NONLINEAR EVOLUTION EQUATIONS AND PRODUCT INTEGRATION IN BANACH SPACES

BY

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Abstract. The method of product integration is used to obtain solutions to the nonlinear evolution equation $g' = Ag$ where $A$ is a function from a Banach space $S$ to itself and $g$ is a continuously differentiable function from $[0, \infty)$ to $S$. The conditions required on $A$ are that $A$ is dissipative on $S$, the range of $(e - \varepsilon A) = S$ for all $\varepsilon \geq 0$, and $A$ is continuous on $S$.

1. Introduction. Let $S$ be a Banach space and let $A$ be a mapping from a subset of $S$ to $S$. An evolution equation is a system $g' = A(g)$, $g(0) = p$, where $g$ is a continuous function from $[0, \infty)$ to $S$ and $p$ is a point in $S$. In [3] F. Browder has considered nonlinear evolution equations in which $S$ is the Hilbert space and $A$ is continuous, bounded, and dissipative on $S$. In recent articles Y. Kömura [12], T. Kato [10], and M. Crandall and A. Pazy [5] have considered nonlinear evolution equations in which $S$ is the Hilbert space and $A$ is maximal dissipative, not necessarily continuous, and is the infinitesimal generator of a semigroup of nonlinear nonexpansive transformations on $S$.

The object of this paper is to obtain solutions to an evolution system in a general Banach space using the method of product integration. A definition of product integration is given as follows:

Suppose that $p$ is in $S$, $x > 0$, and $z$ is a point in $S$ such that if $c > 0$ there exists a chain $\{s_i\}_{i=0}^n$ from 0 to $x$ such that if $\{s_i\}_{i=0}^n$ is a refinement of $\{s_i\}_{i=0}^n$ then

\[ z - \prod_{i=2}^n (e - (t_i - t_{i-1})A)^{-1} p \leq c. \]

(Note that $e$ denotes the identity map on $S$, $(e - (t_i - t_{i-1})A)^{-1}$ denotes the inverse map of $(e - (t_i - t_{i-1})A)$, $\prod_{i=1}^n (e - (t_i - t_{i-1})A)^{-1} p = (e - (t_1 - t_0)A)^{-1} p$, and if $j$ is an integer in $[2, n]$ $\prod_{i=1}^j (e - (t_i - t_{i-1})A)^{-1} p = (e - (t_j - t_{j-1})A)^{-1} \prod_{i=1}^{j-1} (e - (t_i - t_{i-1})A)^{-1} p$,

where the product operation is composition of mappings.) Then $z$ is said to be the product integral of $A$ with respect to $p$ from 0 to $x$ and is denoted by $\prod_{i=0}^n (e - dIA)^{-1} p$. 

Received by the editors March 20, 1969 and, in revised form, August 20, 1969.

AMS Subject Classifications. Primary 3495, 3436; Secondary 3535, 3537.

Key Words and Phrases. Nonlinear evolution equations, product integration, dissipative mapping, semigroup of nonlinear nonexpansive transformations, infinitesimal generator.

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In [1] G. Birkhoff and in [16] J. Neuberger have used product integration to solve evolution systems where the mapping $A$ is Lipschitz continuous. In this paper the product integration method will be extended to mappings not necessarily Lipschitz continuous.

2. An existence theorem. Let $A$ be a mapping from a subset of $S$ to $S$ such that the following are true:

(I) $A$ is dissipative on its domain $D_A$, i.e., if $u, v \in D_A$ and $\epsilon \geq 0$ then $\| (e^{-\epsilon A})u - (e^{-\epsilon A})v \| \geq \| u - v \|$. 

(II) There is an open subset $C$ of $S$ such that $C \subseteq D_A$ and a positive number $\alpha$ such that if $0 \leq \epsilon < \alpha$ then $C \subseteq R(e^{-\epsilon A})$ (where $R(e^{-\epsilon A})$ denotes the range of $(e^{-\epsilon A})$). 

(III) $A$ is continuous on $C$.

Note that by (I) if $\epsilon > 0$ then $(e^{-\epsilon A})$ is 1-1 on $D_A$ and by (II) if $0 \leq \epsilon < \alpha$ and $q \in C$ then $q \in D(e^{-\epsilon A})^{-1} = R(e^{-\epsilon A})$. If $0 \leq \epsilon < \alpha$ and $q \in R(e^{-\epsilon A})$ let $L(\epsilon)q = (e^{-\epsilon A})^{-1}q$. By (I) $L(\epsilon)$ is nonexpansive on $R(e^{-\epsilon A})$, i.e., if $u, v \in R(e^{-\epsilon A})$ then

\[ \| L(\epsilon)u - L(\epsilon)v \| \leq \| u - v \|. \]

**Theorem.** Let $A$ satisfy conditions (I), (II), and (III). If $p \in C$ and

\[ \gamma_p = \min \{ \text{dist}(p, \partial C)/\| Ap \|, \alpha \}, \]

then there is a continuously differentiable function $g_p$ from $[0, \gamma_p)$ to $S$ such that $g_p(0) = p$ and if $0 \leq x < \gamma_p$, $g'_p(x) = Ag_p(x)$ and $g_p(x) = \int_0^x (e^{-dIA})^{-1}p$.

The theorem will be proved by means of a sequence of lemmas each of which is under the hypothesis of the theorem.

**Lemma 1.1.** If $q \in C$ and $0 \leq x, y < \alpha$, then $\| L(x)q - L(y)q \| \leq |x - y| \cdot \| Aq \|$. 

**Proof.** Using (2) we have that

\[ \| L(x)q - L(y)q \| = \| L(x)q - L(x)(e^{-x A})L(y)q \| \]
\[ \leq \| q - (e^{-x A})L(y)q \| \]
\[ = \| q - [(x/y)(e^{-y A})L(y)q + (1 - x/y)L(y)q] \| \]
\[ = |1 - x/y| \| q - L(y)q \| \]
\[ \leq |1 - x/y| \| (e^{-y A})q - q \| \]
\[ = |x - y| \| Aq \|. \]

**Lemma 1.2.** Let $q \in C$, let $0 < x < \gamma_q$, and let $\{ s_i \}_{i=0}^n$ be a chain from 0 to $x$. If $j$ is an integer in $[1, m]$ then

\[ \sum_{i=1}^{j-1} L(s_i - s_{i-1})q \in C, \]

\[ \sum_{i=1}^{j} L(s_i - s_{i-1})q - q \leq s_j \| Aq \|, \]
and

\[ A \left( \sum_{i=1}^{j} L(s_i - s_{i-1})q \right) \leq \|Aq\|. \]

(Note that \( \prod_{i=1}^{j} L(s_i - s_{i-1}) \) denotes the identity map, i.e., \( \prod_{i=1}^{j} L(s_i - s_{i-1})q = q \).)

**Proof.** The proof is by induction. For \( j = 1 \), \( \prod_{i=1}^{1} L(s_i - s_{i-1})q = q \in C \),

\[ \left\| \prod_{i=1}^{1} L(s_i - s_{i-1})q - q \right\| \leq s_1 \cdot \|Aq\| \]

(by Lemma 1.1), and

\[ A \prod_{i=1}^{1} L(s_i - s_{i-1})q = \|1/s_1[L(s_1 - s_0)q - q]\| \leq \|Aq\|. \]

Suppose that \( j \) is an integer in \([1, m-1]\), \( \prod_{i=1}^{j} L(s_i - s_{i-1})q \in C \),

\[ \left\| \prod_{i=1}^{j} L(s_i - s_{i-1})q - q \right\| \leq s_j \cdot \|Aq\|, \]

and \( \|A \prod_{i=1}^{j} L(s_i - s_{i-1})q\| \leq \|Aq\| \). Then,

\[ \prod_{i=1}^{j+1} L(s_i - s_{i-1})q \in C \subseteq D_{L(s_{j+1} - t_j)}, \]

Further,

\[ \left\| \prod_{i=1}^{j+1} L(s_i - s_{i-1})q - q \right\| = \left\| \sum_{k=1}^{j+1} \left[ \prod_{i=1}^{k} L(s_k - s_{k-1})q - \prod_{i=1}^{k} L(s_k - s_{k-1})q \right] \right\| \]

(note that \( \prod_{k=1}^{j+1} L(s_k - s_{k-1}) \) is the identity map)

\[ \leq \sum_{i=1}^{j+1} \|L(s_i - s_{i-1})q - q\| \]

\[ \leq s_{j+1} \cdot \|Aq\|. \]

Moreover,

\[ A \prod_{i=1}^{j+1} L(s_i - s_{i-1})q = \left\| \frac{1}{(s_{j+1} - s_j)} \left[ \prod_{i=1}^{j+1} L(s_i - s_{i-1})q - \prod_{i=1}^{j} L(s_i - s_{i-1})q \right] \right\| \]

\[ \leq \|A \prod_{i=1}^{j} L(s_i - s_{i-1})q\| \]

\[ \leq \|Aq\|. \]

**Lemma 1.3.** Let \( q \in C \), let \( 0 < x < y_q \), and let \( \{t_i\}_{i=0}^{n} \) be a chain from 0 to \( x \). If \( j \) is an integer in \([1, n]\) then

\[ \prod_{i=j}^{n} L(t_i - t_{i-1})q - q = \sum_{i=j}^{n} (t_i - t_{i-1})A \prod_{k=j}^{i} L(t_k - t_{k-1})q. \]
Proof.  
\[
\prod_{t=j}^{n} L(t_{i} - t_{i-1})q - q = \prod_{t=j}^{n} \left[ \prod_{k=j}^{i-1} L(t_{k} - t_{k-1})q - \prod_{k=j}^{n-1} L(t_{k} - t_{k-1})q \right] \\
= \prod_{t=j}^{n} (t_{i} - t_{i-1}) AL(t_{i} - t_{i-1}) \prod_{k=j}^{i-1} L(t_{k} - t_{k-1})q \\
= \sum_{t=j}^{n} (t_{i} - t_{i-1}) A \prod_{k=j}^{i} L(t_{k} - t_{k-1})q.
\]

Let \( p \in C \), let \( c > 0 \), and let \( m \) be a nonnegative integer. The number-sequence \( \{s_{i}\}_{i=0}^{m} \) is said to have property \( P_{c} \) provided that the following are true: (i) \( s_{0} = 0 \), \( s_{m} < \gamma_{r} \) (ii) \( \{s_{i}\}_{i=0}^{m} \) is increasing, and (iii) if \( h \) is an integer in \( [0, m - 1] \), \( s_{h} \leq x \leq s_{h+1} \), \( \{t_{i}\}_{i=0}^{m} \) is a chain from \( s_{h} \) to \( x \), and \( j \) is an integer in \( [0, n] \), then

\[
\left| A \prod_{k=j}^{t} L(t_{k} - t_{k-1}) \prod_{i=1}^{h} L(s_{i} - s_{i-1})p \right| - A \prod_{k=j}^{n} L(t_{k} - t_{k-1}) \prod_{i=1}^{h} L(s_{i} - s_{i-1})p \leq c.
\]

Lemma 1.4. Let \( p \in C \), let \( c > 0 \), and let \( \{s_{i}\}_{i=0}^{m} \) have property \( P_{c} \). There is a nonnegative integer \( s_{m+1} \) such that \( s_{m} < s_{m+1} < \gamma_{r} \) and \( \{s_{i}\}_{i=0}^{m+1} \) has property \( P_{c} \).

Proof. Lemma 1.4 follows from Lemma 1.2 and the continuity of \( A \) at \( \prod_{t=1}^{n} L(s_{t} - s_{t-1})p \).

Lemma 1.5. Let \( p \in C \), let \( c > 0 \), and let \( \{s_{i}\}_{i=0}^{m} \) have property \( P_{c} \). Suppose that \( y \) is a number such that \( s_{m} < y < \gamma_{r} \) and if \( s_{m+1} \) is a nonnegative integer such that \( s_{m} < s_{m+1} < y \) then \( \{s_{i}\}_{i=0}^{m+1} \) has property \( P_{c} \). Then, if \( s_{m+1} = y \), \( \{s_{i}\}_{i=0}^{m+1} \) has property \( P_{c} \).

Proof. Let \( q = \prod_{t=1}^{n} L(s_{t} - s_{t-1})p \), let \( \{t_{i}\}_{i=0}^{n} \) be a chain from \( s_{m} \) to \( y \), and let \( d > 0 \). There is a positive number \( b \) such that if \( u \in C \) and \( \|u - \sum_{t=1}^{n} L(t_{i} - t_{i-1})q\| < b \) then

\[
\left| Au - A \prod_{t=1}^{n} L(t_{i} - t_{i-1})q \right| < d.
\]

There is a positive number \( r \) such that \( t_{n-1} < r < t_{n} = y \) and \( t_{n} - r < b/\|Ap\| \). By Lemmas 1.1 and 1.2

\[
\left| L(r - t_{n-1}) \prod_{t=1}^{n-1} L(t_{i} - t_{i-1})q - \prod_{t=1}^{n} L(t_{i} - t_{i-1})q \right| \leq (t_{n} - r) \cdot \|Ap\| < b.
\]

Then, if \( j \) is an integer in \( [0, n-1] \)

\[
\left| A \prod_{t=1}^{j} L(t_{i} - t_{i-1})q - A \prod_{t=1}^{n} L(t_{i} - t_{i-1})q \right| \\
\leq \left| A \prod_{t=1}^{j} L(t_{i} - t_{i-1})q - AL(r - t_{n-1}) \prod_{t=1}^{n-1} L(t_{i} - t_{i-1})q \right| \\
+ \left| AL(r - t_{n-1}) \prod_{t=1}^{n-1} L(t_{i} - t_{i-1})q - A \prod_{t=1}^{n} L(t_{i} - t_{i-1})q \right| \\
< c + d.
\]
Then, if $j$ is an integer in $[0, n]$

\[ A \left| \sum_{i=1}^{n} L(t_i - t_{i-1})q_m - A \sum_{i=1}^{n} L(t_i - t_{i-1})q_m \right| \leq c \]

and so the lemma is established.

**Lemma 1.6.** Let $p \in C$, let $c > 0$, and suppose that $\{s_i\}_{i=0}^{n_0}$ is an infinite increasing number-sequence such that $\lim \{s_i\}_{i=0}^{n_0} < \gamma_p$ and if $n$ is a nonnegative integer $\{s_i\}_{i=0}^{n_0}$ has property $P_c$. Then there is a positive integer $m$ and a sequence $\{r_i\}_{i=0}^{m+1}$ such that if $i$ is an integer in $[0, m]$ $s_i = r_i$, $r_{m+1} = \lim \{s_i\}_{i=0}^{n_0}$, and $\{r_i\}_{i=0}^{m+1}$ has property $P_c$.

**Proof.** Let $q_0 = p$ and if $n$ is a positive integer let $q_n = L(s_n - s_{n-1})q_{n-1}$. If $n$ is a positive integer then $\|q_n - q_{n-1}\| = \|L(s_n - s_{n-1})q_{n-1} - q_{n-1}\| \leq (s_n - s_{n-1})\| A p \|$. Let $s = \lim \{s_i\}_{i=0}^{n_0}$, let $q = \lim \{q_i\}_{i=0}^{n_0}$, and note that $q \in C$ since $\|q_n - p\| < s \cdot \| A p \|$ and so $\|q - p\| < \text{dist}(p, \partial C)$. There is a positive number $b$ such that if $u \in C$ and $\|u - q\| < b$ then $\|u - Aq\| < c/2$. Let $m$ be a positive integer such that $\|q - q_m\| < b/2$ and $s - s_m < b/2 \| A p \|$. Let $0 < s \leq s_m$, let $\{t_i\}_{i=0}^{n_0}$ be a chain from 0 to $x$, and let $j$ be an integer in $[0, n]$. By Lemma 1.2

\[ A \left| \sum_{i=1}^{j} L(t_i - t_{i-1})q_m - q_m \right| \leq t_j \cdot \| A p \| < b/2 \]

and so

\[ A \left| \sum_{i=1}^{j} L(t_i - t_{i-1})q_m - A q_m \right| < c/2. \]

Then, if $j$ is an integer in $[0, n]$

\[ A \left| \sum_{i=1}^{j} L(t_i - t_{i-1})q_m - A \sum_{i=1}^{j} L(t_i - t_{i-1})q_m \right| \leq A \left| \sum_{i=1}^{j} L(t_i - t_{i-1})q_m - A q_m \right| + \|Aq - A \sum_{i=1}^{j} L(t_i - t_{i-1})q_m\| \]

\[ \leq c \]

and so the lemma is established.

**Lemma 1.7.** Let $p \in C$, let $c > 0$, and let $0 < x < \gamma_p$. There is a chain $\{s_i\}_{i=0}^{n_0}$ from 0 to $x$ such that $\{s_i\}_{i=0}^{n_0}$ has property $P_c$.

**Proof.** By Lemma 1.4 there is an infinite increasing number-sequence $\{s_i\}_{i=0}^{n_0}$ such that $\lim \{s_i\}_{i=0}^{n_0} < \gamma_p$ and if $n$ is a nonnegative integer $\{s_i\}_{i=0}^{n_0}$ has property $P_c$. Let $M$ denote the set of all such sequences. If $s = \{s_i\}_{i=0}^{n_0}$ is in $M$ let $z(s)$ denote the limit of $s$. If each of $s$ and $t$ belongs to $M$ define $s \leq t$ only in case $s$ is $t$ or if $n$ is the greatest nonnegative integer such that if $i$ is an integer in $[0, n]$ $s_i = t_i$, then $z(s) \leq t_{n+1}$. Then, $\leq$ is a partial ordering of $M$.

Assume that there exists no member $s$ of $M$ such that $z(s) > x$. Let $L$ be a linearly ordered subset of $M$ and let $y$ be the smallest positive number such that if $s$ is in
Let \( \{s(t)\}_{t=0}^{\infty}, \{s(t)\}_{t=1}^{\infty}, \ldots \) be an increasing sequence of points in \( L \) such that \( z(s(0)), z(s(1)), \ldots \) converges to \( y \). For each nonnegative integer \( i \) define 
\[ y_i = \sup_k s_i(k) \] 
Then, \( y_i \leq y_{i+1} \) and \( \lim_{i \to \infty} y_i = y \).

Suppose there is a positive integer \( n \) such that \( y_n = y \). Then there is a least positive integer \( n \) such that \( y_n = y \) and there must exist an integer \( k \) such that \( s_n(k) = s_{n+1}(j) \) for each integer \( i \) in \([0, n-1]\) and \( j \geq k \). In this case \( s_n(k), s_n(k+1), \ldots \) converges to \( y \) and so by Lemma 1.5 \( \{s_n\}_{n=0}^{\infty}, s_n = s(k) \) for \( i \) in \([0, n-1]\) and \( s_n = y \), has property \( P_c \). Further, since \( y < y_p \), we have by Lemma 1.4 that \( \{s_i\}_{i=0}^{\infty} \) may be extended to a member \( \{s_i\}_{i=0}^{\infty} \) of \( M \) and so \( \{s_i\}_{i=0}^{\infty} \) is an upper bound for \( L \). If there is no positive integer \( n \) such that \( y_n = y \) then \( y_n < y \) for every \( n \), \( \{y_n\}_{n=0}^{\infty} \) is in \( M \), \( \{y_n\}_{n=0}^{\infty} \) is an upper bound for \( L \).

Thus, if \( L \) is a linearly ordered subset of \( M \), then \( L \) is bounded by a member of \( M \). By Zorn's lemma there exists \( u \in M \) such that \( u \) is maximal. But then we have a contradiction since \( z(u) \leq x < y_p \) and by Lemma 1.6 there exists \( t \in M \) such that \( u < t \). Hence, there exists \( s \in M \) such that \( z(s) > x \) and the lemma is proved.

**Lemma 1.8.** Let \( p \in C \), let \( c > 0 \), and let \( 0 < x < y_p \). There is a chain \( \{s_i\}_{i=0}^{\infty} \) from 0 to \( x \) such that if \( \{t_i\}_{i=0}^{\infty} \) is a refinement of \( \{s_i\}_{i=0}^{\infty} \) then

\[
\left( \prod_{i=1}^{n} L(t_i - t_{i-1})p - \prod_{i=1}^{m} L(s_i - s_{i-1})p \right) < c.
\]

**Proof.** Let \( \{s_i\}_{i=0}^{\infty} \) be a chain from 0 to \( x \) such that \( \{s_i\}_{i=0}^{\infty} \) has property \( P_c \). Let \( \{t_i\}_{i=0}^{\infty} \) be a refinement of \( \{s_i\}_{i=0}^{\infty} \), i.e., there is an increasing sequence \( u \) such that \( u_0 = 0, u_m = n \), and if \( i \) is an integer in \([0, m]\) \( s_i = t_{u_i} \). If \( i \) is an integer in \([1, m]\) let \( K_i = \prod_{j=i}^{u_i-1} L(t_j - t_{j-1}) \), let \( J_i = \prod_{j=1}^{i} L(s_j - s_{j-1}) \), let \( K_{m+1} = e \), and let \( J_0 = e \). Then,

\[
\left( \prod_{i=1}^{n} L(t_i - t_{i-1})p - \prod_{i=1}^{m} L(s_i - s_{i-1})p \right) = \left( \prod_{i=1}^{m} K_i p - J_m \right) 
\]

\[
= \left( \sum_{i=1}^{m} \left[ \prod_{j=1}^{i} K_j p - \prod_{j=i+1}^{m} K_j p \right] \right) 
\]

\[
\leq \sum_{i=1}^{m} \|K_i J_{i-1}p - J_i p\| 
\]

\[
= \sum_{i=1}^{m} \|K_i J_{i-1}p - L(s_i - s_{i-1})J_{i-1}p\| 
\]

\[
\leq \sum_{i=1}^{m} \|K_i J_{i-1}p - L\left((s_i - s_{i-1})A\right)J_{i-1}p - J_{i-1}p\| 
\]

\[
= \sum_{i=1}^{m} \left[ \prod_{j=u_i-1}^{u_i} L(t_j - t_{j-1})J_{i-1}p - J_{i-1}p \right] - (s_i - s_{i-1})A K_i J_{i-1}p 
\]
\[ \begin{align*}
&= \sum_{j=1}^{m} \left[ \sum_{i=1}^{u_j+1} (t_j - t_{j-1}) \left( A \sum_{k=u_{i-1}+1}^{j} L(t_k - t_{k-1}) J_{i-1} - AK_i J_{i-1} p \right) \right] \\
nonumber &\leq \sum_{j=1}^{m} \left[ \sum_{i=1}^{u_j+1} (t_j - t_{j-1}) \left( A \sum_{k=u_{i-1}+1}^{j} L(t_k - t_{k-1}) J_{i-1} p \right) \right] \\
nonumber &\quad - A \sum_{j=1}^{m} \sum_{i=1}^{u_j} (t_j - t_{j-1}) J_{i-1} p \\
nonumber &\leq c \cdot \sum_{i=1}^{m} \sum_{j=1}^{u_j+1} (t_j - t_{j-1}) \\
nonumber &= c \cdot x.
\end{align*}\]

**Proof of the theorem.** Let \( p \in C \). If \( x = 0 \), then \( \prod \delta (e - d I) A p = p \). If \( 0 < x < \gamma_p \), then \( \prod \delta (e - d I) A p \) exists by virtue of Lemma 1.8. If \( 0 \leq x < \gamma_p \) define \( g_p(x) = \prod \delta (e - d I) A p \). By Lemma 1.2 we see that \( g_p \) is Lipschitz continuous on \([0, \gamma_p)\) with Lipschitz constant \( \leq \| A p \| \), \( g_p(x) \in C \) for \( x \in [0, \gamma_p) \), and \( \| A g_p(x) \| \leq \| A p \| \) for \( x \in [0, \gamma_p) \). For \( 0 \leq x < \gamma_p \) we have that \( \text{dist} (p, \partial C) \leq \text{dist} (g_p(x), \partial C) + \| p - g_p(x) \| \leq \text{dist} (g_p(x), \partial C) + x \| A p \| \). Hence,

\[
\text{dist} (p, \partial C) \| A p \| \leq \text{dist} (g_p(x), \partial C) \| A p \| + x
\]

and so \( \gamma_p - x \leq \gamma_{g_p(x)} \). Thus, if \( 0 \leq x < \gamma_p \) and \( 0 \leq y < \gamma_p - x \), one sees that \( g_{g_p(x)}(y) = g_p(x + y) \). To show that \( g_p' = A g_p \), let \( 0 \leq x < \gamma_p \) and let \( c > 0 \). By Lemma 1.2 there is a positive number \( z < \gamma_p - x \) such that if \( 0 < y < z \) and \( \{ s_i \}_{i=0}^n \) is a chain from \( 0 \) to \( y \), then

\[
\left\| A \prod_{i=1}^{n} L(s_i - s_{i-1}) g_p(x) - A g_p(x) \right\| < c/2.
\]

Let \( 0 < y < z \). There is a chain \( \{ t_i \}_{i=0}^n \) from \( 0 \) to \( y \) such that

\[
\left\| \prod_{i=1}^{n} L(t_i - t_{i-1}) g_p(x) - g_{g_p(x)}(y) \right\| < c \cdot y/2.
\]

Then,

\[
\frac{1}{y} [g_p(x+y) - g_p(x)] - A g_p(x) \]

\[
< \frac{c}{2} + \frac{1}{y} \left( \prod_{i=1}^{n} L(t_i - t_{i-1}) g_p(x) - g_p(x) \right) - y A g_p(x) \]

\[
= \frac{c}{2} + \frac{1}{y} \sum_{i=1}^{n} (t_i - t_{i-1}) A \prod_{j=1}^{i} L(t_j - t_{j-1}) g_p(x) - y A g_p(x) \]

\[
\leq \frac{c}{2} + \frac{1}{y} \sum_{i=1}^{n} (t_i - t_{i-1}) A \prod_{j=1}^{i} L(t_j - t_{j-1}) g_p(x) - A g_p(x) \]

\[
< c
\]
and so \( g_p'(x) = Ag_p(x) \). Thus, \( g_p' = Ag_p \) on \([0, \gamma_p)\) and so \( g_p \) has a continuous right derivative on \([0, \gamma_p)\). Then \( g_p \) has a continuous derivative on \([0, \gamma_p)\) and so the theorem is proved.

**Corollary.** Let \( A \) be a mapping from the Banach space \( S \) to \( S \) such that the following are true:

1. \( A^* \) is dissipative on \( S \), i.e., if \( u, v \in D_A \) and \( \varepsilon \geq 0 \) then \( \| (e - eA)u - (e - eA)v \| \geq \| u - v \| \)
2. \( R(e - eA) = S \) for each \( \varepsilon \geq 0 \)
3. \( A \) is continuous on \( S \).

If \( p \in S \) then there is a continuously differentiable function \( g_p \) from \([0, \infty)\) to \( S \) such that \( g_p(0) = p \) and if \( x \geq 0 \) \( g_p'(x) = Ag_p(x) \) and \( g_p(x) = \int_0^x (e - dI_A)^{-1}p \).

**Proof.** The proof follows immediately from the theorem if one observes that \( \alpha = +\infty \) and \( \text{dist} (p, \partial S) = +\infty \).

It may be noted that a result of J. Dorroh \([8]\) can be used to show that the solutions of \( g_p' = Ag_p \), \( g_p(0) = p \) in the corollary are unique. In \([15]\) G. Minty has shown that if \( S \) is the Hilbert space then \((I')\) and \((III')\) imply \((II')\). More generally, it has been shown recently by T. Kato in \([11]\) that \((I')\) and \((III')\) imply \((II')\) in the case that \( S^* \) is uniformly convex. If \( S \) is a general Banach space F. Browder has shown in \([4]\) that \((I')\) and \((III')\) imply \((II')\) in the case that \( A \) is locally uniformly continuous.

By virtue of the corollary one may define for each \( x \geq 0 \) the transformation \( T(x) \) from \( S \) to \( S \) as follows: \( T(x)p = g_p(x) \) for each \( p \in S \). Then \( T \) is a strongly continuous semigroup of nonlinear nonexpansive transformations on \( S \), i.e.,

1. \( T(x+y) = T(x)T(y) \) for \( x, y \geq 0 \),
2. \( T(0) = e \),
3. \( \| T(x)p - T(x)q \| \leq \| p - q \| \) for \( x \geq 0 \) and \( p, q \in S \) and
4. \( g_p(x) = T(x)p \) is continuous for \( p \) fixed and \( x \geq 0 \).

Further, \( A \) is the infinitesimal generator of \( T \), i.e., \( Ap = g_p'(0) \) for each \( p \in S \). In \([2]\), \([14]\), \([17]\), \([18]\), and \([19]\) representations are given for nonlinear nonexpansive semigroups of transformations in terms of their infinitesimal generators using product integrals.

3. **Examples.** In conclusion we give some examples. In \([6]\) a well-known example is given by J. Dieudonné of a continuous mapping \( A \) from a Banach space \( S \) to \( S \) for which there is no solution to the equation \( g' = Ag \) and \( g(0) = 0 \). This example is given in a Banach space which is not reflexive. Recently, J. Yorke \([20]\) has given an example of a continuous mapping \( A \) from a Hilbert space to itself for which no solution exists to \( g' = Ag \), \( g(0) = 0 \).

In the examples below the mapping \( A \) satisfies conditions \((I')\), \((II')\), and \((III')\) of the corollary.

**Example 1.** Let \( S = E_1 \) and let \( A \) be a continuous nonincreasing function from \( E_1 \) to \( E_1 \).
Example 2. Let $S=C_{[0,1]}$, i.e., $S$ is the Banach space of continuous real-valued functions on $[0, 1]$ with supremum norm. Let $F$ be a continuous increasing function from $E_1$ onto $E_1$ such that $F'$ is continuous and nonincreasing on $E_1$. Define the mapping $A$ on $C_{[0,1]}$ as follows:

$$Af = F'[F^{-1}[f]]$$ for each $f \in C_{[0,1]}$.

The solutions $g_f$ of the corollary are then given by $g_f(x) = F[x + F^{-1}[f]]$ for $x \geq 0$.

In both Examples 1 and 2 $A$ may be neither linear nor Lipschitz continuous. In both, however, $A$ is locally uniformly continuous. In Example 3 the mapping $A$ is not locally uniformly continuous.

Example 3. Let $S=(c_0)$, i.e., $S$ is the Banach space of real-number sequences $x=(x_n)$ converging to 0 with $||x|| = \sup_n |x_n|$. If each of $(a, b)$ and $(c, d)$ is a point in the plane define the function $F_{(a, b),(c, d)}$ from $[a, c]$ to $[b, d]$ by

$$F_{(a, b),(c, d)}(x) = b + \frac{d-b}{c-a}(x-a)$$ for $x \in [a, c]$.

For each positive integer $n$ define the function $A_n$ from $E_1$ to $E_1$ as follows:

$$A_n(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } x \geq 0 \\ \frac{1}{k+1} & \text{if } x \in \left( -\frac{1}{k} + \frac{1}{k+1}, \frac{1}{k} - \frac{1}{k+1} \right) \\ \frac{1}{k} & \text{if } x \in \left( -\frac{1}{k}, \frac{1}{k} \right) \end{cases}$$

Define the mapping $A$ from $(c_0)$ to $(c_0)$ by $Ax=(A_n(x_n))$ for each $x=(x_n) \in (c_0)$. One sees that $A$ satisfies conditions (I'), (II'), and (III'), since for each positive integer $n$ $A_n$ is nonincreasing and continuous. Moreover, there is no neighborhood about 0 on which $A$ is uniformly continuous.

References


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