

STABLE MAPS AND SCHWARTZ MAPS

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1. **Introduction.** In the present paper M and N will denote two von Neumann algebras where $N \subset M$. If \mathcal{A} is any von Neumann algebra, \mathcal{A}' will denote the commutant of \mathcal{A} . N^c will denote the relative commutant of N in M , i.e. $N^c = N' \cap M$. $U(N)$ will denote all unitary operators of N . Let G be a group of unitaries of M . Let ϕ be a linear map of M into M . ϕ is called G -stable if $\phi(UXU^{-1}) = \phi(X)$ for all X in M and all U in G . $S(G, M)$ will denote all Schwartz maps which are G -stable. The purpose of this paper is to study the existence and properties of G -stable expectations. The main results contained here are:

THEOREM 1. *Let Tr be a faithful, semifinite trace on M . Let L be a von Neumann subalgebra of M such that Tr restricted to L is semifinite. Then there exists a normal, faithful, $U(L^c)$ stable expectation ϕ of M on L such that $\text{Tr}(A\phi(X)) = \text{Tr}(AX)$ for all X in M and all A in L for which $\text{Tr}|A| < \infty$.*

THEOREM 2. *Suppose M has a faithful, normal, semifinite trace, call it Tr . Suppose $S(G, M)$ is sufficiently large, then there exists a faithful, normal, $U(N^{cc})$ stable expectation of M on N^c .*

As corollary to the above theorem, it follows that with the hypothesis of Theorem 2, N is finite, N^c can not be purely infinite. Moreover if M is of type I so is N^c . Another corollary to Theorem 2 is that if $S(G, L(h))$ has sufficiently many maps, then the von Neumann algebra N generated by G is atomic.

Next a notion of equivalence of two unitary groups will be defined. Two groups of unitary operators are equivalent if they generate the same von Neumann algebra.

THEOREM 3. *Assume $S(G, L(h))$ contains a normal map, then G is equivalent to a countable direct sum of finite groups.*

The next result is a sort of converse to Theorem 3.

THEOREM 4. *If G has the property (F), then G is a countable direct sum of finite groups and $S(G, M)$ has sufficiently many maps.*

A corollary to Theorem 3 is that if N is a finite atomic von Neumann algebra, then N is generated by a direct sum of finite unitary groups.

Next uniqueness of expectations of certain type will be considered. The main result of this section is:

THEOREM 5. *Assume that*

- (1) $N^c \subset N$,
- (2) N is finite,
- (3) M is semifinite.

Then there exists at most one normal expectation ϕ of M on N .

2. Preliminaries.

DEFINITION. Let ϕ be a map of M into N which preserves the identity. Assume that ϕ is a positive linear map and that $\phi(AX) = A\phi(X)$ for all A in N and X in M . ϕ will then be called an expectation of M in N .

It is trivial to see that ϕ is onto N and that ϕ is a bounded map. The notion of expectations in von Neumann algebras was studied in [2], [7], and [9].

DEFINITION. Let ϕ be an expectation of M on N , ϕ is called normal if $\phi(\text{Sup } A_\alpha) = \text{Sup } \phi(A_\alpha)$ for any increasing net of uniformly bounded selfadjoint operators.

ϕ is called faithful if, given a positive operator A such that $\phi(A) = 0$, then $A = 0$.

Let ϕ_α be a set of expectations of M onto N . The set ϕ_α is called complete if given a positive operator A such that $\phi_\alpha(A) = 0$ then $A = 0$.

DEFINITION. Let G be a subgroup of $U(M)$. By a Schwartz map relative to (G, M) one means a linear map of M into itself such that

- (1) $P(X) = UP(X)U^{-1}$ for all U in G and all X in M ,
- (2) $P(X)$ is in $C_G[X]$ where $C_G[X]$ denotes the weak closure of the convex hull generated by elements of the type UXU^{-1} as U ranges over G .

For more information on Schwartz maps see [6].

$S(G, M)$ will denote all Schwartz maps relative to (G, M) which are G -stable, i.e. $P(X) = P(VXV^{-1})$ for all V in G . $S(G, M)$ will be called sufficient if for any positive operator X in M such that $P(X) = 0$ for all P in $S(G, M)$ then $X = 0$.

DEFINITION. A group G is said to be amenable as a discrete group if there exists a finitely additive probability measure μ on the field of all subsets of G such that $\mu(xE) = \mu(E)$. For more information on amenable groups see [4] and [5].

3. Stable maps and Schwartz maps.

LEMMA 1. *If there exists a complete set of U -stable expectations of M on N then U is in N^c .*

Proof. Let V be a unitary of N . Let ϕ_α be a complete set of U -stable expectations, then $\phi_\alpha(UVU^{-1}V^{-1}) = \phi_\alpha(VU^{-1}V^{-1}U) = V\phi_\alpha(U^{-1}V^{-1}U)$. This by stability. Also $V\phi_\alpha(V^{-1}) = VV^{-1} = I$. Let $W = UVU^{-1}V^{-1}$. Then $\phi_\alpha[(W-I)^*(W-I)] = 0$. By completeness $W = I$ or $UV = VU$. So U is in N^c .

LEMMA 2. *A normal $U(N)$ stable expectation of M on N^c is faithful.*

Proof. Let ϕ be the expectation. Let $I = \{A/A \in M, \phi(A^*A) = 0\}$ clearly as $\phi[(XA)^*(XA)] \leq \|X\|^2\phi(A^*A) = 0$ and $(A+B)^*(A+B) \leq 2(A^*A + B^*B)$. I is a left ideal.

Now to show that I is ultra-weakly closed. The ultra-weak closure of I coincides with its ultra-strong closure. Let X_α be a net in I converging ultra-strongly to X , then $(X_\alpha - X)^*(X_\alpha - X)$ converges ultra-weakly to 0. Hence $\phi[(X_\alpha - X)^*(X_\alpha - X)]$ converges to 0 ultra-weakly (normality). As

$$\phi(X_\alpha - X)^*\phi(X_\alpha - X) \leq \phi(X_\alpha - X)^*(X_\alpha - X)$$

it follows that $\phi(X_\alpha)$ converges to $\phi(X)$. X^*X_α and $X_\alpha X^*$ have the same limit so $\phi(X^*X) = 0$. Hence I is a left ultra-weakly closed ideal. So there exists a unique projection E in M such that $I = \{T/TE = T\}$. $UTU^{-1} \in I$ for all $U \in U(N)$ by stability. So $UEU^{-1} = E$. So $E \in N^c$. So $E = \phi(E) = 0$. So if $\phi(X^*X) = 0$ then $X = XE = 0$, so ϕ is faithful.

Now let G be a subgroup of $U(M)$. Let N be the von Neumann algebra generated by G .

LEMMA 3. *A Schwartz map relative to (G, M) is an expectation onto N^c .*

Proof. Let P be the Schwartz map. As $P(X)$ commutes with all unitaries of G , $P(X)$ is in N^c . Now if A is in N^c , $C_G[A]$ reduces to the element A . So $P(A) = A$. So $P^2 = P$. N^c is hence the range of P and $P(I) = I$. Now to show that $\|P\| < 1$. Let $T = \sum_{i=1}^n \alpha_i U_i A U_i^{-1}$ where $\alpha_i > 0$ and $\sum \alpha_i = 1$. Then $\|T\| < \|A\|$. Because $P(A)$ is in $C_G[A]$ this means that there exists a net T_α of the same form as T such that T_α converges strongly to $P(A)$. Let X be a vector of norm one. $\|T_\alpha X\|$ converges to $\|P(A)X\|$ but $\|T_\alpha X\| < \|A\|$. So $\|P(A)\| < \|A\|$. By a result of J. Tomiyama [7], this implies that P is an expectation.

LEMMA 4. *If G is amenable, $S(G, M)$ is nonvoid.*

Proof. Let λ be a mean. Let ξ and η be 2 vectors. Considering U as the variable, $\lambda(U^{-1}XU\xi, \eta)$ is a bounded hermitian form. By the Riez Lemma there exists an operator E_λ such that $\lambda(U^{-1}XU\xi, \eta) = (E_\lambda(X)\xi, \eta)$. It was shown in [1] that E_λ is in $S(G, M)$.

LEMMA 5. *Let M be finite and countably decomposable, let G be any subgroup of $U(N)$, then $S(G, M)$ is nonvoid. (In particular if N is any von Neumann subalgebra of M , then $S(U(N), M)$ is nonvoid.)*

Proof. Let Tr be a faithful, normal, finite trace on M [3]. By finiteness there exists a faithful, normal expectation ϕ of M on N^c such that $\text{Tr}(XB) = \text{Tr}(\phi(X)B)$ for all X in M and all B in N^c . Hence $\phi(VXV^{-1}) = \phi(X)$ for all X in M and all V in $U(N^c) \supset U(N)$. Now to show $\phi(X)$ is in $C_G[X]$. $C_G[X]$ intersects N^c [3]. Let T be in $C_G[X] \cap N^c$, then by normality $T = \phi(T) = \phi(X)$. Hence ϕ is in $S(G, M)$.

Let G be a subgroup of $U(M)$. Let N be the von Neumann algebra generated by G .

LEMMA 6. *If $S(G, M)$ contains a normal map ϕ , then $S(G, M)$ reduces to ϕ , and so does $S(U(N), M)$. Moreover $C_G[X]$ intersects N^c in just one point.*

Proof. Let T be in $C_G[X]$, by normality $\phi(T) = \phi(X)$. Now let T be in $C_G[X] \cap N^c$. Then $T = \phi(T)$ by Lemma 3. So T is the unique point in $C_G[X] \cap N^c$. By normality ϕ is $U(N)$ stable, so $S(U(N), M) = \{\phi\}$.

LEMMA 7. *Let Tr be a faithful, normal, semifinite trace on M . Let G be a subgroup of $U(M)$ and N the von Neumann algebra generated by G . Suppose $S(G, M)$ is sufficient, then the restriction of Tr to N^c is semifinite.*

Proof. In this proof the notation of [3] will be used. Let \mathcal{M} be the ideal whose positive part consists of positive operators A such that $\text{Tr } A < \infty$. Consider $\mathcal{M}^{1/2}$. If A is in $\mathcal{M}^{1/2}$, $C_G[A] \subset \mathcal{M}^{1/2}$ and $C_G[A] \cap N^c$ is nonvoid [3]. Let S be a positive operator in N^c , $S \neq 0$. To show that there exists $S_1 \neq 0$, $S_1 \leq S$ where S_1 is a positive operator of $N^c \cap M$. Let A be in \mathcal{M} such that $0 \leq A \leq I$. Let P_α be in $S(G, M)$. Then $S \geq \sqrt{S} P_\alpha(A) \sqrt{S} = P_\alpha(\sqrt{S} A \sqrt{S})$. A can be picked such that $\sqrt{S} A \sqrt{S} \neq 0$ or else $A \sqrt{S} = 0$ for all A positive in \mathcal{M} . By semifiniteness there would exist a net A_α converging weakly to I so $I \sqrt{S} = 0$. So $S = 0$, a contradiction. Pick A then so that $\sqrt{S} A \sqrt{S} \neq 0$. Let $H = \sqrt{S} A \sqrt{S}$ then H is in $\mathcal{M}^{1/2}$. $P_\alpha(\sqrt{H})$ is in $\mathcal{M}^{1/2} \cap N^c$. So $[P_\alpha(\sqrt{H})]^2$ is in $\mathcal{M} \cap N^c$. So $(P_\alpha(\sqrt{H}))^2 \leq P_\alpha(H) \leq S$. By sufficiency, there exists an α_0 such that $P_{\alpha_0}(\sqrt{H}) \neq 0$. Choose $S_1 = (P_{\alpha_0}(\sqrt{H}))^2$.

THEOREM 1. *Let Tr be a faithful, semifinite trace of M . Let N be a von Neumann subalgebra of M and assume that the restriction of Tr to N is semifinite, then there exists a normal, faithful $U(N^c)$ -stable expectation ϕ of M on N such that $\text{Tr}(A\phi(X)) = \text{Tr}(AX)$ for all X in M and all A in N such that $\text{Tr} |A| < \infty$.*

Proof. Using the notations of the above lemma, let A and B be in $\mathcal{M}^{1/2} \cap N$ (that intersection is nonvoid), define $(A, B) = \text{Tr}(AB^*)$. Choose X positive in M and define $[A, B] = \text{Tr}(AB^*X)$. $[,]$ is a bounded hermitian form respectively to $(,)$. Let k be the completion of $\mathcal{M}^{1/2}$ under $(,)$. By the Riez Lemma there exists an operator $\phi(X)$ in $L(k)$ such that $[A, B] = (\phi(X)(A), B)$. Now: Let R_c denote the right multiplication by C , where C is in $\mathcal{M}^{1/2}$.

$$\begin{aligned} (R_c\phi(X)(A), B) &= (\phi(X)(A), BC^*) = [A, BC^*] = \text{Tr}(ACB^*X), \\ (\phi(X)R_c(A), B) &= [R_c(A), B] = [AC, B] = \text{Tr}(ACB^*X) \end{aligned}$$

so $R_c\phi(X) = \phi(X)R_c$.

By the commutation theorem [3] this implies that $\phi(X)(A)$ is a left multiplication by an element of N . Call that element $\phi(X)$. Then $\text{Tr}(AB^*X) = \text{Tr}(\phi(X)AB^*) = \text{Tr}(AB^*\phi(X))$ for all A and B in $\mathcal{M}^{1/2} \cap N$ and all X positive in M . ϕ can then be extended in the obvious fashion to all of M . As Tr is faithful, normal, it is easy

to see that ϕ is faithful, normal, and $U(N^c)$ stable. For example to check that ϕ is $U(N^c)$ stable; let V be in $U(N^c)$, let A be in N , then:

$$\text{Tr}(A\phi(VXV^{-1})) = \text{Tr}(AVXV^{-1}) = \text{Tr}(VAXV^{-1}) = \text{Tr}(AX) = \text{Tr}(A\phi(X)).$$

So $\text{Tr}[A(\phi(VXV^{-1}) - \phi(X))] = 0$ for all A in $N \cap \mathcal{M}$. Since Tr is semifinite on N , let P_α be a family of orthogonal projections of N such that $\text{Tr} P_\alpha < \infty$ and $\sum P_\alpha = I$. Make $A = (\phi(VXV^{-1}) - \phi(X)) * P_\alpha$. One has $P_\alpha(\phi(VXV^{-1}) - \phi(X)) = 0$ for all α , i.e. $\phi(VXV^{-1}) = \phi(X)$.

THEOREM 2. *Suppose M has a faithful, normal, semifinite trace Tr . Suppose $S(G, M)$ is sufficient, then there exists a faithful, normal $U(N^c)$ stable expectation of M on N^c . (N is the algebra generated by G .)*

Proof. By Lemma 7 the restriction of Tr to N^c is semifinite. By Theorem 1 there exists a normal, faithful, $U(N^c)$ stable expectation ϕ of M on N^c such that $\text{Tr}(AX) = \text{Tr}(A\phi(X))$ for all A in N^c such that $\text{Tr}|A| < \infty$. Now ϕ is in $S(G, M)$. Indeed ϕ is G -stable and if P is in $S(G, M)$ then $\phi(P(X)) = P(X)$ (as ϕ is the identity on N^c). By normality $\phi(P(X)) = \phi(X)$. So $P = \phi$. Hence ϕ is a normal, faithful, $U(N^c)$ stable expectation of M on N^c by Lemma 3.

The above theorem says that if there is a sufficient number of G -stable expectations of M on N^c , there is a faithful, normal one which in fact is more than G -stable it is $U(N^c)$ stable.

COROLLARY 1. *With the above hypothesis N^{cc} is finite.*

Proof. By the above theorem $S(U(N^{cc}), M)$ is nonvoid. Let P be in $S(U(N^{cc}), M)$. Let A be in N^{cc} , let $C(A)$ be the norm closure of the convex hull K_A of points of the form UAU^{-1} as U ranges over $U(N^{cc})$. Consider $C(A) \cap Z$ where Z is the center of N^{cc} . $C(A) \cap Z$ is nonvoid [3]. By [3] it is sufficient to show that $C(A) \cap Z$ reduces to one point. P is constant on K_A hence on $C(A)$. Let T_1 and T_2 be in $C(A) \cap Z$, then $T_1 = P(T_1) = P(T_2) = T_2$, so N^{cc} is finite. In particular N is finite.

COROLLARY 2. *With the above hypothesis N^c can not be purely infinite.*

Proof. In [7] J. Tomiyama proved that if π is an expectation from a semifinite algebra M onto a purely infinite subalgebra \mathcal{A} , then π is always singular, i.e. π is not normal. Since there exists a normal expectation from M on N^c , N^c is not purely infinite.

COROLLARY 3. *With the above hypothesis if M is of type I, so is N^c .*

Proof. In [7] it has been shown that if there exists an expectation from M of type I to a subalgebra of type II, that expectation is not normal. By the above corollary N^c has no part of type III and hence no part II or III are present, so N^c is of type I.

Let G be a subgroup of $U(M)$. Let N be generated by G .

COROLLARY 4. *Let M be a countably decomposable von Neumann algebra and consider the following conditions:*

- (1) N is finite and there exists a faithful, normal expectation ϕ of M and N .
 - (2) There exists a faithful, normal state ρ of M such that $\rho(UXU^{-1}) = \rho(X)$ for all U in G .
 - (3) There exists a faithful, normal expectation ψ of M on N^c such that $\psi(VXV^{-1}) = \psi(X)$ for all V in $U(N)$.
 - (4) $S(G, M)$ is sufficient and M has a faithful, semifinite normal trace Tr .
- Then (1) and (2) are equivalent. If $S(G, M)$ is nonvoid, (2) and (3) are equivalent. Finally (4) always implies (3).

Proof. Assume (1), then there exists a faithful, normal finite trace λ on N . Let $r(X) = \lambda[\phi(X)]$. Clearly r is faithful, normal and bounded. Let U be in G , then $r(UXU^{-1}) = \lambda\phi(UXU^{-1}) = \lambda\phi(X) = r(X)$. Normalizing r , (2) is established.

Assume (2). By a classical Hilbert algebra argument one can show that there exists an expectation ϕ such that $\rho(AX) = \rho(A\phi(X))$ for all A in N and all X in M . ϕ will satisfy (1).

Assume now (2) together with the fact that $S(G, M)$ is nonvoid. Let P be in $S(G, M)$. ρ is constant on $C_G[A]$. Hence $\rho(A) = \rho(P(A))$. This shows that P is faithful, normal and satisfies $P(VAV^{-1}) = P(A)$, for all V in $U(N)$. For example to check that $P(A) = P(VAV^{-1})$:

Let B be any element of N^c .

$$\begin{aligned} \rho(BVAV^{-1}) &= \rho(P(BVAV^{-1})) = \rho(BP(VAV^{-1})), \\ \rho(BVAV^{-1}) &= \rho(VBAV^{-1}) = \rho(BA) = \rho(BP(A)). \end{aligned}$$

Choose $B = (P(VAV^{-1}) - P(A))^*$, by faithfulness of ρ , $P(VAV^{-1}) = P(A)$.

Assume now (3). By countable decomposability there exists a faithful, normal state σ of M (get a maximal set of orthogonal projections P_n of M where each P_n is the projection on $[M'x_n]$, and let $\sigma = \sum W_{x_n}$ (notation of [3]). Let $\rho(X) = \sigma\psi(X)$. ρ in the state of (2).

Finally to show that (4) implies (3). By Theorem 2 there exists a faithful, normal expectation of M on N^c , call it Ψ such that $\text{Tr}(XA) = \text{Tr}(\Psi(X)A)$ for all A in $\mathcal{M} N^c$. As above one shows that $\Psi(VXV^{-1}) = \Psi(X)$.

COROLLARY 5. *If $S(G, L(h))$ is sufficient, N , the algebra generated by G , is atomic.*

Proof. By Theorem 2 there exists a faithful, normal, expectation of $L(h)$ on N' which is $U(N)$ stable. By Corollary 3, N' is of Type I, hence so is N [3]. Also N is finite by Corollary 1. Let Z be the center of N . Any projection of N dominates an abelian projection in N , call it $P \neq 0$. If Q is a projection of N such that $Q \leq P$, then $Q = PC$ where C is a projection of Z . Since Z is atomic [3], Q and hence P dominates a minimal projection. So N is atomic.

REMARKS. The following statements are trivial to see:

- (1) If $S(G, L(h))$ is sufficient then there exists a normal expectation ϕ from $L(h)$ to N' such that $\phi(UXU^{-1}) = \phi(X)$ for all U in G , this is part of Corollary 4.
- (2) Assuming $S(G, L(h))$ contains a normal map, then $S(G, L(h))$ is sufficient.

Let π be a normal map, then π is faithful. Indeed: by normality $\pi(UXU^{-1}) = \pi(X)$ for all U in $U(N)$. Assume that P is a projection such that $\pi(P) = 0$. Let $Q = \text{Sup } UPU^{-1}$ is $U \in U(N)$. Then $Q = VQV^{-1}$ for all V in $U(N)$, so Q is in N' . Hence $Q = \pi(Q) = 0$. So $P = 0$.

DEFINITION. Two groups of unitaries are equivalent if they generate the same von Neumann algebra.

THEOREM 3. Assume $S(G, L(h))$ contains a normal map π , then G is equivalent to a countable direct sum of finite groups.

Proof. By Lemma 6, $S(G, L(h)) = \{\pi\}$. By the above remark π is faithful and by normality π is in $S(U(N), L(h))$. By Corollary 1, N is finite. Let Z be the center of N . By Corollary 5, Z is atomic. Pick a maximal set of orthogonal minimal projections, C_n of Z such that $N = \bigoplus N_{c_n}$. N_{c_n} is a factor of Type I_n . N_{c_n} is isomorphic to $n \times n$ matrices, so N_{c_n} is generated by a finite group K_n of unitaries. Let $K = \bigoplus K_n$ (all components are the identity except a finite number). The algebra generated by K contains all N_{c_n} , so it contains N . Each K_n is a subgroup of $U(N)$. So the algebra generated by K is N .

Let M be a von Neuman algebra and let G be a subgroup of $U(M)$.

DEFINITION. G will satisfy condition (F) if

- (1) There exists orthogonal projections C_α of N' (N is the algebra generated by G) such that $I = \sum C_\alpha$ and $|GC_\alpha| < \infty$.
- (2) For every U in G , $UC_\alpha = C_\alpha$ for all but a finite number of α .

THEOREM 4. If G has property (F), then G is a countable direct sum of finite groups, and $S(G, M)$ is sufficient.

Proof. Define a map π_α on G by $\pi_\alpha(U) = UC_\alpha$. π_α is clearly a homomorphism of G and $\pi_\alpha(G)$ is finite. Also the intersection of all kernels of π_α is I . Let $F_\alpha = \pi_\alpha(G)$, then by definition of condition (F), $G = \bigoplus F_\alpha$. As each F_α is finite G is amenable since it is locally finite. So $S(G, M)$ is nonvoid by Lemma 4. Now let A be a positive operator in M , let P be in $S(G, M)$ and suppose $P(A) = 0$ for all P in $S(G, M)$. If $A \neq 0$, $C_\alpha AC_\alpha \neq 0$ for some C_α , call α_0 such an α . $C_{\alpha_0} P(A) C_{\alpha_0} = P(C_{\alpha_0} A C_{\alpha_0}) \in C_G[C_{\alpha_0} A C_{\alpha_0}]$. Let H be all elements of G where the α_0 component is the identity. Then $G = HF_{\alpha_0}$. Let U be in G , then U is uniquely written as $U = VW$ where V is in H and W in F_{α_0} . $UC_{\alpha_0} A C_{\alpha_0} U^{-1} = WC_{\alpha_0} A C_{\alpha_0} W^{-1}$ but there is only a finite number of $UC_{\alpha_0} A C_{\alpha_0} U^{-1}$. Hence $C_G[C_{\alpha_0} A C_{\alpha_0}]$ is the convex hull of $WC_{\alpha_0} A C_{\alpha_0} W^{-1}$ as W ranges in F_{α_0} . So $0 = P(C_{\alpha_0} A C_{\alpha_0}) = \sum_{i=1}^n \alpha_i W_i C_{\alpha_0} A C_{\alpha_0} W_i^{-1}$. So $C_{\alpha_0} A C_{\alpha_0} = 0$, a contradiction. So $A = 0$ and $S(G, M)$ is sufficient.

REMARK. While proving Theorem 3 it has been shown that if N is a finite atomic von Neumann algebra, then N is generated by a direct sum of finite groups K_n .

4. Uniqueness properties.

LEMMA 8. If there exists only one faithful, normal expectation Φ of M on N then $N^c \subset N$.

Proof. Let $\varepsilon > 0$. Let H be a positive operator in N^c such that $H > \varepsilon I > 0$, $\Phi(H)$ is in N , $\Phi(H) > \varepsilon I$. Let X be in N , then $X\Phi(H) = \Phi(XH) = \Phi(H)X$. So $\Phi(H)$ is in N' , so in $N \cap N' = Z_N$. Define $\pi(X) = \Phi(H)^{-1}\Phi(H^{1/2}XH^{1/2})$.

Clearly π is another expectation of M on N , by uniqueness $\pi = \Phi$. So $\Phi(H^{1/2}XH^{1/2}) = \Phi(H)\Phi(X)$, in particular if $X = H$ then $\Phi(H)^2 = \Phi(H)^2$; this holds for any self-adjoint operator in N^c which is positive. Let H be any selfadjoint operator in N^c . Pick $C > 0$ such that $CI + H > \varepsilon I$ then $[\Phi(CI + H)]^2 = \Phi[(CI + H)^2]$ so $\Phi(H)^2 = \Phi(H)^2$. Let P be a projection in N^c , then $(P - \Phi(P))^2 > 0$ so $\Phi(P - \Phi(P))^2 = (\Phi(P) - \Phi(P))^2 = 0$. By faithfulness $P = \Phi(P)$ i.e. P is in N so $N^c \subset N$.

LEMMA 9. *Let N be normal in M (i.e. $N^{cc} = N$) if a faithful expectation exists from M to N then $N^c \subset N$. If $N^c \subset N$ and if a normal expectation Φ exists from M to N then Φ is the only such normal expectation.*

Proof. The first part was shown in Lemma 1. Now to show the second part: As $N^c \subset N$, N^c is the center of N in particular N^c is abelian. Hence, $U(N^c)$ is amenable, so $S(U(N^c), M)$ is nonvoid by Lemma 4. Let P be in $S(U(N^c), M)$, then P is an expectation on $N^{cc} \cap M$ by Lemma 3. So P is an expectation on N . Let $\Phi(P(X)) = \Phi(X)$. Also $\Phi(P(X)) = P(X)$, so $\Phi = P$. This shows that if Φ exists, it is unique.

THEOREM 5. *Assume the following conditions:*

- (1) $N^c \subset N$,
- (2) N is finite,
- (3) M is semifinite.

Then there exists at most one normal expectation ϕ .

Proof. N^c is the center of N , by finiteness the map (notation of [3]) is defined from N to N^c . If X is in M , define $\Psi(X) = (\phi(X))^{\natural}$. Ψ is a normal map and $S(U(N^c), M)$ is nonvoid. Let P be in $S(U(N^c), M)$. If X is in $\mathcal{M}^{1/2}$, $C_{U(N)}[X] \cap \mathcal{M}^{1/2}$ and $C_{U(N)}[X]$ intersect N^c . Let T be in $C_{U(N)}[X] \cap N^c$. Ψ is invariant under $U(N)$, so $T = \Psi(T) = \Psi(X)$. So $N^c \cap C_{U(N)}[X] = \{\Psi(X)\}$. If ϕ_1 is another normal expectation of M on N , then define $\Psi_1(X) = [\phi_1(X)]^{\natural}$. Also $N^c \cap C_{U(N)}[X] = \{\Psi_1(X)\}$, so $\Psi = \Psi_1$ on $\mathcal{M}^{1/2}$, hence on M .

Let λ be any normal, finite trace on N . Then: $\lambda\phi(X) = \lambda\Psi(X) = \lambda\Psi_1(X) = \lambda\phi_1(X)$. Since the λ form a complete set $\phi = \phi_1$.

In conclusion consider the following problem. Let N be a von Neumann algebra. Suppose there exists sufficiently many expectations of M on N . Is N relatively semifinite? An answer to that problem was given when the expectations are of a certain type (Lemma 7).

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