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LÉVY MEASURES FOR A CLASS OF MARKOV SEMIGROUPS IN ONE DIMENSION

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Abstract. Given a Markov semigroup of linear operators in the space of real-valued continuous functions on the line vanishing at infinity, we prove that the Lévy measure exists if the domain of the infinitesimal generator contains $\mathfrak{D}_K(D_m D_s^+)$, the domain of William Feller's generalized second order differential operator restricted to functions with compact supports. We give estimate of singularity of the Lévy measure and representation of the infinitesimal generator. Conversely, given Lévy measure or the form of infinitesimal generator, existence of the corresponding Markov semigroup is shown under some conditions. The case of circles is also discussed.

1. **Introduction.** Let $C_0(R)$ be the Banach space of real-valued continuous functions on the real line R vanishing at infinity with norm $\|f\| = \max_{x\in R} |f(x)|$, and let $C_K(R)$ be the subset of $C_0(R)$ of functions with compact supports. A family of linear operators $\{T_t;\ t\geqq 0\}$ on $C_0(R)$ is called *Markov semigroup* or *M-semigroup* if it is a strongly continuous semigroup and if each T_t is nonnegative with norm $\leqq 1$. A family of measures $\{n_x(dy);\ x\in R\}$ is called *Lévy measure* for $\{T_t\}$ if n_x is a measure on $R\backslash\{x\}$(2) finite for compact sets in $R\backslash\{x\}$ and if, for each $x\in R$ and $f\in C_K(R)$ such that $x\notin S(f)$(3), we have

$$(1.1)\qquad \lim_{t\to 0+} t^{-1}(T_t f)(x) = \int_{R\backslash\{x\}} f(y)n_x(dy).$$

Given an M-semigroup, the Lévy measure is unique if it exists. In this paper we will, under the assumption that the domain $\mathfrak{D}(\mathfrak{G})$ of the infinitesimal generator $\mathfrak{G}$ of $\{T_t\}$ contains $\mathfrak{D}_K(D_m D_s^+)$ for a generalized second order differential operator $D_m D_s^+$ of Feller [2](4), prove the existence of the Lévy measure and make an

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(2) $E\backslash F$ is the intersection of E and the complement of F.

(3) $S(f)$ is the support of f, i.e., the closure of the set $\{x;\ f(x)\neq 0\}$.

(4) $D_m D_s^+$ is $D_v D_u$ in [2] where $s(x)=u(x)$ and $m(x_1, x_2]=v(x_2)-v(x_1)$. For rigorous definition, see §2.

estimation of its singularity. Here $\mathfrak{D}_K(D_m D_s^+)$ is the set of functions with compact supports in the domain of $D_m D_s^+$. Let Γ (possibly void) be the set of discontinuity points for m. Let

$$\varphi_x(y) = \int_x^y s(dz)m(x,z] \quad \text{for } y \geqq x, \qquad \int_y^x s(dz)m(z,x] \quad \text{for } y < x, \tag{1.2}$$

$$s_x(y) = s(y) - s(x). \tag{1.3}$$

It will be proved that n_x is finite outside a neighborhood of x and that the singularity of n_x at x is such that $\varphi_x(y)$ or $|s_x(y)|$ is integrable with respect to n_x in a neighborhood of x according as $x \in R\backslash\Gamma$ or $x \in \Gamma$ (§3). Using this, we will get representation of $\mathfrak{G}f$ for $f \in \mathfrak{D}_K(D_m D_s^+)$ in the following integro-differential form (§4):

$$\begin{aligned} \mathfrak{G}f(x) &= a(x)D_m D_s^+ f(x) + b(x)D_s f(x) + c(x)f(x) \\ &\quad + \int_{R\backslash\{x\}} [f(y) - f(x) - \chi_U(y)s_x(y)D_s f(x)]n_x(dy) \end{aligned} \tag{1.4}$$

for $x \in R\backslash\Gamma$ where

$$a(x) \geqq 0, \qquad c(x) \leqq 0 \tag{1.5}$$

and U is a bounded open interval containing x(5), and

$$\begin{aligned} \mathfrak{G}f(x) &= (m\{x\})^{-1}(a^+(x)D_s^+ f(x) - a^-(x)D_s^- f(x)) + c(x)f(x) \\ &\quad + \int_{R\backslash\{x\}} [f(y) - f(x)]n_x(dy) \end{aligned} \tag{1.6}$$

for $x \in \Gamma$ where

$$a^+(x) \geqq 0, \qquad a^-(x) \geqq 0, \qquad c(x) \leqq 0. \tag{1.7}$$

It is known that the transition semigroups of processes continuous in probability with stationary independent increments are M-semigroups in $C_0(R)$, that the domains of their infinitesimal generators $\mathfrak{G}$ contain all C^2 functions with compact supports, and that they have (translation invariant) Lévy measures $n_x(dy)$, which are finite outside a neighborhood of x and make $(y-x)^2$ integrable near x. $\mathfrak{G}$ has the form

$$\begin{aligned} \mathfrak{G}f(x) &= a\frac{d^2 f}{dx^2}(x) + b\frac{df}{dx}(x) + cf(x) \\ &\quad + \int_{R\backslash\{x\}} \left[f(y) - f(x) - \chi_{(x-1,x+1)}(y)\frac{df}{dx}(x)\right] n_x(dy) \end{aligned}$$

where $a \geqq 0$, $c \leqq 0$ and b are constants. Our results are extension of these facts. In case all C^2 (or C^∞) functions with compact supports belong to $\mathfrak{D}(\mathfrak{G})$, the results are already known for R^n [1], [12], [13], [15].

(5) χ_U is the indicator function of U.

Let us call an operator A in $C_0(R)$ *dispersive*, if, whenever $f \in \mathfrak{D}(A)$ attains its positive maximum at x_0, $Af(x_0)$ is nonpositive. Dispersiveness is a necessary condition for an operator to generate an M-semigroup(6). An operator defined by the right-hand sides of (1.4) and (1.6) on $R\backslash\Gamma$ and Γ, respectively, is dispersive if the coefficients satisfy the sign conditions (1.5) and (1.7). Although we can prove the representation (1.4) for *all* $x \in R$ including Γ with $D_s^+ f$ or $D_s^- f$ replacing $D_s f$, the condition (1.5) for points in Γ does not guarantee dispersiveness of an operator defined by the right-hand side. This is the reason why we ought to use (1.6) on Γ.

In §§3 and 4 we investigate Lévy measures and infinitesimal generators, given M-semigroups. Turning to the converse direction, we will give in §5 sufficient conditions for operators of the form (1.4)–(1.7) to generate M-semigroups in $C_0(R)$ by using perturbation theory for semigroup generators. The M-semigroup thus generated has Lévy measure equal to the given n_x. In case m is continuous and D_s and D_m are ordinary differentiation with smooth coefficients, related results are found in [6], [10].

In §6 we will show that all the results can be carried over to the case of circles from the real line.

To every M-semigroup in $C_0(R)$ there corresponds a Markov process on the line and the probabilistic meaning of the Lévy measure is investigated in [5], [6], [8], [14]. The problem of finding Markov processes which have Lévy measures equal to (or greater than) a certain given measure has a special importance connected with the study of behavior near boundary for Markov processes. Thus Motoo [9] essentially proves that in order to find all those Markov processes on the closed disk which have continuous trajectories, stay on the boundary only for a set of times of Lebesgue measure zero, and behave in the interior in the same manner as the Brownian motion, it is necessary and sufficient to find all the Markov processes on the boundary whose Lévy measures are identical with that of the Cauchy process wound around on it. It is for this reason that, in examples, we are especially interested in the existence of those M-semigroups which have Lévy measure $n_x(dy) = \pi^{-1}(y-x)^{-2}\,dy$ (i.e., that of the Cauchy process) in case of the line or $n_x(dy) = (2\pi)^{-1}(1-\cos(y-x))^{-1}\,dy$ (i.e., that of the Cauchy process wound around) in case of the unit circle.

Motoo (unpublished) introduced integro-differential operators of the form (1.4) in 1966 and found the result stated in Example 6.1 with (6.3) replaced by a stronger condition

$$\lim_{\delta\to 0+} \sup_{x\in S} \int_{\{y;\theta_x(y)<\delta\}} |s_x(y)|m(x,y)n_x(dy) = 0.$$

Our Theorem 6.4 is merely an extension of this result of his.

(6) We say that A *generates* an M-semigroup if A is its infinitesimal generator.

2. **Lemmas concerning $D_mD_s^+$.** Suppose we are given a strictly increasing continuous function $s(x)$ on R and a measure m on R finite for compact sets and positive for nonempty open sets(7). s also induces a measure, which we denote by the same letter. Functions $\varphi_x(y)$ and $s_x(y)$ are defined by (1.2) and (1.3) and $\psi_x(y)$ is defined by

$$(2.1)\quad \psi_x(y) = \int_x^y s(dz)m[x, z] \quad \text{for } y \geqq x, \qquad \int_y^x s(dz)m(z, x) \qquad \text{for } y < x.$$

Obviously, $\varphi_x = \psi_x$ if $x \notin \Gamma$ where Γ is the set of discontinuity points for m. For any open set U, let $\mathfrak{D}(D_mD_s^+; U)$ be the set of functions f on U such that

(i) f is continuous;

(ii) $D_s^+f(x) = \lim_{h\to 0+} ((f(x+h)-f(x))/(s(x+h)-s(x)))$ exists and is right continuous with bounded variation on any compact subset of U;

(iii) the induced signed measure $D_s^+f(dx)$ on U is absolutely continuous with respect to m and its Radon-Nikodym derivative has a continuous version, which we denote by $D_mD_s^+f$.

Note that $f \in \mathfrak{D}(D_mD_s^+; U_1 \cup U_2)$ if $[f]_{U_1} \in \mathfrak{D}(D_mD_s^+; U_1)$ and $[f]_{U_2} \in \mathfrak{D}(D_mD_s^+; U_2)$(8). We denote by $\mathfrak{D}_0(D_mD_s^+)$ the set of f such that $f \in C_0(R) \cap \mathfrak{D}(D_mD_s^+; R)$ and $D_mD_s^+f \in C_0(R)$, and by $\mathfrak{D}_K(D_mD_s^+)$ the set $C_K(R) \cap \mathfrak{D}(D_mD_s^+; R)$.

We also use the left derivative $D_s^-f(x)$, the limit of $(f(x)-f(x-h))/(s(x)-s(x-h))$ as $h \to 0+$. If $D_s^+f(x) = D_s^-f(x)$, we write the value as $D_sf(x)$.

We will prove some lemmas.

Lemma 2.1. *Let f be continuous on $[x_1, x_2]$ and let D_s^+f exist in (x_1, x_2). Then*

$$D_s^+f(x_0) \leqq \frac{f(x_2)-f(x_1)}{s(x_2)-s(x_1)} \leqq D_s^+f(x_0')$$

for some x_0 and x_0' in (x_1, x_2).

Proof is easy and omitted. One consequence is that if $D_s^+f(x-)$, the limit of $D_s^+f(x-h)$ as $h \to 0+$, exists, then $D_s^-f(x) = D_s^+f(x-)$. Another consequence is that if f is continuous in (x_1, x_2) and $D_s^+f = 0$ in (x_1, x_2), then f is constant. More generally, we have

Lemma 2.2. *Let $f \in \mathfrak{D}(D_mD_s^+; (x_1, x_2))$ and $D_mD_s^+f = 0$. If $f(x_0) = D_s^+f(x_0) = 0$ at some $x_0 \in (x_1, x_2)$, or $f(x_1+) = D_s^+f(x_1+) = 0$, or $f(x_2-) = D_s^+f(x_2-) = 0$, then $f = 0$.*

Proof is immediate, since we get $D_s^+f = 0$.

(7) We sometimes write mE instead of $m(E)$ for a set $E \subset R$.

(8) $[f]_U$ stands for the restriction of f to U.

LEMMA 2.3. *Let $x_1 < x_0 < x_2$ and let g be continuous on (x_1, x_2). If*

$$\text{(2.2)}\qquad \begin{aligned} f(x) &= a+bs_{x_0}(x)+\int_{x_0}^{x} s(dy)\int_{(x_0,y]} g(z)m(dz), \qquad x \geqq x_0,\\ f(x) &= a+bs_{x_0}(x)+\int_{x}^{x_0} s(dy)\int_{(y,x_0]} g(z)m(dz), \qquad x < x_0, \end{aligned}$$

then, we have $f \in \mathfrak{D}(D_mD_s^+; (x_1, x_2))$ and

$$\text{(2.3)}\qquad f(x_0) = a, \qquad D_s^+f(x_0) = b, \qquad D_mD_s^+f = g.$$

Similarly, if

$$\text{(2.4)}\qquad \begin{aligned} f(x) &= a+bs_{x_0}(x)+\int_{x_0}^{x} s(dy)\int_{[x_0,y]} g(z)m(dz), \qquad x \geqq x_0,\\ f(x) &= a+bs_{x_0}(x)+\int_{x}^{x_0} s(dy)\int_{(y,x_0)} g(z)m(dz), \qquad x < x_0, \end{aligned}$$

then, $f \in \mathfrak{D}(D_mD_s^+; (x_1, x_2))$ and

$$\text{(2.5)}\qquad f(x_0) = a, \qquad D_s^-f(x_0) = b, \qquad D_mD_s^+f = g.$$

Conversely, every $f \in \mathfrak{D}(D_mD_s^+; (x_1, x_2))$ has the expressions (2.2) *and* (2.4) *where a, b, g are determined by* (2.3) *and* (2.5), *respectively.*

Proof. It is easy to check $f \in \mathfrak{D}(D_mD_s^+; (x_1, x_2))$, (2.3), and (2.5). In order to get the converse part, we have only to note Lemma 2.2.

REMARK 2.1. The above lemma is also true for $x_1 = x_0 < x_2$ or $x_1 < x_0 = x_2$ under suitable modification. Thus, if $f \in \mathfrak{D}(D_mD_s^+; (x_1, x_2))$ and if $D_mD_s^+f(x_1+)$ exists[9], then

$$f(x) = a+bs_{x_1}(x)+\int_{x_1}^{x} s(dy)\int_{(x_1,y]} g(z)m(dz)$$

where $f(x_1+) = a$, $D_s^+f(x_1+) = b$, and $D_mD_s^+f = g$. (Existence of a and b is proved from Lemma 2.3.)

The following three lemmas deal with some extension of functions in $\mathfrak{D}(D_mD_s^+; U)$ to a wider domain $U' \supset U$.

LEMMA 2.4. *Let $x_1 < x_2 < x_3 < x_4$ and let f be a function on $(x_1, x_2) \cup (x_3, x_4)$ which is constant in each interval: $f = a$ on (x_1, x_2), $f = a'$ on (x_3, x_4). Then, f can be extended to a function $\tilde{f} \in \mathfrak{D}(D_mD_s^+; (x_1, x_4))$ such that $a \wedge a' \leqq \tilde{f} \leqq a \vee a'$*[10].

Proof. Suppose $a < a'$. Let $g \in C[x_2, x_3]$, $g(x_2) = g(x_3) = 0$. Define $\tilde{f}$ by $\tilde{f} = f$ on $(x_1, x_2) \cup (x_3, x_4)$ and

$$\tilde{f}(x) = a+\int_{x_2}^{x} s(dy)\int_{(x_2,z]} g(z)m(dz) \quad \text{for } x \in [x_2, x_3].$$

(9) We use the word *existence* of a limit letting its finiteness be included.

(10) $a \wedge a'$ is the smaller of a and a' while $a \vee a'$ is the larger.

If $\tilde{f}$ satisfies $\tilde{f}(x_3)=a'$ and $D_s^+\tilde{f}(x_3-)=0$, then $\tilde{f}$ belongs to $\mathfrak{D}(D_mD_s^+; (x_1, x_4))$. Fix $x_0 \in (x_2, x_3)$ and let $g_1, g_2 \in C[x_2, x_3]$ be such that $g_1=0$ at x_2, >0 on (x_2, x_0), $=0$ on $[x_0, x_3]$ and $g_2=0$ on $[x_2, x_0]$, >0 on (x_0, x_3), $=0$ at x_3. We can choose positive constants c_1 and c_2 in such a way that $g=c_2(g_1-c_1g_2)$ satisfies $\int_{(x_2,x_3]} g(y)m(dy)=0$ and $\int_{x_2}^{x_3} s(dy) \int_{(x_2,y]} g(z)m(dz)=a'-a$. If we use this g, $\tilde{f}$ satisfies all the requirements. The case $a=a'$ is trivial and the case $a>a'$ is similar.

REMARK 2 2. In the above lemma, $\tilde{f}$ can be chosen to satisfy $\|D_mD_s^+\tilde{f}\| \leqq k|a-a'|$, where k is a constant depending on x_2 and x_3. This is clear from the proof.

LEMMA 2.5. *Let $x_1<x_2<x_3<x_4$ and let $f \in \mathfrak{D}(D_mD_s^+; (x_1, x_2) \cup (x_3, x_4))$. Let $D_mD_s^+f(x_2-)$ and $D_mD_s^+f(x_3+)$ exist. Then, for any $\varepsilon>0$ we can find $\tilde{f} \in \mathfrak{D}(D_mD_s^+; (x_1, x_4))$ which is an extension of f and satisfies*

$$(a \wedge a')-\varepsilon < \tilde{f} < (a \vee a')+\varepsilon \tag{2.6}$$

on $[x_2, x_3]$ where $a=f(x_2-)$ and $a'=f(x_3+)$.

Proof. Let $b=D_s^+f(x_2-)$ and $c=D_mD_s^+f(x_2-)$. Choose $x_2<\xi_1<\xi_2<x_3$ such that $(|b|+|c|m[x_2, \xi_1])s_{x_2}(\xi_1)<\varepsilon$, and then, choose a continuous function g on (x_1, ξ_2) in such a way that $g=D_mD_s^+f$ on (x_1, x_2), $g=0$ on $[\xi_1, \xi_2)$, and that if we define $h(x)=b+\int_{[x_2,x]} g(y)m(dy)$, then $|h(x)| \leqq |b|+|c|m[x_2, \xi_1]$ on $[x_2, \xi_1]$ and $h(\xi_1)=0$. Define $\tilde{f}$ on (x_1, ξ_2) by the right-hand side of (2.4) with $x_0=x_2$. By Lemma 2.3 and Remark 2.1 we see that $\tilde{f}$ is an extension of $[f]_{(x_1,x_2)}$ and belongs to $\mathfrak{D}(D_mD_s^+; (x_1, \xi_2))$. $\tilde{f}$ satisfies (2.6) and is flat on $[\xi_1, \xi_2)$. In a similar way we can get ξ_3, ξ_4 $(\xi_2<\xi_3<\xi_4<x_3)$, and $\tilde{f}$ on (ξ_3, x_4) which is flat on $(\xi_3, \xi_4]$. Finally, $\tilde{f}$ on $[\xi_2, \xi_3]$ is obtained by Lemma 2.4.

REMARK 2.3. In the above proof, if $|b| \leqq k_1$ and $|c| \leqq k_1$, then g can be chosen to satisfy $\|g\| \leqq k_2$, where k_2 is a constant which depends on x_2, x_3, ε, and k_1. For, choose ξ_1 such as $k_1(1+m[x_2, \xi_1])s_{x_2}(\xi_1)<\varepsilon$, let $\xi_0 \in (x_2, \xi_1)$ and let g_0 be a continuous function on $[\xi_0, \xi_1]$ vanishing at ξ_0 and ξ_1 and positive on (ξ_0, ξ_1). Given f, we can choose g as follows: $g(x)=c(\xi_0-x)/(\xi_0-x_2)$ for $x \in [x_2, \xi_0]$ and

$$g(x) = -h(\xi_0)\left(\int_{(\xi_0,\xi_1]} g_0(y)m(dy)\right)^{-1} g_0(x) \quad \text{for } x \in [\xi_0, \xi_1].$$

This remark will be useful later.

LEMMA 2.6. *Let $x_1<x_2<x_3<x_4$ and let $f(x)=c\varphi_{x_0}(x)$ on (x_1, x_2) and $f(x)=c'\varphi_{x_0}(x)$ on (x_3, x_4). Then, there is a function $\tilde{f} \in \mathfrak{D}(D_mD_s^+; (x_1, x_4))$ such that $\tilde{f}=f$ on $(x_1, x_2) \cup (x_3, x_4)$ and*

$$(c \wedge c')\varphi_{x_0}(x) \leqq \tilde{f}(x) \leqq (c \vee c')\varphi_{x_0}(x). \tag{2.7}$$

The same is true if we replace φ_{x_0} by ψ_{x_0} in the above statement.

Proof. Assume $x_0 < x_2 < x_3$ or $x_2 < x_3 < x_0$. The other case is readily reduced to this case. Let $c < c'$. Let $x_2 < \xi_1 < \xi_2 < x_3$ and let $\tilde{f} = f$ on $(x_1, x_2) \cup (x_3, x_4)$, and

$$\tilde{f}(x) = c\varphi_{x_0}(x) + \int_{x_2}^{x} s(dy) \int_{(x_2, y]} g(z)m(dz) \quad \text{for } x \in [x_2, \xi_1),$$

$$\tilde{f}(x) = c'\varphi_{x_0}(x) - \int_{x}^{x_3} s(dy) \int_{(y, x_3]} g(z)m(dz) \quad \text{for } x \in (\xi_2, x_3],$$

where g is positive continuous on $(x_2, \xi_1) \cup (\xi_2, x_3)$ and $g(x_2+) = g(x_3-) = 0$. Then $\tilde{f} \in \mathfrak{D}(D_m D_s^+; (x_1, \xi_1) \cup (\xi_2, x_4))$ by Lemma 2.3. If we choose ξ_1 close to x_2 and ξ_2 close to x_3, we have $c\varphi_{x_0}(x) < \tilde{f}(x) < c'\varphi_{x_0}(x)$ on $(x_2, \xi_1) \cup (\xi_2, x_3)$. In order to obtain $\tilde{f}$ on (x_1, x_4), we have only to connect the two separate parts of $\tilde{f}$ obtained in this way first by a step function, and then using Lemmas 2.4 and 2.5 appropriately. The case $c = c'$ is trivial and the case $c > c'$ is treated similarly.

Using these lemmas, we can prove some assertions concerning how large the domain $\mathfrak{D}_K(D_m D_s^+)$ is.

LEMMA 2.7. *Let $f \in C_K(R)$ and $S(f) \subset [x_1, x_2]$. For any $\varepsilon > 0$, we can find a function $f' \in \mathfrak{D}_K(D_m D_s^+)$ such that $\|f - f'\| < \varepsilon$ and $S(f') \subset [x_1, x_2]$.*

Proof. First approximate f by a step function vanishing near x_1 and x_2, and then use Lemma 2.4.

LEMMA 2.8. *Let x_0 be fixed. Let $f \in C_K(R)$ and $S(f) \subset [x_1, x_2]$. Given $\varepsilon > 0$, we can find $f' \in C_K(R)$ such that $f' \cdot \varphi_{x_0} \in \mathfrak{D}_K(D_m D_s^+)$, $\|f - f'\| < \varepsilon$, and $S(f') \subset [x_1, x_2]$. The statement remains true if we replace φ_{x_0} by ψ_{x_0}.*

Proof. Suppose $x_0 \notin (x_1, x_2)$. Approximate f by a step function g vanishing on $(-\infty, x_1 + \delta) \cup (x_2 - \delta, +\infty)$ for some $\delta > 0$ and having an even number of jump points $\xi_1, \xi_2, \ldots, \xi_n$. Suppose $x_1 < \xi_1 < \cdots < \xi_n < x_2$ and $g(x) = c_i$ on (ξ_i, ξ_{i+1}). Choose a function $h \in \mathfrak{D}_K(D_m D_s^+)$ such that $h(x) = 0$ on $(-\infty, \xi_1] \cup [\xi_n, +\infty)$, $h(x) = c_i \varphi_{x_0}(x)$ on (ξ_i, ξ_{i+1}) for even i, and

$$(c_{i-1} \wedge c_{i+1})\varphi_{x_0}(x) \leqq h(x) \leqq (c_{i-1} \vee c_{i+1})\varphi_{x_0}(x)$$

on (ξ_i, ξ_{i+1}) for odd i, letting $c_0 = c_n = 0$. Such an h exists by Lemma 2.6. Then $f'(x) = h(x)/\varphi_{x_0}(x)$ is the desired function. In case $x_0 \in (x_1, x_2)$, the proof is the same if only we choose g and f' flat in a neighborhood of x_0.

Using Lemmas 2.4 and 2.5 combined with Remarks 2.2 and 2.3, we can prove another extension lemma.

LEMMA 2.9. *Given positive constants k_1 and k_2 $(k_1 > k_2)$, let $f \in \mathfrak{D}(D_m D_s^+; (x_1, x_2))$ have $\|f\|$, $\|D_s^+ f\|$, and $\|D_m D_s^+ f\|$*(11) *not exceeding k_1, let $D_m D_s^+ f(x_2-)$ (hence also $f(x_2-)$ and $D_s^+ f(x_2-)$) exist and let $f(x_2-) \geqq k_2$. Then we can find a constant k_3 such that there is an extension $\tilde{f}$ of f to $(x_1, +\infty)$ which is in $\mathfrak{D}(D_m D_s^+; (x_1, +\infty))$,*

(11) We use the notation $\|g\| = \sup_x |g(x)|$ for any function g.

positive on $[x_2, x_3)$ *and zero on* $[x_3, +\infty)$ *for some* x_3, *and satisfies* $\|\tilde{f}\|$, $\|D_m D_s^+ \tilde{f}\| \leqq k_3$. k_3 *is determined by* x_2, k_1, *and* k_2. *The similar assertion is true for extension to* $(-\infty, x_2)$, *too.*

3. **Lévy measures.** The following theorem gives a sufficient condition for the existence of the Lévy measure and describes the order of its singularity and some other properties.

THEOREM 3.1. *Let* $\{T_t\}$ *be an M-semigroup in* $C_0(R)$ *with infinitesimal generator* $\mathfrak{G}$ *and assume that* $\mathfrak{D}(\mathfrak{G}) \supset \mathfrak{D}_K(D_m D_s^+)$. *Then,*

(i) $\{T_t\}$ *has the Lévy measure* $\{n_x(dy); x \in R\}$.

(ii) n_x *is continuous with respect to* x *in the sense that*

$$\lim_{x \to x_0} \int_{R \setminus \{x\}} f(y) n_x(dy) = \int_{R \setminus \{x_0\}} f(y) n_{x_0}(dy)$$

for each x_0 *and* $f \in C_K(R)$ *such that* $S(f) \not\ni x_0$.

(iii) $\lim_{|x| \to \infty} \int_{R \setminus \{x\}} f(y) n_x(dy) = 0$ *for each* $f \in C_K(R)$.

(iv) *If* U *is open,* V *is bounded open, and* $\overline{V} \subset U$, *then* $\sup_{x \in V} n_x(R \setminus U) < \infty$.

(v) (1.1) *holds for all* $x \in R$ *and* $f \in C_0(R)$ *such that* $S(f) \not\ni x$.

(vi) *Let* E *be compact. We have*

$$\int_{E \setminus \{x\}} \varphi_x(y) n_x(dy) < \infty \quad \text{if } x \notin \Gamma, \tag{3.1}$$

$$\int_{E \setminus \{x\}} |s_x(y)| n_x(dy) < \infty \quad \text{if } x \in \Gamma. \tag{3.2}$$

(vii) *If* $x_0 \notin \Gamma$, *then we have*

$$\lim_{t \to 0+} t^{-1} T_t(f \varphi_{x_0})(x_0) = \int_{R \setminus \{x_0\}} f(y) \varphi_{x_0}(y) n_{x_0}(dy) \tag{3.3}$$

for every $f \in C_K(R)$ *such that* $f(x_0) = 0$. *If* $x_0 \in \Gamma$, *then*

$$\lim_{t \to 0+} t^{-1} T_t(f s_{x_0})(x_0) = \int_{R \setminus \{x_0\}} f(y) s_{x_0}(y) n_{x_0}(dy) \tag{3.4}$$

for every $f \in C_K(R)$ *such that* $f(x_0) = 0$.

Proof. Let $x_0 \in R$. For each $f \in \mathfrak{D}_K(D_m D_s^+)$ such that $f(x_0) = 0$, $t^{-1} T_t f(x_0)$ tends to $\mathfrak{G} f(x_0)$ as $t \to 0+$. Given $E = [x_1, x_2] \not\ni x_0$, let g be such that $g \in \mathfrak{D}_K(D_m D_s^+)$, $g \geqq 0$, $[g]_E = 1$, and $g(x_0) = 0$. Such g is found by Lemma 2.4. For each $f \in C_K(R)$ such that $S(f) \subset E$, $t^{-1} T_t f(x_0)$ is convergent as $t \to 0+$. In fact, given $\varepsilon > 0$, we can pick $f' \in \mathfrak{D}_K(D_m D_s^+)$ such that $\|f - f'\| < \varepsilon$ and $S(f') \subset E$ (Lemma 2.7), and we have

$$t^{-1} T_t f(x_0) = t^{-1} T_t f'(x_0) + t^{-1} T_t (f - f')(x_0),$$

the first term of which is convergent as $t \to 0+$ and the second term has absolute value $\leqq \varepsilon t^{-1} T_t g(x_0)$, which tends to $\varepsilon \mathfrak{G} g(x_0)$. Consequently, there exists a finite measure $n^E_{x_0}$ on E such that $t^{-1}T_t f(x_0)$ tends to $\int_E f(y)n^E_{x_0}(dy)$. $E\subset E'$ implies $n^E_{x_0}=n^{E'}_{x_0}$ on E. Hence $n^E_{x_0}$ is extended to a measure n_{x_0} on $R\backslash\{x_0\}$, which satisfies the conditions of Lévy measure. The assertion (ii) is proved through approximation of f by functions of $\mathfrak{D}_K(D_m D_s^+)$. Note that the assertion is immediate if $f\in\mathfrak{D}_K(D_m D_s^+)$, since

$$\int_{R\backslash\{x\}} f(y)n_x(dy) = \mathfrak{G}f(x), \qquad x\notin S(f). \tag{3.5}$$

(iii) is also obvious for $f\in\mathfrak{D}_K(D_m D_s^+)$. For general $f\in C_K(R)$, use $f'\in\mathfrak{D}_K(D_m D_s^+)$ such that $f'\geqq |f|$. In proving (iv), we may assume U to be bounded. Let g be such that $g\in\mathfrak{D}_K(D_m D_s^+)$, $g\geqq 0$, and $[g]_U=1$ (Lemma 2.4). For each compact set $E\supset U$, choose f such that $f\in\mathfrak{D}_K(D_m D_s^+)$, $0\leqq f\leqq 1$, and $f=1$ on E (Lemma 2.4). We have, for $x\in V$,

$$\begin{aligned}\mathfrak{G}f(x) &= \int_{R\backslash U}(f-g)(y)n_x(dy)+\mathfrak{G}g(x)\\ &\geqq n_x(E\backslash U)-\int_{R\backslash U} g(y)n_x(dy)+\mathfrak{G}g(x)\end{aligned}$$

by (3.5). It follows from $\mathfrak{G}f(x)\leqq 0$ that

$$n_x(E\backslash U) \leqq \int_{R\backslash U} g(y)n_x(dy)-\mathfrak{G}g(x).$$

The right-hand side is independent of E and bounded on V, because the first term is $\leqq\mathfrak{G}g'(x)$ if we choose $g'\in\mathfrak{D}_K(D_m D_s^+)$ satisfying $g'\geqq 0$, $g'=0$ on V, and $g'\geqq g$ on $R\backslash U$ (Lemma 2.4). Hence we obtain (iv). Let $P_{t,x}(dy)$ be the measure such that $T_t f(x)=\int f(y)P_{t,x}(dy)$. By the same idea we get finiteness of $\sup_t t^{-1}P_{t,x}(R\backslash U)$ for $x\in U$, and hence, (v) is proved through approximation by functions of $C_K(R)$.

Proceeding to the proof of (vi) and (vii), let $x\in R$ be fixed, and let

$$t^{-1}\varphi_x(y)P_{t,x}(dy) = Q_{t,x}(dy).$$

If $f\in C_K(R)$ satisfies $f\cdot\varphi_x\in\mathfrak{D}_K(D_m D_s^+)$, then $\int f(y)Q_{t,x}(dy)$ tends to $\mathfrak{G}(f\varphi_x)(x)$ as $t\to 0+$. We claim that $\int f(y)Q_{t,x}(dy)$ converges for all $f\in C_K(R)$. Let $S(f)\subset[x_1,x_2]$. Choose $g\in C_K(R)$ such that $g\cdot\varphi_x\in\mathfrak{D}_K(D_m D_s^+)$, $g\geqq 0$, and $g=1$ on $[x_1,x_2]$, using Lemma 2.6. Further, we can choose, for any $\varepsilon>0$, a function $f'\in C_K(R)$ such that $f'\cdot\varphi_x\in\mathfrak{D}_K(D_m D_s^+)$, $\|f-f'\|<\varepsilon$, and $S(f')\subset[x_1,x_2]$ by using Lemma 2.8. Then, $\int f'Q_{t,x}(dy)$ converges and $\int(f-f')Q_{t,x}(dy)$ has absolute value $\leqq\varepsilon\int gQ_{t,x}(dy)$. Hence, $\int fQ_{t,x}(dy)$ converges as $t\to 0+$. There is a measure $R_x(dy)$ on R finite for compact sets such that the limit is represented by $\int fR_x(dy)$. If f vanishes in a neighborhood of x, then the limit also equals $\int f\varphi_x n_x(dy)$. Hence, $R_x(dy)=\varphi_x(y)n_x(dy)$ on $R\backslash\{x\}$. This proves (3.1) for all x and (3.3) for all x_0. The same fact can be proved for ψ_x instead of φ_x, and we get (3.2) and (3.4), since $\psi_x(y)-\varphi_x(y)=m\{x\}s_x(y)$. The proof of Theorem 3.1 is complete.

COROLLARY. *Let* $f \in \mathfrak{D}_K(D_m D_s^+)$ *and let* U *be a bounded open interval. The integral*

$$(3.6) \qquad \int_{R\setminus\{x\}} [f(y)-f(x)-\chi_U(y)s_x(y)D_s^+f(x)]n_x(dy)$$

exists for $x \in U$ *and is measurable*([12]) *with respect to* x. *The same is true with* D_s^+f *replaced by* D_s^-f.

Proof. Existence of the integral is a consequence of (iv), (vi), and Lemma 2.3. If E and E' are disjoint compact sets and if $g(x, y)$ is a bounded jointly measurable function on $E\times E'$, then $\int_{E'} g(x, y)n_x(dy)$ is measurable on E by a standard argument. Let $g(x, y)$ be the integrand in (3.6), $E=[x_0, x_0+1/n]$, and $E'=[x_0+2/n, x_0+n]$. Then we see that $\int_{[x+2/n,x+n-1/n]} g(x, y)n_x(dy)$ is measurable on E, hence on R. Therefore $\int_{(x,+\infty)} g(x, y)n_x(dy)$ is measurable on U, and so is the integral on $(-\infty, x)$.

THEOREM 3.2. *Under the same assumption as in Theorem* 3.1, *suppose, further,* $\mathfrak{D}(\mathfrak{G})\supset\mathfrak{D}_0(D_m D_s^+)$. *Then,*

$$(3.7) \qquad \sup_{x\in E} \int_{E\setminus\{x\}} (\varphi_x(y)+\psi_x(y))n_x(dy) < \infty$$

for any compact set E.

Proof. Let $A=D_m D_s^+$ with domain $\mathfrak{D}(A)=\mathfrak{D}_0(D_m D_s^+)$. We claim A is a closed operator in $C_0(R)$. Suppose $f_n \in \mathfrak{D}(A)$, $Af_n=g_n$, $\|f-f_n\| \to 0$, and $\|g-g_n\| \to 0$. We will prove $f \in \mathfrak{D}(A)$ and $Af=g$. Let $x_1<x_2$ and $x_1, x_2 \notin \Gamma$. As Feller [3] shows, there are the Green function $G(x, y)$ and the minimal harmonic functions $H_1(x)$ and $H_2(x)$ relative to $D_m D_s^+$ on $[x_1, x_2]$, and f_n is represented as

$$f_n(x) = \int_{x_1}^{x_2} G(x, y)g_n(y)m(dy)+H_1(x)f_n(x_1)+H_2(x)f_n(x_2).$$

It follows that the same equality holds with f and g replacing f_n and g_n, which implies $f \in \mathfrak{D}(A)$ and $Af=g$. Hence, A is closed. $\mathfrak{G}$ is also closed since it is an infinitesimal generator. From this closedness of two operators, we get an estimate

$$(3.8) \qquad \|\mathfrak{G}f\| \leqq k_1\|D_m D_s^+f\|+k_2\|f\|, \qquad f \in \mathfrak{D}_0(D_m D_s^+),$$

by application of the closed graph theorem (see Yosida [16, Chapter II, §6]). Let E be a compact set and U be a bounded open interval, $U\supset E$. Applying Lemma 2.9 to $f=[\varphi_x]_U$ with $x \in E$, we find extensions $\tilde{\varphi}_x$ of $[\varphi_x]_U$ such that $\tilde{\varphi}_x \in \mathfrak{D}_K(D_m D_s^+)$,

([12]) We mean Borel measurable by *measurable*.

$\tilde{\varphi}_x \geqq 0$, $\|\tilde{\varphi}_x\| \leqq k_3$, and $\|D_m D_s^+ \tilde{\varphi}_x\| \leqq k_3$. k_3 is a constant independent of $x \in E$. Let $g_x(y) = \tilde{\varphi}_x(y)/\varphi_x(y)$. We have

$$\int_{E\setminus\{x\}} \varphi_x(y) n_x(dy) \leqq \int_{R\setminus\{x\}} g_x(y)\varphi_x(y) n_x(dy) \leqq \int_R g_x(y) R_x(dy)$$

$$= \lim_{t\to 0+} \int_R g_x(y) Q_{t,x}(dy) = \mathfrak{G}(\tilde{\varphi}_x)(x) \leqq (k_1+k_2)k_3$$

by (3.8). A similar argument can be made for ψ_x and the proof is complete.

4. **Representation of infinitesimal generators.**

THEOREM 4.1. *Suppose that $\{T_t\}$ is an M-semigroup in $C_0(R)$ with infinitesimal generator $\mathfrak{G}$ such that $\mathfrak{D}(\mathfrak{G}) \supset \mathfrak{D}_K(D_m D_s^+)$. Then,*

(i) *For each bounded open interval U, there are unique functions $a(x)$, $b(x)$, and $c(x)$ on $U\setminus\Gamma$ such that for every $f \in \mathfrak{D}_K(D_m D_s^+)$ and $x \in U\setminus\Gamma$ $\mathfrak{G}f(x)$ is represented by* (1.4), *where $n_x(dy)$ is the Lévy measure for $\{T_t\}$. a, b, and c are measurable, a and c are independent of U and satisfy* (1.5).

(ii) *For each $x \in \Gamma$ there are unique $a^+(x)$, $a^-(x)$, and $c(x)$ such that for every $f \in \mathfrak{D}_K(D_m D_s^+)$ $\mathfrak{G}f(x)$ is represented by* (1.6). *They satisfy* (1.7).

REMARK 4.1. Actually, a and b are also defined on the whole R and U, respectively, and we have

$$\begin{aligned} \mathfrak{G}f(x) &= a(x)D_m D_s^+ f(x) + b(x) D_s^+ f(x) + c(x) f(x) \\ &\quad + \int_{R\setminus\{x\}} [f(y) - f(x) + \chi_U(y) s_x(y) D_s^+ f(x)] n_x(dy) \end{aligned} \tag{4.1}$$

for *all* $x \in U$. (Note that $D_s^+ f = D_s^- f = D_s f$ outside of Γ.) c is lower semicontinuous on R, a is upper semicontinuous at points $\notin \Gamma$, and b is continuous at x such that $n_x(\{x_1\}) = n_x(\{x_2\}) = 0$ where x_1 and x_2 are the boundary points of U. Using any function $\rho \in C_K(R)$ which is 1 on U, we can also have

$$\begin{aligned} \mathfrak{G}f(x) &= a(x) D_m D_s^+ f(x) + b'(x) D_s^+ f(x) + c'(x) f(x) \\ &\quad + \int_{R\setminus\{x\}} [f(y) - \rho(y)(f(x) + s_x(y) D_s^+ f(x))] n_x(dy) \end{aligned} \tag{4.2}$$

for all $x \in U$. This time, b' and c' are continuous. A similar remark can be made with $D_s^+ f(x)$ replaced by $D_s^- f(x)$.

Proof. Let $U = (x_1, x_2)$, $x \in U$, and let $\tilde{s}_x$, $\tilde{\varphi}_x$, and $\tilde{\rho}$ be functions in $\mathfrak{D}_K(D_m D_s^+)$ which coincide on U with s_x, φ_x, and 1, respectively, and $\tilde{\varphi}_x \geqq 0$, $1 \geqq \tilde{\rho} \geqq 0$. Given $f \in \mathfrak{G}_K(D_m D_s^+)$ define $g_x(y)$ by

$$f(y) = \tilde{\rho}(y) f(x) + \tilde{s}_x(y) D_s^+ f(x) + \tilde{\varphi}_x(y) D_m D_s^+ f(x) + g_x(y).$$

Since $g_x \in C_K(R)$ and $g_x(y)/\varphi_x(y)$ tends to 0 as $y \to x$ (use Lemma 2.3), we have

$$\mathfrak{G}f(x) = \mathfrak{G}\tilde{\rho}(x)\cdot f(x)+\mathfrak{G}(\tilde{s}_x)(x)\cdot D_s^+f(x)+\mathfrak{G}(\tilde{\varphi}_x)(x)\cdot D_mD_s^+f(x)+\int_{R\setminus\{x\}} g_x(y)n_x(dy)$$

by Theorem 3.1 (vii). We obtain (4.1) from this because

$$\begin{aligned}\int g_x(y)n_x(dy) = &\int [f(y)-f(x)-\chi_U(y)s_x(y)D_s^+f(x)]n_x(dy)\\ &+f(x)\int[1-\tilde{\rho}(y)]n_x(dy)+D_s^+f(x)\int[\chi_U(y)s_x(y)-\tilde{s}_x(y)]n_x(dy)\\ &-D_mD_s^+f(x)\int\tilde{\varphi}_x(y)n_x(dy).\end{aligned}$$

It follows from

$$\mathfrak{G}\tilde{\rho}(x) = c(x)+\int_{R\setminus\{x\}}[\tilde{\rho}(y)-1]n_x(dy)$$

that c is unique and measurable. This formula also implies that c is independent of the choice of U. In fact, if U' is another bounded open interval containing x and if $\tilde{\rho}'$ is a function in $\mathfrak{D}_K(D_mD_s^+)$ equal to 1 on U', then the above formula is valid with $\tilde{\rho}'$ replacing $\tilde{\rho}$. We see that $c(x)\leqq 0$ by choosing U large and noting $\mathfrak{G}\rho(x)\leqq 0$, and, at the same time, c turns out to be lower semicontinuous, being the increasing limit of continuous functions. If $\tilde{s}\in\mathfrak{D}_K(D_mD_s^+)$ is an extension of $[s]_U$, we have

$$\mathfrak{G}\tilde{s}(x) = b(x)+c(x)s(x)+\int[\tilde{s}(y)-s(x)-\chi_U(y)s_x(y)]n_x(dy),$$

which implies uniqueness and measurability of b. Let us write (4.1) into (4.2). Then, c' and b' are continuous on U, since

$$\mathfrak{G}\tilde{\rho}(x) = c'(x)+\int_{R\setminus\{x\}}[\tilde{\rho}(y)-\rho(y)]n_x(dy),$$

$$\mathfrak{G}\tilde{s}(x) = b'(x)+c'(x)s(x)+\int_{R\setminus\{x\}}[\tilde{s}(y)-\rho(y)s(y)]n_x(dy).$$

Since $\mathfrak{G}(\tilde{\varphi}_x)(x)=a(x)+\int_{R\setminus\{x\}}\tilde{\varphi}_x(y)n_x(dy)$, a is unique and is independent of the choice of U. We have $a(x)\geqq 0$ for fixed x by choosing U and $S(\tilde{\varphi}_x)$ very small and noting $\mathfrak{G}(\tilde{\varphi}_x)(x)\geqq 0$. Fix a point $x_0<x_1$ and let $\tilde{\varphi}=\tilde{\varphi}_{x_0}$. By the equality

$$\begin{aligned}\mathfrak{G}\tilde{\varphi}(x) = &\ a(x)+b'(x)m(x_0,x]+c'(x)\tilde{\varphi}(x)\\ &+\int_{R\setminus\{x\}}[\tilde{\varphi}(y)-\rho(y)(\tilde{\varphi}(x)+s_x(y)m(x_0,x])]n_x(dy),\end{aligned}\tag{4.3}$$

a is measurable on U and upper semicontinuous at points $\notin\Gamma$. In fact, let $\xi\notin\Gamma$ and let $\{\sigma_n\}$ be an increasing sequence of continuous functions such that $1\geqq\sigma_n\geqq 0$, $\sigma_n=1$ outside of $(\xi-2/n,\ \xi+2/n)$ and $\sigma_n=0$ on $(\xi-1/n,\ \xi+1/n)$. Let $g(x)$ be the integral in (4.3) and $g_n(x)$ be the same integral with the integrand multiplied by $\sigma_n(y)$. Then, g_n is continuous at ξ, $g_n\leqq g$, and $g_n(\xi)$ increases to $g(\xi)$ as $n\to\infty$.

Hence, g is lower semicontinuous at ξ, and the upper semicontinuity of a at ξ follows. The continuity of b at x such that $n_x(\{x_1\})=n_x(\{x_2\})=0$ is proved from

$$\mathfrak{G}\tilde{s}(x) = b(x)+c'(x)s(x)+\int_{R\setminus U}[\tilde{s}(y)-\rho(y)s(x)]n_x(dy).$$

The proof of (ii) is as follows. Let $x \in \Gamma$. Noting that

$$D_mD_s^+f(x) = (m\{x\})^{-1}(D_s^+f(x)-D_s^-f(x))$$

and using (3.2), we can rewrite (4.1) into (1.6), and we have $a^-(x)=a(x)\geqq 0$ and $c(x)\leqq 0$. The uniqueness of a^+, a^-, c in (1.6) follows from that of a, b, c in (4.1), since (1.6) can be written into (4.1) conversely. Let $\tilde{\psi}_x$ be a nonnegative function in $\mathfrak{D}_K(D_mD_s^+)$ which equals ψ_x in a neighborhood of x. Then,

$$0 \leqq \mathfrak{G}(\tilde{\psi}_x)(x) = a^+(x)+\int\tilde{\psi}_x(y)n_x(dy),$$

and hence we get $a^+(x)\geqq 0$, letting $S(\tilde{\psi}_x)$ be small enough. The proof of Theorem 4.1 and Remark 4.1 is complete.

REMARK 4.2. In Theorem 4.1, $\lim_{t\to 0+} t^{-1}(T_tf(x)-f(x))$ exists for each point $x \in R$ if $f \in \mathfrak{D}(D_mD_s^+; R) \cap C_0(R)$. The limit is represented in the same way. In fact, such f is the sum of a function in $\mathfrak{D}_K(D_mD_s^+)$ and a function in $C_0(R)$ which vanishes in a neighborhood of x (Lemma 2.5). Theorem 3.1 (v) applies to the latter.

Adding some observations to the above proof as we did in the proof of Theorem 3.2, we get the following result.

THEOREM 4.2. *Suppose, further,* $\mathfrak{D}(\mathfrak{G})\supset\mathfrak{D}_0(D_mD_s^+)$ *in Theorem* 4.1. *Then,* a, a^+, a^-, *and* c *are bounded on any compact set in* R *and* b *is bounded on any compact set in* U.

5. **Generation of M-semigroups.** In this section we refer to the following restrictions on m and s.

PROPERTY I. $D_mD_s^+$ with domain $\mathfrak{D}_0(D_mD_s^+)$ generates an M-semigroup in $C_0(R)$.

PROPERTY II. There is a positive constant α such that for each $x \in R\setminus\Gamma$ there exist y_x and y_x' which satisfy $y_x\leqq x\leqq y_x'$, $(s(y_x')-s(y_x))^{-1}(\varphi_x(y_x)+\varphi_x(y_x'))\leqq\alpha$, and $\sup_{x\in R\setminus\Gamma}(s(y_x')-s(y_x))^{-1}<\infty$.

PROPERTY III. The constant α in Property II can be chosen arbitrarily small.

PROPERTY IV. m is continuous, that is, Γ is void.

LEMMA 5.1. *Suppose that* m *and* s *have Property* II. *Then, there is a constant* β *such that*

$$\|D_s^+f\| = \|D_s^-f\| \leqq \alpha\|D_mD_s^+f\|+\beta\|f\| \tag{5.1}$$

for all $f \in \mathfrak{D}_0(D_mD_s^+)$.

Proof. The first equality is clear from $D_s^- f(x) = D_s^+ f(x-)$. We have, by Lemma 2.3,

$$f(y')-f(y) = s_y(y')D_s^+f(x)+\int_x^{y'} s(dz)\int_{(x,z]} g(w)m(dw)-\int_y^x s(dz)\int_{(z,x]} g(w)m(dw)$$

where $g = D_m D_s^+ f$ and $y \leqq x \leqq y'$. Hence

$$(5.2)\quad |D_s^+f(x)| \leqq s_y(y')^{-1}|f(y')-f(y)|+s_y(y')^{-1}(\varphi_x(y)+\varphi_x(y'))\sup_{z\in(y,y')}|g(z)|$$

and we get (5.1) by Property II. Note that $R\backslash\Gamma$ is dense.

The next lemma is a special case of a result [4] in general Banach lattices (see [11] in case $k_1 < 1/2$).

LEMMA 5.2. *Suppose that A generates an M-semigroup in $C_0(R)$ and B is a linear operator with domain $\mathfrak{D}(B) \supset \mathfrak{D}(A)$. Let $\mathfrak{G} = A+B$. If $\mathfrak{G}$ is dispersive and if there are constants $k_1 < 1$ and $k_2 < +\infty$ such that*

$$\|Bf\| \leqq k_1\|Af\|+k_2\|f\| \quad \textit{for } f \in \mathfrak{D}(A),$$

then $\mathfrak{G}$ generates an M-semigroup.

We will give three theorems.

THEOREM 5.1. *Suppose that m and s have Properties* I *and* II. *Let $\mathfrak{G}$ be an operator in $C_0(R)$ with domain $\mathfrak{D}(\mathfrak{G}) = \mathfrak{D}_0(D_m D_s^+)$ such that, for each $x \in R$, there are a measure $n_x(dy)$ on $R\backslash\{x\}$ and an open interval U_x containing x and the following two conditions are satisfied*:

(i) *There are functions b and c on $R\backslash\Gamma$ such that* (1.4) *holds on $R\backslash\Gamma$ for $f \in \mathfrak{D}(\mathfrak{G})$ with $a(x)$ replaced by* 1 *and U replaced by U_x, and they satisfy*

$$(5.3)\qquad \alpha\|b\|+\sup_{x\in R\backslash\Gamma}\int_{U_x}\varphi_x(y)n_x(dy) < 1,$$

$$(5.4)\qquad \sup_{x\in R\backslash\Gamma} n_x(R\backslash U_x) < \infty,$$

$$(5.5)\qquad \|b\| < \infty, \quad \|c\| < \infty, \quad \textit{and} \quad c \leqq 0.$$

(ii) *There are functions a^+, a^-, and c on Γ such that* (1.6) *holds on Γ for $f \in \mathfrak{D}(\mathfrak{G})$ and they satisfy* (1.7),

$$(5.6)\quad \sup_{x\in\Gamma}(m\{x\})^{-1}(|a^+(x)-1|+|a^-(x)-1|)+\sup_{x\in\Gamma}\int_{U_x}|s_x(y)|n_x(dy) < \alpha^{-1},$$

$$(5.7)\qquad \sup_{x\in\Gamma} n_x(R\backslash U_x) < \infty,$$

$$(5.8)\qquad \|c\| < \infty.$$

Then $\mathfrak{G}$ generates an M-semigroup in $C_0(R)$.

Proof. The conditions in Lemma 5.2 are checked for $A=D_mD_s^+$ and $B=\mathfrak{G}-A$. In fact, since A generates an M-semigroup (Property I) and $\mathfrak{G}$ is dispersive, it is enough to establish the estimation in Lemma 5.2. Let $x\notin\Gamma$. We have

$$|Bf(x)| \leqq \|b\|\ \|D_s^+f\|+\|c\|\ \|f\|+\int_{U_x}\varphi_x(y)n_x(dy)\|D_mD_s^+f\|+2n_x(R\backslash U_x)\|f\|$$

and, by using Property II and Lemma 5.1, we get $|Bf(x)|\leqq\alpha'\|Af\|+\beta'\|f\|$ where α' is the left-hand side of (5.3) and β' is some constant. If $x\in\Gamma$, then we have

$$Bf(x) = (m\{x\})^{-1}[(a^+(x)-1)D_s^+f(x)-(a^-(x)-1)D_s^-f(x)]+c(x)f(x)$$
$$+\int_{R\backslash\{x\}}[f(y)-f(x)]n_x(dy),$$

and hence, $|Bf(x)|\leqq\alpha''\|Af\|+\beta''\|f\|$ where α'' is α times the left-hand side of (5.6) and β'' is some constant. Since α' and α'' are both less than 1, the proof is complete.

THEOREM 5.2. *Assume Properties* I, III, *and* IV *for m and s. Let* $\mathfrak{G}$ *be the same as in Theorem* 5.1 *except that condition* (ii) *is dropped and that* $\Gamma=\varnothing$ *and* $\alpha=0$ *in condition* (i). *Then,* $\mathfrak{G}$ *generates an M-semigroup in* $C_0(R)$.

Proof. This is easily reduced to Theorem 5.1.

THEOREM 5.3. *Assume Property* I *for m and s. Let* $\mathfrak{G}$ *be a dispersive operator in* $C_0(R)$ *with domain* $\mathfrak{D}(\mathfrak{G})=\mathfrak{D}_0(D_mD_s^+)$. *Suppose there are a measure* $n_x(dy)$ *on* $R\backslash\{x\}$ *for each* x, *an open interval* U_x *containing* x *for each* x, *a bounded function* $c(x)$, *and a subset* Γ' *of* Γ *such that we have*

$$\mathfrak{G}f(x) = D_mD_s^+f(x)+c(x)f(x)+\int_{R\backslash\{x\}}[f(y)-f(x)-\chi_{U_x}(y)s_x(y)D_s^\pm f(x)]n_x(dy) \tag{5.9}$$

for $f\in\mathfrak{D}(\mathfrak{G})$, *where* $D_s^\pm f(x)$ *stands for* $D_s^+f(x)$ *or* $D_s^-f(x)$ *according as* $x\in R\backslash\Gamma'$ *or* $x\in\Gamma'$, *and such that*

$$\sup_{x\in R\backslash\Gamma'}\int_{U_x}\varphi_x(y)n_x(dy) < 1,\quad \sup_{x\in\Gamma'}\int_{U_x}\psi_x(y)n_x(dy) < 1, \tag{5.10}$$

and

$$\sup_{x\in R} n_x(R\backslash U_x) < \infty. \tag{5.11}$$

Then, $\mathfrak{G}$ *generates an M-semigroup in* $C_0(R)$.

Proof. Letting $A=D_mD_s^+$ and $B=\mathfrak{G}-A$, one can readily check the conditions in Lemma 5.2.

REMARK 5.1. Instead of dispersiveness of $\mathfrak{G}$ it suffices to assume $c\leqq 0$ on R and

$$(m\{x\})^{-1}\mp\int_{U_x}s_x(y)n_x(dy)\geqq 0$$

on Γ. Here $\mp$ stands for $-$ or $+$ according as $x\in\Gamma\backslash\Gamma'$ or $x\in\Gamma'$.

We make some remarks on Properties I–IV.

REMARK 5.2. If we use Feller's classification of the boundary points $-\infty$ and $+\infty$ into four sorts: regular, exit, entrance, and natural ([3], [7], or [16]), Property I is equivalent to that neither $-\infty$ nor $+\infty$ is an entrance boundary. For the proof, see Itô [7, §62].

REMARK 5.3. If the measure m is finite, then m and s have Property II with $\alpha=$ the total measure of m. This is because

$$(5.12)\quad (s(y')-s(y))^{-1}(\varphi_x(y')+\varphi_x(y)) \leqq m(y, y'] \quad \text{for } y \leqq x \leqq y', y \neq y'.$$

REMARK 5.4. The following fact is useful in checking Property II: Let K be compact and let α_0 be the supremum of $m\{x\}$ for $x \in K \cap \Gamma$. Given any $\varepsilon>0$, we can find y_x and y'_x for each $x \in K$ such that $y_x \leqq x \leqq y'_x$,

$$(s(y'_x)-s(y_x))^{-1}(\varphi_x(y_x)+\varphi_x(y'_x)) < \alpha_0+\varepsilon$$

and $\sup_{x\in K}(s(y'_x)-s(y_x))^{-1}<\infty$. In fact, we can choose $y_x=x$ and $y'_x=x+\delta$, δ being independent of x.

The following two remarks are consequences of the preceding remark and (5.12).

REMARK 5.5. If m is finite and continuous, then it has Property III for every s.

REMARK 5.6. Let $s(x)=x$ and assume $\limsup_{x\to-\infty} m(x-\delta, x]=\alpha^-<\infty$ and $\limsup_{x\to+\infty} m[x, x+\delta)=\alpha^+<\infty$ for some $\delta>0$. Then, they have Property II for any $\alpha>\alpha^+ \vee \alpha^- \vee \sup_{x\in\Gamma} m\{x\}$. Such is the case if $s(x)=x$ and $m=\text{const}\times$ Lebesgue measure.

EXAMPLE 5.1. Let $s(x)=x$ and let $n_x(dy)$ be $\text{const}\times|y-x|^{\lambda-1}\,dy$, $\lambda>-1$, for $0<|y-x|<1$ and 0 for $|y-x|\geqq 1$. Assume Properties I and II, and assume, further, that we can choose

$$(5.13)\qquad y_x = x \text{ near } +\infty \quad\text{and}\quad y'_x = x \text{ near } -\infty$$

in Property II. It follows from (5.2) that $D_s^+f(x)$ tends to 0 as $|x|\to\infty$ for $f\in \mathfrak{D}_0(D_mD_s^+)$. Let

$$\mathfrak{G}f(x) = D_mD_s^+f(x)+\int_{0<|y-x|<1}[f(y)-f(x)]n_x(dy)$$

for $f\in\mathfrak{D}_0(D_mD_s^+)$. Then, $\mathfrak{G}$ carries $\mathfrak{D}_0(D_mD_s^+)$ into $C_0(R)$. If moreover,

$$\sup_{x\in R\setminus\Gamma'}\int_{0<|y-x|<1}\varphi_x(y)n_x(dy) < 1 \quad\text{and}\quad \sup_{x\in\Gamma'}\int_{0<|y-x|<1}\psi_x(y)n_x(dy) < 1$$

for some $\Gamma'\subset\Gamma$, then $\mathfrak{G}$ generates an M-semigroup in $C_0(R)$ by virtue of Theorem 5.3.

The measure $n_x(dy)$ above has a mild singularity: it satisfies (3.2) for *all* x. Let us consider a more singular case: $\lambda=-1$. As the next example shows, the condition in Theorem 5.1 or 5.3 that $\mathfrak{G}$ maps $\mathfrak{D}_0(D_mD_s^+)$ into $C_0(R)$ is a fairly strong restriction in case m is not continuous. In case m is continuous, we will give in Example 5.3 a sufficient condition for application of Theorem 5.2.

EXAMPLE 5.2. Suppose $s(x)=x$ and $m=$the Lebesgue measure added to a unit mass at the origin, and let

$$Bf(x) = \int_{0<|y-x|<1} [f(y)-f(x)-(y-x)D_s f(x)]|y-x|^{-2}\,dy$$

for $f\in\mathfrak{D}_0(D_mD_s^+)$ and $x\neq 0$. Then, B cannot carry $\mathfrak{D}_0(D_mD_s^+)$ into $C_0(R)$, no matter how $Bf(0)$ is defined. In fact, if f is such that $D_mD_s^+f=1$ near the origin, then $Bf(0+)=+\infty$.

EXAMPLE 5.3. Assume Properties I, III, and IV for m and s. For every x, let $n_x(dy)=\pi^{-1}|y-x|^{-2}\,dy$ on $R\backslash\{x\}$. Assume that

$$\sup_{x\in R}\int_{0<|y-x|<h}\varphi_x(y)n_x(dy) < 1$$

for some $h>0$ and that $\int_{0<|y-x|<\delta}\varphi_x(y)n_x(dy)$ tends to 0 uniformly on any compact set as $\delta\to 0+$. Let b and c be bounded continuous, $c\leqq 0$, and $\lim_{|x|\to\infty}b(x)=0$, and let $\mathfrak{G}f(x)$ be defined by (1.4) with $a(x)$ and U replaced by 1 and $(x-h, x+h)$, respectively. Then, $\mathfrak{G}$ is an operator in $C_0(R)$ which satisfies the condition in Theorem 5.2, and hence generates an M-semigroup in $C_0(R)$. If we can have (5.13) in Property III, the assumption of the vanishing at infinity for b can be dropped.

6. **The case of circles.** Let S be a circle, let $C(S)$ be the Banach space of continuous functions on S, and define M-semigroups and Lévy measures in the same manner as in $C_0(R)$. Suppose that we are given a finite continuous measure s and a finite measure m on S, both of which are positive for nonvoid open sets. Let Γ be the discontinuity set for m. All the results in the preceding sections are carried over to this set-up. The situation is even simpler, since the state space S is compact and we do not need any consideration relating to the boundary. Thus, any choice of m and s satisfies Property I. Also, Properties II and III are meaningless. For distinct points $x, y\in S$, we denote by (x, y) the open connected set in S with endpoints x and y such that if a point moves from x to y in the set (x, y), it goes counterclockwise. $(x, y]$, $[x, y)$, and $[x, y]$ are defined in like manner. The totality of nonempty open connected sets U in S which are different from S is denoted by $\mathscr{U}$. We say that $x<y$ in U if $(x, y)\subset U$. For each $U\in\mathscr{U}$, we define $\mathfrak{D}(D_mD_s^+; U)$ and $D_mD_s^+f$ on U as we did in §2. $\mathfrak{D}(D_mD_s^+)$ is defined to be the set of f such that $[f]_U$ belongs to $\mathfrak{D}(D_mD_s^+; U)$ for every $U\in\mathscr{U}$. For each $U\in\mathscr{U}$, we define $s_x^U(y)$, $\varphi_x^U(y)$, and $\psi_x^U(y)$ for $x, y\in U$ as follows:

$$\begin{aligned}
s_x^U(y) &= s(x, y),\ 0,\ -s(y, x),\\
\varphi_x^U(y) &= \int_{(x,y)} s(dz)m(x, z],\ 0,\ \int_{(y,x)} s(dz)m(z, x],\\
\psi_x^U(y) &= \int_{(x,y)} s(dz)m[x, z],\ 0,\ \int_{(y,x)} s(dz)m(z, x)
\end{aligned}$$

for $y>x$, $y=x$, and $y<x$, in U, respectively.

We get the following theorems:

THEOREM 6.1. *Let $\{T_t\}$ be an M-semigroup in $C(S)$ generated by $\mathfrak{G}$ such that $\mathfrak{D}(\mathfrak{G})\supset\mathfrak{D}(D_mD_s^+)$. Then,*

(i) *$\{T_t\}$ has Lévy measure $\{n_x(dy); x\in S\}$.*

(ii) *If $f\in C(S)$ and $x_0\notin S(f)$, then $\int_{S\setminus\{x\}} f(y)n_x(dy)$ is continuous at x_0.*

(iii) *For any $U\in\mathscr{U}$ containing x, $\int_{U\setminus\{x\}}\varphi_x^U(y)n_x(dy)$ or $\int_{U\setminus\{x\}}|s_x^U(y)|n_x(dy)$ is finite according as $x\notin\Gamma$ or $x\in\Gamma$. Moreover, we have*

$$\sup_{x\in U}\int_{U\setminus\{x\}}(\varphi_x^U(y)+\psi_x^U(y))n_x(dy) < \infty$$

for any $U\in\mathscr{U}$.

(iv) *Let $U\in\mathscr{U}$, $f\in C(S)$, $S(f)\subset U$, $x_0\in U$, and $f(x_0)=0$. If $x_0\notin\Gamma$, then we have*

$$\lim_{t\to 0+} t^{-1}T_t(f\varphi_{x_0}^U)(x_0) = \int_{U\setminus\{x_0\}} f(y)\varphi_{x_0}^U(y)n_{x_0}(dy).$$

If $x_0\in\Gamma$, then we have

$$\lim_{t\to 0+} t^{-1}T_t(fs_{x_0}^U)(x_0) = \int_{U\setminus\{x_0\}} f(y)s_{x_0}^U(y)n_{x_0}(dy).$$

THEOREM 6.2. *Let $\{T_t\}$ be an M-semigroup in $C(S)$ with infinitesimal generator $\mathfrak{G}$ such that $\mathfrak{D}(D_mD_s^+)\subset\mathfrak{D}(\mathfrak{G})$. Then,*

(i) *For each $U\in\mathscr{U}$ there are unique functions $a(x)$, $b(x)$, and $c(x)$ on $U\setminus\Gamma$ such that, for every $f\in\mathfrak{D}(D_mD_s^+)$ and $x\in U\setminus\Gamma$, we have*

$$\begin{aligned}\mathfrak{G}f(x) &= a(x)D_mD_s^+f(x)+b(x)D_sf(x)+c(x)f(x)\\ &\quad+\int_{S\setminus\{x\}}[f(y)-f(x)-\chi_U(y)s_x^U(y)D_sf(x)]n_x(dy).\end{aligned}\tag{6.1}$$

a and c are independent of U. a is bounded measurable, c is continuous[13]*, and they satisfy* (1.5). *b is measurable and bounded on any compact set in U.*

(ii) *For each $x\in\Gamma$ there are unique $a^+(x)$, $a^-(x)$, and $c(x)$ such that*

$$\begin{aligned}\mathfrak{G}f(x) &= (m\{x\})^{-1}(a^+(x)D_s^+f(x)-a^-(x)D_s^-f(x))+c(x)f(x)\\ &\quad+\int_{S\setminus\{x\}}[f(y)-f(x)]n_x(dy)\end{aligned}\tag{6.2}$$

for all $f\in\mathfrak{D}(D_mD_s^+)$. a^+, a^-, and c are bounded and satisfy (1.7).

THEOREM 6.3. *Let $\alpha=\sup_{x\in S}m\{x\}$. Let $\mathfrak{G}$ be an operator in $C(S)$ with domain $\mathfrak{D}(\mathfrak{G})=\mathfrak{D}(D_mD_s^+)$ such that, for each $x\in S$, there are a measure $n_x(dy)$ on $S\setminus\{x\}$ and a set $U_x\in\mathscr{U}$ containing x satisfying the two conditions below. Then, $\mathfrak{G}$ generates an M-semigroup in $C(S)$.*

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[13] Actually, c, defined on $R\setminus\Gamma$ in (i) and on Γ in (ii), is continuous on R.

(i) *There are functions b and c on* $S\backslash\Gamma$ *such that* (6.1) *holds on* $S\backslash\Gamma$ *for* $f\in\mathfrak{D}(\mathfrak{G})$ *with* $a(x)$ *replaced by* 1 *and U replaced by* U_x, *and that* (5.3), (5.4), *and* (5.5) *hold with R and* φ_x *replaced by S and* $\varphi_x^{U_x}$, *respectively.*

(ii) *There are functions* a^+, a^-, *and c on* Γ *such that* (6.2) *holds on* Γ *for* $f\in\mathfrak{D}(\mathfrak{G})$ *and that* (1.7), (5.6), (5.7), *and* (5.8) *hold with R and* s_x *replaced by S and* $S_x^{U_x}$, *respectively.*

THEOREM 6.4. *Suppose that m is continuous. Let* $\mathfrak{G}$ *be the same as in Theorem* 6.3 *except that the condition* (ii) *is dropped and that* $\Gamma=\varnothing$ *and* $\alpha=0$ *in the condition* (i). *Then,* $\mathfrak{G}$ *generates an M-semigroup in* $C(S)$.

THEOREM 6.5. *Let* $\mathfrak{G}$ *be a dispersive operator in* $C(S)$ *with domain* $\mathfrak{D}(\mathfrak{G})=\mathfrak{D}(D_mD_s^+)$ *satisfying the condition described in Theorem* 5.3 *with* S, $s_x^{U_x}$, $\varphi_x^{U_x}$, *and* $\psi_x^{U_x}$ *replacing* R, s_x, φ_x, *and* ψ_x, *respectively. Then,* $\mathfrak{G}$ *generates an M-semigroup in* $C(S)$.

EXAMPLE 6.1. Let S be a unit circle, $\lambda(dy)$ be the Lebesgue measure (the ordinary length) on S, and $\theta_x(y)$ be the distance of y from x along the circle (hence $0\leqq\theta_x(y)\leqq\pi$). Assume that m is continuous. Let $\mathfrak{G}f$ be defined by (6.1) with $a(x)=1$, $b(x)$ continuous, $c(x)$ continuous and $\leqq 0$, $n_x(dy)=(2\pi)^{-1}(1-\cos\theta_x(y))^{-1}\lambda(dy)$, and $U=U_x$ where U_x is S minus a point y such that $\theta_x(y)=\pi$. Let $U_x^\delta=\{y;\ \theta_x(y)<\delta\}$. If we have

$$\lim_{\delta\to 0+}\sup_{x\in S}\int_{U_x^\delta}\varphi_x^{U_x}(y)n_x(dy) = 0, \tag{6.3}$$

then $\mathfrak{G}$ is defined on $\mathfrak{D}(D_mD_s^+)$ and generates an M-semigroup in $C(S)$. In fact, Theorem 6.4 applies to this case.

EXAMPLE 6.2. Let $\mathfrak{G}$ be such as in the preceding example. Let us impose more restriction on s and m: suppose $s(dy)=\theta_{x_0}(y)^\alpha\lambda(dy)$, $\alpha>-1$, and $m(dy)=\theta_{x_0}(y)^\beta\lambda(dy)$, $\beta>-1$, x_0 being a fixed point of S. Then, $\mathfrak{G}$ is defined on $\mathfrak{D}(D_mD_s^+)$ and generates an M-semigroup in $C(S)$ if and only if $\alpha+\beta>-1$. In fact, we have

$$k_1\theta_x(y)^{-2}\lambda(dy) \leqq n_x(dy) \leqq k_2\theta_x(y)^{-2}\lambda(dy) \tag{6.4}$$

for some positive constants k_1 and k_2, and if $\mathfrak{G}$ is defined on $\mathfrak{D}(D_mD_s^+)$ and generates an M-semigroup, then we have $\alpha+\beta>-1$ by virtue of Theorem 6.1 (iii), noting that $\varphi_x^{U_x}(y)=\text{const}\times\theta_x(y)^{\alpha+\beta+2}$ for $x=x_0$. It can be proved that there is a constant $k_{\alpha,\beta}$ such that $\varphi_x^{U_x}(y)\leqq k_{\alpha,\beta}\theta_x(y)^{(\alpha+\beta+2)\wedge 2}$ for all x and y. Hence, conversely, if $\alpha+\beta>-1$ holds, then (6.3) is proved by (6.4) and $\mathfrak{G}$ is defined and generates an M-semigroup.

EXAMPLE 6.3. Again let $\mathfrak{G}$ be the same as in Example 6.1, m being continuous. Suppose $s(dy)=\lambda(dy)$. If m satisfies

$$\lim_{\delta\to 0+}\int_{U_x^\delta}|\log\theta_x(y)|m(dy) = 0 \tag{6.5}$$

uniformly on S, then $\mathfrak{G}$ is defined on $\mathfrak{D}(D_m D_s^+)$ and generates an M-semigroup. In fact, we have $\varphi_x^{U_x}(y)=\int_{(x,y)} \theta_y(z)m(dz)$ or $\int_{(y,x)} \theta_y(z)m(dz)$ according as $x<y$ or $y<x$ in U respectively, and hence,

$$(6.6) \quad \int_{U_x^\delta} \varphi_x^{U_x}(y)\theta_x(y)^{-2}\lambda(dy) = \int_{U_x^\delta} (\log \delta - \log \theta_x(y) + \delta^{-1}\theta_x(y) - 1)m(dy),$$

which tends to 0 as $\delta \to 0$ uniformly on S. Thus we have (6.3) by using (6.4), and Example 6.1 applies. Conversely, if $\mathfrak{G}$ is defined on $\mathfrak{D}(D_m D_s^+)$ and generates an M-semigroup, then we have (6.5) for each $x \in S$, because $\int_{U_x^\delta} |\log \theta_x(y)| m(dy)$ is finite for fixed δ by Theorem 6.1 (iii) combined with (6.4) and (6.6).

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