such that $p_*(\pi_1(B_r)) = \langle \gamma \rangle$ and $\tilde{f}$ is a $K$ quasiconformal map of degree $q$. From Theorem 4 it follows that $|q| \leqq K$. If $\deg f = -K$, then $\tilde{f}$ has a fixed point and thus so does $f$. Q.E.D.

Let $M$ be a hyperbolic Riemann surface. Then for each nontrivial closed curve $\gamma$ in $M$ we obtain a covering surface $p_\gamma$: $\tilde{M}(\gamma) \to M$ with the property that

$p_{\gamma^*}(\pi_1(\tilde{M}(\gamma)))=\langle\gamma\rangle$. If $\tilde{M}(\gamma)=B_r$, then we say that $\gamma$ is of type $r$. The corollary to Theorem 4 implies that if $\gamma$ is of type $r$ and $f(\gamma)$ is of type $t$, then $t\leqq r^{1/K}$ if $f\colon M\to M'$ is $K$ quasiconformal.

7. **Harmonic, $K$ quasiconformal maps.** In remark (3) following Theorem 2, it was observed that in general $K$ quasiconformal self-mappings of the unit disc $D$ are not distance decreasing with respect to the Poincaré-Bergman metric. In this section we show that for a special class of $K$ quasiconformal maps, the situation is different.

Before doing this, it is necessary to introduce some notation and to prove a general lemma. Let $f\colon M\to M'$ be a map of Riemannian manifolds. The map $f$ induces a vector bundle $W=f^*(T(M'))$ over $M$, where $T(M')$ is the tangent bundle of $M'$. The differential $df$ can be considered as a 1-form with coefficients in the bundle $W$. Let $\varphi=\|df\|^2$. Using a method of Bochner (see [4, p. 121] or [16, Chapter II]), we will compute a formula for the Laplacian $\Delta\varphi$ which involves the curvatures of $M$ and $M'$. By examining a point where $\varphi$ has a maximum, this formula will imply that under some additional assumptions, harmonic $K$ quasiconformal maps are distance decreasing. To avoid confusion on signs, we shall use the following notation.

$M=$ Riemannian manifold with metric given in a coordinate neighborhood $U=(u_1,\ldots,u_n)$ by $ds^2=g_{ij}\,du_i\,du_j$ and with metric connection $\nabla$ in $T(M)$.

$W=$ Riemannian vector bundle over $M$ with local trivializing sections $w_\alpha$ over $U$. The metric is given by $\langle w_\alpha, w_\beta\rangle=a_{\alpha\beta}$ and the metric connection in $W$ is denoted by $\nabla'$.

$\nabla$ is the connection induced by $\nabla$ and $\nabla'$. This defines the covariant derivative of elements of $A^p(M, W)=\{p\text{-forms with coefficients in } W\}$.

$d\colon A^p(M, W)\to A^{p+1}(M, W)$ is the exterior derivative.

$\delta\colon A^p(M, W)\to A^{p-1}(M, W)$ is the adjoint of $d$.

$\Delta=d\delta+\delta d$ is the Laplacian. Thus for a function $h$ on $M$, we have $\Delta h=-g^{ij}\nabla_i\nabla_j h$ and $\Delta h\geqq 0$ at a maximum of $f$.

$R^i_{jkl}$ is defined by $\nabla_l\nabla_k X_j-\nabla_k\nabla_l X_j=R^i_{jkl}X_i$ where $X_i=\partial/\partial u_i$.

$R'^{\alpha}{}_{\beta kl}$ is defined by $\nabla'_l\nabla'_k w_\beta-\nabla'_k\nabla'_l w_\beta=R'^{\alpha}{}_{\beta kl}w_\alpha$.

We shall follow the usual conventions of tensor calculus (e.g. see [16]). However, it should be noted that the definition of $R^i_{jkl}$ in [16] differs from ours by a minus sign. Finally we observe that if $M$ has constant curvature $G$, then

$$R_{ijkl} = G(g_{jk}g_{il}-g_{jl}g_{ik}).$$

LEMMA. *Let $\psi\in A'(M, W)$ be given in $U$ by $\psi=\psi^\alpha_b w_\alpha$. Then*

$$\Delta\langle\tfrac{1}{2}\psi, \psi\rangle = \langle\Delta\psi, \psi\rangle-\|\nabla\psi\|^2-R^{hj}\psi^\alpha_h\psi^\beta_j a_{\alpha\beta}-R'^{\alpha h}{}_{\beta i}\psi^\beta_h\psi^\gamma_k g^{ik}a_{\alpha\gamma}. \tag{9}$$

**Proof.** Computing, we have

$$\begin{aligned}(\psi^\alpha_b w_\alpha)_{;i;c}-(\psi^\alpha_b w_\alpha)_{;c;i} &= \{\psi^\alpha_{b;i;c}-\psi^\alpha_{b;c;i}\}w_\alpha+\psi^\alpha_b\{w_{\alpha;i;c}-w_{\alpha;c;i}\}\\ &= (-\psi^\alpha_h R^h{}_{bic})w_\alpha+(\psi^\beta_b R'^{\alpha}{}_{\beta ic})w_\alpha.\end{aligned}$$

Now

$$g^{bc}\psi_{i;b;c}+[-g^{bc}(\psi_{i;b}-\psi_{b;i})_{;c}-g^{bc}\psi_{b;c;i}] = g^{bc}\{-\psi_h^\alpha R^h{}_{bic}+\psi_b^\beta R'^\alpha{}_{\beta ic}\}w_\alpha.$$

Since the expression in [ ] is $\Delta\psi$ and since

$$g^{bc}\psi_{i;b;c} = g^{bc}\nabla_b\nabla_c\psi_i = \nabla^k\nabla_k\psi_i,$$

we have

$$\Delta\psi_i = -\nabla^k\nabla_k\psi_i+\psi_h^\alpha R_i^h w_\alpha+R'^{\alpha h}{}_{\beta i}\psi_h^\beta w_\alpha. \tag{10}$$

But $\Delta\langle\frac{1}{2}\psi, \psi\rangle=\langle-\nabla^k\nabla_k\psi, \psi\rangle-\langle\nabla^k\psi, \nabla_k\psi\rangle$. The desired result now follows by substituting (10) in the last equation. Q.E.D.

Let $f$: $M\to M'$ be a map, where $M$ and $M'$ are Riemann surfaces with hermitian metrics. (For convenience, from now on all maps will be assumed to be sufficiently differentiable.) As noted above, $\psi=df$ is an element of $A'(M, W)$ where $W=f^*(T(M'))$. We shall interpret the previous lemma for this case.

Choose local coordinates $z=x_1+ix_2$ in $U\subset M$ and $w=u_1+iu_2$ in $V\subset M'$ where $f(U)\subset V$. In $U$ and $V$, the metrics are given by $ds_u^2=g(dx_1^2+dx_2^2)$ and $ds_V^2=g'(du_1^2+du_2^2)$. The vector fields $\partial/\partial u_1$ and $\partial/\partial u_2$ induce, in a natural way, trivializing sections $w_1$ and $w_2$ of $W|_U$. In terms of the notation introduced above, we have $a_{\alpha\beta}=\langle w_\alpha, w_\beta\rangle=\delta_{\alpha\beta}g'$ and $R'^\alpha{}_{\beta Kl}$ is essentially the Riemannian tensor of $M'$. Since

$$\psi = df = \frac{\partial u_\alpha}{\partial x_j}\,dx_j\otimes\frac{\partial}{\partial u_\alpha} = \left(\frac{\partial u_\alpha}{\partial x_j}\,dx_j\right)w_\alpha,$$

we have $\psi_j^\alpha=\partial u_\alpha/\partial x_j$. Letting $G_p$ and $G_q'$ denote the Gaussian curvature at $p\in M$ and $q\in M'$ respectively, the lemma implies:

$$\begin{aligned}\Delta\langle\tfrac{1}{2}\psi, \psi\rangle = \langle\Delta\psi, \psi\rangle-\|\nabla\psi\|^2-G_p\{\|f_*(X_1)\|^2+\|f_*(X_2)\|\}\\ +2G'_{f(p)}\|f_*(X_1)\wedge f_*(X_2)\|^2\end{aligned} \tag{11}$$

where $X_i=\partial/\partial x_i$.

Following Eells-Sampson [4] we say that $f$ is harmonic if $\Delta\psi=0$. For Riemann surfaces this concept is independent of the choice of the hermitian metric in the domain. Let $f$ be harmonic and let $p$ be a maximum point of the function $\langle\frac{1}{2}\psi, \psi\rangle$. Then (11) implies that

$$0 \leqq -G_p\{\|f_*(X_1)\|^2+\|f_*(X_2)\|^2\}+2G'_{f(p)}\|f_*(X_1)\wedge f_*(X_2)\|^2. \tag{12}$$

THEOREM 6. *Let $f$: $M\to M'$ be a harmonic $K$ quasiconformal map where $M$ and $M'$ are as above. Assume that $G_p\geqq -A$ for every $p\in M$ and $G_q'\leqq -B<0$ for every $q\in M'$, where $A$ and $B$ are positive constants. If the function $\langle\frac{1}{2}\psi, \psi\rangle$ has a maximum on $M$, then*

$$\|f_*(X)\|_{M'}^2 \leqq (2A/B)K^2\|X\|_M^2 \quad \textit{for every } X\in T(M).$$

*In particular, if $M$ is compact and the curvature conditions are satisfied, then every harmonic, $K$ quasiconformal map $f$: $M\to M'$ is distance decreasing up to a fixed constant.*

**Proof.** Let $p$ be a maximum point of $\langle \frac{1}{2}\psi, \psi\rangle$. Using the curvature assumptions, (12) yields:

$$2\|f_*(X_1) \wedge f_*(X_2)\|^2 \leqq (A/B)\{\|f_*(X_1)\|^2 + \|f_*(X_2)\|^2\}. \tag{13}$$

Assume that $X_1$ and $X_2$ are orthonormal at $p$ and let $J$ denote the Jacobian of $f$ at $p$ with respect to the local coordinates. A simple calculation shows that $J = \|f_*(X_1) \wedge f_*(X_2)\|$, and since $f$ is $K$ quasiconformal, $\|f_*(X_i)\|^2 \leqq JK$. By (13), $J^2 \leqq (A/B)KJ$ and therefore $J \leqq (A/B)K$. Now

$$\langle \psi, \psi\rangle_p = \|f_*(X_1)\|^2 + \|f_*(X_2)\|^2 \leqq 2JK \leqq (2A/B)K^2.$$

Since $\langle \psi, \psi\rangle$ has a maximum at $p$, the result follows. Q.E.D.

COROLLARY 1. *Let $f\colon M \to M'$ be holomorphic and let everything else be as above. Then*

$$\|f_*(X)\|^2_{M'} \leqq (A/B)\|X\|^2_M.$$

**Proof.** Let $u = \|f_*(X_1)\|^2$. Then (13) becomes $2u^2 \leqq (A/B)2u$ or $u \leqq A/B$. Since $\langle \frac{1}{2}\psi, \psi\rangle \leqq u \leqq A/B$ and since $df$ at any point is a rotation followed by an expansion, the proof is complete. Q.E.D.

COROLLARY 2. *Let $M = D$ with the Poincaré-Bergman metric. Let everything else be as in Theorem 6, except possibly for the assumption $\langle \psi, \psi\rangle$ has a maximum. Then the conclusions of the theorem are still valid.*

**Proof.** Let $D_r = \{z \mid |z| < r < 1\}$ with the metric $ds_r^2 = (r^2/(r^2 - |z|^2)^2)\, dz\, d\bar{z}$ and let $i_r\colon D_r \to D$ be the inclusion. If $f_r = f \circ i_r$ and $\psi_r = df_r$, then it is easy to see that $\langle \psi_r, \psi_r\rangle$ has a maximum in $D_r$. Since $f_r$ is harmonic and $K$ quasiconformal, we can apply Theorem 6 to $f_r$. By letting $r \to 1$, we obtain the desired result for $f$. Q.E.D.

It is clear that we can use the same trick to replace $M$ in Corollary 1 by $D$. Since any holomorphic map between Riemann surfaces is harmonic, the corollary reduces to the generalized Schwarz-Pick lemma when $K = 1$ (see Kobayashi [10]).

THEOREM 7. *Let $M$ and $M'$ be Riemann surfaces and let $f\colon M \to M'$ be harmonic and $K$ quasiconformal. Then*

$$d_{M'}(f(x), f(y)) \leqq \sqrt{2K}\, d_M(x, y), \quad \textit{for every } x, y \in M.$$

**Proof.** If $d_{M'} \equiv 0$, there is nothing to prove and if $d_M \equiv 0$, Theorem 3 implies that $f$ is constant. Thus we can assume that $M$ and $M'$ are hyperbolic. We have the following commutative diagram:

$$\begin{tikzcd} D \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & D \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

where $\pi$ and $\pi'$ are holomorphic and $f$ and $\tilde{f}$ are harmonic, $K$ quasiconformal maps. The Poincaré-Bergman metric on $D$ has constant curvature $-4$ and is invariant under the group of holomorphic automorphisms of $D$. Thus $\pi$ and $\pi'$ induce hermitian metrics on $M$ and $M'$ which have constant curvature $-4$ and which induce the metrics $d_M$ and $d_{M'}$. Using Corollary 2 and the fact that $\pi_*$ is an isometry, we obtain the infinitesimal version of the desired result. Q.E.D.

## References


1. L. Ahlfors, *On quasiconformal mappings*, J. Analyse Math. **3** (1954), 1–58; correction, 207–208. MR **16**, 348.

2. ———, *Lectures on quasiconformal mappings*, Van Nostrand, Princeton, N. J., 1966. MR **34** #336.

3. L. Bers, *On a theorem of Mori and the definition of quasiconformality*, Trans. Amer. Math. Soc. **84** (1957), 78–84. MR **18**, 646.

4. J. Eells, Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160. MR **29** #1603.

5. F. Gehring, *Definitions for a class of plane quasiconformal mappings*, Nagoya Math. J. **29** (1967), 175–184. MR **35** #4410.

6. H. Huber, *Über analytische Abbildungen Riemannscher Flächen in sich*, Comment. Math. Helv. **27** (1953), 1–73. MR **14**, 862.

7. K. Ikoma and K. Shibata, *On distortions in certain quasiconformal mappings*, Tôhoku Math. J. (2) **13** (1961), 241–247. MR **26** #6405.

8. O. Lehto and K. I. Virtanen, *Quasikonforme Abbildungen*, Springer-Verlag, Berlin, 1965. MR **32** #5872.

9. S. Kobayashi, *Invariant distances on complex manifolds and holomorphic mappings*, J. Math. Soc. Japan **19** (1967), 460–480. MR **38** #736.

10. ———, *Distance, holomorphic mappings and the Schwarz lemma*, J. Math. Soc. Japan **19** (1967), 481–485. MR **36** #1705.

11. ———, *Lecture notes on hyperbolic manifolds*, Lecture Notes in Math., Springer-Verlag, New York (to appear).

12. A. Marden, I. Richards and B. Rodin, *Analytic self-mappings of Riemann surfaces*, J. Analyse Math. **18** (1967), 197–225. MR **35** #3057.

13. J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591. MR **28** #2356.

14. A. Mori, *On quasiconformality and psuedo-analyticity*, Trans. Amer. Math. Soc. **84** (1957), 56–77. MR **18**, 646.

15. C. B. Morrey, Jr., *On the solutions of quasilinear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.

16. K. Yano and S. Bochner, *Curvature and Betti numbers*, Ann. of Math. Studies, no. 32, Princeton Univ. Press, Princeton, N. J., 1953. MR **15**, 989.

17. H. Wu, *Normal families of holomorphic mappings*, Acta Math. **119** (1967), 193–233. MR **37** #468.


University of California,
Berkeley, California