COMPACT IMBEDDING THEOREMS FOR QUASIBOUNDED DOMAINS

BY

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1. Introduction. Let $G$ be an unbounded open set in Euclidean $n$-space $E_n$. In this paper we investigate (for a large class of such domains) the problem of determining for which values of $m$, $p$, $j$ and $r$ the Sobolev space imbedding

$$ W^{m,p}_0(G) \rightarrow W^{j,r}_0(G) $$

is or is not compact. Provided $j < m$ continuous imbeddings of this type are known to exist for $p \leq r \leq np(n-mp+jp)^{-1}$ if $n > mp - jp$ or for $p \leq r < \infty$ if $n \leq mp - jp$ (the Sobolev Imbedding Theorem, e.g. [5, Lemma 5]). If $G$ were bounded Kondrashov's compactness theorem [9] would yield the complete continuity of these imbeddings except in the extreme case $r = np(n-mp+jp)^{-1}$. Such compactness theorems are useful for studying existence and spectral theory for partial differential operators on $G$.

In a sequence of recent papers the writer [1]-[4] and C. W. Clark [6], [7], [8] have studied such compactness problems for various unbounded domains. It is clear that the imbedding (1) cannot be compact if $G$ contains infinitely many disjoint congruent balls, for if a fixed $C^\infty$ function has support in one of these balls then the set of its translates with supports in the other balls is bounded in any space $W^{m,p}_0(G)$ but is not precompact in any such space. Thus a necessary condition for the compactness of imbedding (1) is that $G$ should be quasibounded, i.e. that dist $(x, \text{bdry } G) \rightarrow 0$ whenever $|x| \rightarrow \infty$, $x \in G$. In [1] the writer has shown that if $n > 1$ then quasiboundedness is not sufficient for compactness.

The dimension of the boundary of $G$ is a critical factor in determining whether or not (1) is compact. If $G$ is quasibounded and bounded by smooth "reasonably unbroken" $(n-1)$-dimensional manifolds then (1) is compact [3, Theorem 1] for any $m$ and $p$ and for the same values of $j$ and $r$ as in the case of bounded $G$. However if $G$ has discrete (0-dimensional) boundary then [2, Theorem 1] no such imbedding can be compact unless $mp > n$.

Our purpose in this paper is to study the compactness of imbedding (1) for quasibounded domains $G$ whose boundaries are comprised of smooth manifolds...
of various dimensions. Roughly speaking our results are as follows. If $k$ is the smallest integer for which those boundary manifolds of $G$ having dimension not less than $n-k$ bound a quasibounded domain then no imbedding of type (1) can be compact when $mp<k$. On the other hand, if, in addition, the boundary manifolds are "reasonably unbroken" and if $mp>n+p-np/k$ then (1) is compact for the same values of $j$ and $r$ as in the case of bounded $G$. Our results thus interpolate between the extreme cases mentioned above. We consider first domains $G$ with flat (planar) boundaries, establishing in §2 a necessary condition for the compactness of (1) for such $G$, and in §3 a slightly stronger sufficient condition. If $m=1$ these conditions are equivalent for certain domains. In §4 similar results are obtained for nonflatly-bounded domains $G$.

As usual, in this paper $W_0^{m,p}(G)$ denotes, for $p \geq 1$ and $m=0, 1, 2, \ldots$, the Sobolev space obtained by completing with respect to the norm

$$
\|u\|_{m,p,G} = \left\{ \sum_{j=0}^{\infty} |u|_{j,p,G}^p \right\}^{1/p}
$$

the space $C_0^\infty(G)$ of all infinitely differentiable, complex functions having compact support in $G$ where

$$
|u|_{j,p,G} = \sum_{|\alpha|=j} \int_G |D^\alpha u(x)|^p \, dx.
$$

$\alpha$ denotes an $n$-tuple of nonnegative integers $(\alpha_1, \ldots, \alpha_n)$; $|\alpha|=\alpha_1+\cdots+\alpha_n$; $D^\alpha=D_1^{\alpha_1} \cdots D_n^{\alpha_n}$; $D_j=\partial/\partial x_j$. Note that $|u|_{0,p,G} = \|u\|_{0,p,G}$ is the norm of $u$ in $L^p(G)$. $W^{m,p}(G)$ represents the completion with respect to the norm (2) of the space of all infinitely differentiable functions on $G$ for which (2) is finite. Provided the boundary of $G$ satisfies certain mild regularity conditions [5, Lemma 5] the Sobolev Imbedding Theorem referred to above, and also (provided $G$ is bounded) the Kondrašov Compactness Theorem, remain valid for imbeddings of $W^{m,p}(G)$. No compactness theorems of this sort are yet known if $G$ is unbounded.

2. Flatly-bounded domains—noncompact imbeddings. Let $H$ be a $k$-dimensional plane ($0 \leq k \leq n-1$) in $E_n$ and let $a$ be a point on $H$. With respect to a new system of rectangular coordinates $z$ in $E_n$ having origin at $a$ and obtained from the usual coordinates by an affine transformation, $H$ has equations $z_1=z_2=\cdots=z_{n-k}=0$, or more simply $r=0$ where $r=\sum_{j=1}^{n-k} z_j^2$. The coordinate $r$, together with $n-k-1$ angle coordinates collectively denoted $\sigma$ and the coordinates $z'=(z_{n-k+1}, \ldots, z_n)$ form a system of cylindrical polar coordinates in $E_n$ with origin at $a$ and cylindrical axis $H$.

The $k$-tube $T_\delta(H)$ of radius $\delta$ and axis $H$ is the set $\{x \in E_n : \text{dist} (x, H)=r<\delta\}$. By a tube function for the tube $T_\delta(H)$ we mean a $C^\alpha$ function $\theta: E_n \to [0, 1]$ whose value at $x$ depends only on $r=\text{dist} (x, H)$ and which vanishes identically near $H$ and is identically unity outside $T_\delta(H)$. 
Lemma 1. Let \( H \) be a \( k \)-plane in \( E_n \) and \( a \in H \). Let \( 1 \leq p < \infty \). If \( u(x) = v(r) \) where \( r = \text{dist} \ (x, \ H) \) and \( v \in C^{|a|}(0, \infty) \) then for all \( x \notin H \)
\[
|D^a u(x)|^p \leq \text{const} \sum_{j=1}^{|a|} |\partial v(r)|^{|a|} r^{j-|a|-p}
\]
where the constant depends only on \( a \), \( p \) and \( k \).

Proof. Since \( z_i = \sum_{j=1}^n c_i(x_j - a_j) \) and so \( \partial z_i z_j = \sum_{j=1}^n c_i \partial_i \partial z_j \) we may assume with no loss of generality that \( H \) is a coordinate plane and \( z = x \). We show that there exist homogeneous polynomials \( P_{a,i}(x) \) of degree \( |a| \) (possibly the zero polynomial) such that for \( r > 0 \)
\[
D^a u(x) = \sum_{j=1}^{|a|} P_{a,j}(x) v^{(j)}(r) r^{j-2|a|}.
\]
Since \( |P_{a,i}(x)| \leq \text{const} r^{|a|} \) the conclusion of the lemma for \( p = 1 \) follows at once. The result for general \( p \) then follows from the well-known inequality
\[
\left| \sum_{j=1}^N A_j \right|^p \leq \text{const} \sum_{j=1}^N |A_j|^p
\]
where the constant depends only on \( p \) and \( N \).

Note that \( D^a u(x) = 0 \) unless \( a_{n-k+1} = \cdots = a_n = 0 \). If \( 1 \leq i \leq n-k \) then \( D_i u(x) = v'(r)x_i/r \) which is of the required form. Assume (3) holds for all \( a \) with \( |a| \leq m \). If \( |\beta| = m+1 \) then \( D^\beta = D_i D^a \) for some \( i, a \) where \( |a| = m \). Applying the induction hypothesis and the chain rule we obtain
\[
D^\beta u(x) = \sum_{j=1}^{m+1} \left\{ D_i P_{a,\beta}(x) v^{(j)}(r) r^{j-2} + P_{a,\beta}(x) v^{(j+1)}(r) x_i r^{j-2m-1} - (2m-j) P_{a,\beta}(x) v^{(j)}(r) x_i r^{j-2m-2} \right\}
\]
where \( P_{a,\beta} \) is given by
\[
P_{a,1}(x) = r^2 D_i P_{a,1}(x) - (2m-1) x_i P_{a,1}(x),
\]
\[
P_{a,j}(x) = r^2 D_i P_{a,j}(x) - (2m-j) x_i P_{a,j}(x) + x_i P_{a,j-1}(x) \quad \text{if} \ 2 \leq j \leq m,
\]
\[
P_{a,m+1}(x) = x_i P_{a,m}(x).
\]
Clearly \( P_{a,\beta}(x) \) is a polynomial of the desired type and the proof is complete.

Lemma 2. Let \( \lambda \) be a positive integer and let \( r = s^\lambda \), \( s > 0 \). If \( f \in C^\lambda((0, \infty)) \) and \( 1 \leq p < \infty \) then
\[
|d/ds f(r^{1/j})|^p \leq \text{const} \sum_{i=1}^\lambda \lambda^{-i p \gamma} \lambda^{-j \lambda} |f^{(\lambda)}(s)|^p
\]
where the constant depends only on \( j \) and \( p \).
Proof. Again the case of general \( p \) follows from the special case \( p = 1 \) via (4). For \( p = 1 \) (5) is an immediate consequence of the formula

\[
(d/dr)^i = \lambda^{-j} \sum_{i=1}^{j} P_{j-i,i}(\lambda)s^{i-\lambda}(d/ds)^i
\]

where \( P_{i,j} \) is a polynomial of degree \( i \) depending on \( j \). We prove (6) by induction on \( j \). Note that \( d/dr = \lambda^{-1}s^{1-\lambda}d/ds \) which is of the required form. Assuming (6) we have

\[
(d/dr)^{i+1} = \lambda^{-j} \sum_{i=1}^{j} P_{j-i,i}(\lambda)s^{i-\lambda}(d/ds)^i + (i-\lambda)s^\lambda(d/ds)^i
\]

where the polynomials \( P_{i,j+1} \) are given by

\[
P_{0,i+1}(\lambda) = P_0, i(\lambda),
\]

\[
P_{i,i+1}(\lambda) = P_{i,i}(\lambda) + (j+1-i-\lambda)P_{i-1,i}(\lambda) \quad \text{for } 1 \leq i \leq j-1,
\]

\[
P_{j,i+1}(\lambda) = (1-\lambda)P_{j-1,i}(\lambda),
\]

which are of the desired form.

Lemma 3. Let \( T \) be a \( k \)-tube in \( E_n \) with axis \( H \) and radius \( \delta \leq 1 \). Let \( 1 \leq p < \infty \) and let \( \lambda \) be a positive integer. Then there exists a tube function \( \theta \) for \( T \) satisfying for \( |\alpha| > 0, \)

\[
|D^\alpha \theta(x)|^p \leq \text{const} \lambda^{-p s^\lambda-\lambda p |\alpha|}
\]

where \( s^\lambda = r = \text{dist} (x, H) \) and the constant depends only on \( \alpha, n, p \) and \( k \) and not on \( \lambda \).

Proof. Let \( f: [0, \infty) \to [0, 1] \) be a fixed \( C^\infty \) function such that \( f(s) = 0 \) near \( s = 0 \) and \( f(s) = 1 \) for \( s^\lambda \geq \delta \). Define \( \theta \) by \( \theta(x) = r = f(s) \). Clearly \( \theta \) is a tube function for \( T \). By Lemmas 1 and 2 we have

\[
|D^\alpha \theta(x)|^p \leq \text{const} \sum_{j=1}^{[\alpha]} r^{|p-j| \alpha} |f(j)(r)|^p
\]

\[
\leq \text{const} \sum_{j=1}^{[\alpha]} \sum_{i=1}^{j} \lambda^{-1+p p^\lambda-\lambda p |\alpha|} |f(0)(s)|^p
\]

\[
\leq \text{const} \lambda^{-p s^\lambda-\lambda p |\alpha|}.
\]

The final inequality follows because whenever \( D^\alpha \theta(x) \neq 0, s < \delta^{1/\lambda} \leq 1, \) and also \( |f(0)(s)| \leq \text{const} \) for \( 1 \leq i \leq |\alpha| \).

In the following lemma we consider several \( (n-k) \)-tubes in \( E_n \) simultaneously. Hence all the related quantities \( \theta, r, \sigma, z', s, H \) carry subscripts ranging from 1 to \( N \).
Lemma 4. Let $S$ be a bounded open set in $E_n$. Let $H_1, \ldots, H_N$ be a finite collection of $(n-k)$-planes which intersect $S$. Let $m$ be a positive integer and let $\epsilon > 0$. If either $p > 1$ and $mp \leq k$ or $p = 1$ and $m < k$ then there exists a function $\psi \in C^\infty(E_n)$ with the properties:

(i) $\psi(x) = 0$ for $x$ near $\bigcup_{i=1}^N H_i$, 
(ii) $0 \leq \psi(x) \leq 1$ for all $x$, 
(iii) $\psi(x) = 1$ for $x$ in $E_n - \bigcup_{i=1}^N T_\delta(H_i)$, $\delta > 0$, 
(iv) $\|D^\alpha \psi\|_{0,p,S} \leq \epsilon$ for $0 < |\alpha| \leq m$.

Proof. We first consider the case that no two of the planes $H_i$ intersect in $S$. It is then possible to choose $\delta \leq 1$ small enough so that if $T_i = T_\delta(H_i)$ then $T_i \cap T_j \cap S$ is empty if $i \neq j$. By Lemma 3 there exist tube functions $\theta_i$ for $T_i$ satisfying

$$|D^\alpha \theta_i(x)|^p \leq \text{const} \lambda^{-p} s_i^{p - \lambda p|\alpha|}$$

where $s_i = \text{dist} (x, H_i)$ and the constant is independent of $\lambda$ and $\alpha$ for $0 < |\alpha| \leq m$. Let $\psi(x) = \theta_1(x) \theta_2(x) \cdots \theta_N(x)$. Clearly $\psi$ satisfies (i)-(iii). Note that $D^\alpha \psi = 0$ outside $\bigcup_{i=1}^N T_i$ and that $D^\alpha \psi(x) = D^\alpha \theta_i(x)$ in $T_i$. We have

$$\|D^\alpha \psi\|_{0,p,S} = \sum_{i=1}^N \int_{T_i} |D^\alpha \theta_i(x)|^p \, dx$$

$$\leq \text{const} \lambda^{-p} \sum_{i=1}^N \int_{T_i} s_i^{p - \lambda p|\alpha| + \lambda k} \, dr \, d\alpha_i \, ds_i$$

$$\leq \text{const} \lambda^{1-p} \int_0^1 s_i^{p - \lambda p|\alpha| + \lambda k} \, ds_i.$$ 

The final constant depends on $\alpha, p, n, k, N$ and $\text{diam} S$ but not on $\lambda$. If $|\alpha| \leq m$ and $mp \leq k$ then $p - \lambda p|\alpha| + \lambda k > 0$ and so

$$\|D^\alpha \psi\|_{0,p,S} \leq \text{const} \lambda^{1-p}(p + \lambda k - \lambda p|\alpha|)^{-1}.$$ 

The expression on the right can be made arbitrarily small for sufficiently large $\lambda$ provided either $p > 1$ or $m < k$. This establishes (iv).

The case of intersecting $H_i$ remains to be considered. Again pick $\delta < 1$ small enough so that $T_i \cap T_j \cap S$ is empty whenever $H_i \cap H_j \cap S$ is empty. Define $\theta_i$ and $\psi$ as above. The general Leibniz formula states

$$D^\alpha \psi(x) = \sum_{\beta_1 + \cdots + \beta_N = \alpha} \left( \beta_1, \ldots, \beta_N \right) D_{\beta_1} \theta_1(x) \cdots D_{\beta_N} \theta_N(x).$$

For estimates of $|D^\alpha \psi(x)|$ we may drop from terms in the Leibniz expression any factor $D^\beta \theta_i(x)$ for which $\beta_i = 0$ because $|\theta_i(x)| \leq 1$. For simplicity consider a term $D^\beta_1 \theta_1(x) \cdots D^\beta_N \theta_N(x)$ where no $\beta_i$ is zero. Decompose $S$ into the union of $N$ subregions $S_i$ such that in $S_i$ we have $s_i \leq s_i$ for $i \neq j$. We now obtain via Lemma 3 in the manner above, noting that whenever $D^\beta_1 \theta_1(x) \neq 0$ then $s_i < \delta \leq 1$. 


\[
\int_S |D^\beta \theta_1(x) \cdots D^\beta \theta_n(x)|^p \, dx = \sum_{j=1}^N \int_{S_j} |D^\beta \theta_1(x) \cdots D^\beta \theta_n(x)|^p \, dx
\]
\[
\leq \text{const } \lambda^{-Np} \sum_{j=1}^N \int_{S_j} s_j^{p-\lambda p |\beta_1|} \cdots s_j^{p-\lambda p |\beta_n|} \, dx
\]
\[
\leq \text{const } \lambda^{-Np} \sum_{j=1}^N \int_{S_j} s_j^{p-\lambda p |\alpha|} \, dx
\]
\[
\leq \text{const } \lambda^{1-Np} \int_0^1 s^{Np-\lambda p |\alpha| + \lambda k - \lambda} \, ds
\]
\[
\leq \text{const } \lambda^{1-Np}(Np + \lambda k - \lambda p |\alpha|)^{-1}.
\]

Similar estimates can be found for all terms in (7) and (iv) follows once more by taking \(\lambda\) sufficiently large.

**Definition.** Let \(G\) be an open set in \(E_n\). \(G\) is called a regular domain if \(\text{bdry } G = \bigcup_{k=0}^n G_k\) where \(G_k\) is the union of a locally finite collection of smooth manifolds of dimension \(k\) in \(E_n\). A regular domain \(G\) whose boundary manifolds are all segments of planes of various dimensions will be called a regular flatly-bounded domain. An unbounded regular domain \(G\) is called 0-quasibounded if it is quasi-bounded, i.e. if there exist at most finitely many disjoint congruent balls \(Q\) in \(G\), having any specified positive radius, which do not intersect the boundary of \(G\). \(G\) is called \(k\)-quasibounded \((1 \leq k \leq n-1)\) if there exist at most finitely many disjoint congruent balls \(Q\) in \(G\), having any specified positive radius, such that \(Q \cap \text{bdry } G \subset \bigcup_{i=0}^{n-1} G_i\); i.e. if dist \((x, \bigcup_{i=0}^{n-1} G_i) \to 0\) as \(|x| \to \infty\), \(x \in G\). For \(1 \leq k \leq n-1\) the condition of \(k\)-quasiboundedness is stronger than that of \((k-1)\)-quasiboundedness.

**Theorem 1.** Let \(G\) be a regular, quasibounded, flatly-bounded domain in \(E_n\). Let \(k\) be the smallest integer \((1 \leq k \leq n)\) for which \(G\) is \((n-k)\)-quasibounded. If either \(mp \leq k\) and \(p > 1\) or \(m < k\) and \(p = 1\) then no imbedding of the form
\[
W^{m,p}_C(G) \to W^{k,p}_C(G)
\]
can be compact.

**Proof.** For \(2 \leq k \leq n\) since \(G\) is not \((n-k+1)\)-quasibounded there is a sequence of congruent open balls \(\{Q_i\}_{i=1}^\infty\) in \(G\) such that \(Q_i \cap \text{bdry } G\) is contained in the union of finitely many \((n-k)\)-planes. If \(k=1\) balls \(Q_i\) with this property exist trivially since \(G\) is regular and flatly-bounded. Let \(Q\) denote any one of these balls and let \(H_1, \ldots, H_N\) be the corresponding \((n-k)\)-planes. Let \(\varphi \in C_0^\infty(Q)\) be a function for which
\[
\|\varphi\|_{0,r,Q} = 2C > 0, \quad \|\varphi\|_{m,p,Q} = K < \infty.
\]
There exists a constant \(M\) such that for all \(x \in E_n\) and for all \(\alpha\) with \(0 \leq |\alpha| \leq m\), \(|D^\alpha \varphi(x)| \leq M\). Choose \(\delta_0 > 0\) small enough so that the sum of the volumes of the intersections with \(Q\) of the \((n-k)\)-tubes \(T_{\delta_0}(H_i), i=1, \ldots, N\), does not exceed \((C/M)^\gamma\). Let
\[
e = K \left[ M \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \left( \frac{\alpha}{\beta} \right) \right]^{-1}.
\]
By Lemma 4 there exists for some $\delta \leq \delta_0$ a function $\psi \in C_0^\infty(E_n - \bigcup_{i=1}^n H_i)$ satisfying $0 \leq \psi(x) \leq 1$, $\psi(x) = 1$ outside $\bigcup_{i=1}^n T_\delta(H_i)$ and $\|D^\alpha \psi\|_{0, p, q} \leq \varepsilon$ for $0 < |\alpha| \leq m$. Let $\gamma = \varphi \cdot \psi = \varphi - \varphi(1 - \psi)$. Clearly $\gamma \in C_0^\infty(Q \cap G)$. Putting $T_i = T_\delta(H_i)$ we have

$$\|\gamma\|_{0, r, G} \geq \|\varphi\|_{0, r, q} - \|\varphi\|_{0, r, q \cap (T_1 \cup \cdots \cup T_n)} \geq 2C - M[\text{vol } Q \cap (T_1 \cup \cdots \cup T_n)]^{\frac{1}{r'}} \geq C.$$ 

Moreover, for $0 < |\alpha| \leq m$

$$D^\alpha \gamma = \psi D^\alpha \varphi + \sum_{\beta < \alpha} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha - \beta} \psi,$$

$$\|D^\alpha \gamma\|_{0, p, q} \leq \|D^\alpha \varphi\|_{0, p, q} + \sum_{\beta < \alpha} \binom{\alpha}{\beta} M \varepsilon \leq 2K.$$

Thus if $K_\alpha = \sum_{|\alpha| \leq m} 1$ we have $\|\gamma\|_{m, p, q} \leq 2KK_\alpha$.

Now let $\varphi_i, i = 1, 2, \ldots$ be a translate of $\varphi$ with support in $Q_i$ and let $\gamma_i$ be constructed from $\varphi_i$ as $\gamma$ from $\varphi$ above, so that

$$\|\gamma_i\|_{0, r, G} \geq C, \quad \|\gamma_i\|_{m, p, q} \leq 2KK_\alpha.$$

The sequence $(\gamma_i)$, though bounded in $W_0^{m, p}(G)$ has no subsequence converging in $L'(G)$. In fact if $i \neq j$ then $\|\gamma_i - \gamma_j\|_{0, r, G} \geq 2^{1/r' \kappa}$. Thus the imbedding

$$W_0^{m, p}(G) \rightarrow W_0^{\mu, p}(G) = L'(G)$$

if it exists cannot be compact. Neither can the imbedding $W_0^{m, p}(G) \rightarrow W_0^{\mu, p}(G)$ for if this latter imbedding were compact then so would be the composition

$$W_0^{m, p}(G) \rightarrow W_0^{\mu, p}(G) \rightarrow L'(G).$$

3. Flatly-bounded domains—compact imbeddings. Let $H$ be a $k$-plane in $E_n$ and let $a \in H$. We denote by $T_{\delta, \rho}(H, a)$ the tube segment of radius $\delta$ and length $2\rho$ having axis $H$ and centre $a$. Thus, if $P$ is the orthogonal projection operator on $H$ then

$$T_{\delta, \rho}(H, a) = \{x \in E_n : \text{dist } (x, H) < \delta, \text{dist } (Px, a) < \rho\}.$$

**Lemma 5.** Let $H$ be an $(n-k)$-plane in $E_n$ ($1 \leq k \leq n$) and let $a \in H$. If either $p > k$ or $p = k = 1$ then there exists a constant depending only on $p$ and $k$ such that for each $\delta, \rho > 0$

$$\|\gamma\|_{0, p, \tau} \leq \text{const } \delta |\gamma|_{1, p, \tau}$$

for all $\gamma \in C_0^\infty(E_n - H \cap T)$ where $T = T_{\delta, \rho}(H, a)$.

**Proof.** First consider the case $p > k$. Let $(r, \sigma, z')$ denote cylindrical polar coordinates in $E_n$ with origin at $a$ and cylindrical axis $H$. By H"{o}lder's inequality for $(r, \sigma, z') \in T$
Integrating over $\Sigma$ and $Z$ the domains of the variables $\sigma$ and $z'$ respectively in $T$ we obtain

$$\|y\|_{\delta,p,T} = \int_{\Sigma} d\sigma \int_{Z} dz' \int_{0}^{\delta} |y(\sigma, z')|^{p} \, d\tau,$$

$$\leq \left( \frac{p - 1}{p - k} \right)^{p - 1} \delta^{p - 1} \int_{0}^{\delta} d\tau \int_{\Sigma} d\sigma \int_{Z} dz' \int_{0}^{\delta} |D_{\gamma}(t, \sigma, z')|^{p} t^{k - 1} \, dt.$$

For the special case $p = k = 1$ we have

$$|y(x)| = |y(r, z')| \leq \delta|y|_{1,1,T}.$$

**Corollary.** Under the conditions of the lemma, if $1 \leq q \leq p$ then there exists a constant depending only on $p$, $q$, $k$, $n$ such that for all $\delta > 0$

$$\|y\|_{0,q,T} \leq \delta^{1 + n/q - n/p} |y|_{1,1,T}.$$

for all $y \in C^{q}_{0}(E_{n} - H \cap T)$ where $T = T_{\delta,0}(H, a)$.

**Proof.** By Hölder's inequality and since $\text{vol } T = \text{const } \delta^{n}$

$$\|y\|_{0,q,T} \leq \|y\|_{0,p,T} [\text{vol } T]^{1/q - 1/p} \leq \text{const } \delta^{1 + n/q - n/p} |y|_{1,1,T}.$$

**Definition.** Let $G$ be an unbounded, regular, flatly-bounded domain in $E_{n}$. We shall say that $G$ has the $k$-tube property if for every sufficiently large positive number $R$ there exists a positive number $\delta = \delta(R)$ with the properties:

(i) $\delta(R) \to 0$ as $R \to \infty$,

(ii) for each $x \in G_{R} = \{ y \in G : |y| > R \}$ there exists a $k$-plane $H$ and a point $a \in H$ such that $x \in T_{\delta,0}(H, a)$ and $H \cap T_{2\delta,2\delta}(H, a) \subset \text{bdry } G$.

It is clear that if $G$ has the $k$-tube property then $G$ is $k$-quasibounded. Of course the converse is not true as the planar segments comprising the boundary of $G$ may have too many gaps to satisfy condition (ii). For domains $G$ whose boundaries consist only of whole planes the $k$-tube property is equivalent to $k$-quasibounded-
ness. Other examples of domains with the \( k \)-tube property are not difficult to construct—for example Clark’s “spiny urchin” [7] has the 1-tube property in \( E_2 \).

**Lemma 6** (A variant on Poincaré’s Inequality). Let \( G \) be an unbounded, regular, flatly-bounded domain in \( E_n \) having the \((n-k)\)-tube property for some \( k \) (\( 1 \leq k \leq n \)). If \( 1 \leq r \leq p \) where either \( p > k \) or \( p = k = 1 \) then there exists a constant depending only on \( n, k, p \) and \( r \) such that for all \( u \in W^1_{0,p}(G) \) and all sufficiently large \( R \)

\[
\|u\|_{0,r,G_\delta} \leq \text{const } [\delta(R)]^{1 + \frac{n}{r} - \frac{n}{p}} |u|_{1,p,G}.
\]

**Proof.** Fix \( R \) large enough so that \( \delta = \delta(R) \) exists. If \( \alpha \) is an \( n \)-tuple of integers (not necessarily nonnegative) let \( Q_\alpha = \{x \in E_n : \alpha x^2 = (\alpha_1 + 1)n^{-1/2}\delta \}. \) Then \( E_\delta = \bigcup \alpha Q_\alpha. \) If \( x \in G_\delta \) then \( x \in Q_\alpha \) for some \( \alpha \) and there exists an \((n-k)\)-plane \( H \) and a point \( a \in H \) such that \( x \in T_{a,H}(a) = T \) and \( H \cap T' = \text{bdry} G \) where \( T' = T_{2a,2a}(H, a) \). Clearly \( Q_\alpha \cap T' \). For any \( \gamma \in C_0(G) \) since \( \gamma \) vanishes near \( H \cap T' \) we have by the corollary of Lemma 5

\[
\|\gamma\|_{0,r,G_\delta} \leq \|\gamma\|_{0,r,T'} \leq \text{const } (2\delta)^{1 + \frac{n}{r} - \frac{n}{p}} |\gamma|_{1,p,T'} \leq \text{const } \delta^{1 + \frac{n}{r} - \frac{n}{p}} |\gamma|_{1,p,G_\delta}
\]

where \( Q'_\alpha \) is the union of all the cubes \( Q_\delta \) which intersect \( T' \). There is a number \( N \) depending only on \( n \) such that any \( N + 1 \) of the sets \( Q'_\alpha \) have empty intersection. Summing the above inequality over all \( \alpha \) for which \( Q_\alpha \) meets \( G_\delta \) we obtain

\[
\|\gamma\|_{0,r,G_\delta} \leq \text{const } N \cdot \delta^{1 + \frac{n}{r} - \frac{n}{p}} |\gamma|_{1,p,G}
\]

and this inequality extends by completion from \( C_0(G) \) to \( W^1_{0,p}(G) \).

**Theorem 2.** Let \( G \) be an unbounded, regular, flatly-bounded domain in \( E_n \) having the \((n-k)\)-tube property (\( 1 \leq k \leq n \)). If either \( p > k \) or \( p = k = 1 \) then the imbedding \( W^1_{0,p}(G) \rightarrow W^1_{0,p}(G) \) (exists and) is compact for \( m = 0, 1, 2, \ldots \) and \( 1 \leq r < \infty \) if \( p \geq n \) or for \( 1 \leq r < np(n-p)^{-1} \) if \( p < n \).

**Proof.** First consider the case \( 1 \leq r \leq p, \, m = 0 \). To prove that the imbedding \( W^1_{0,p}(G) \rightarrow L'(G) \) is compact we use the following compactness criterion for sets in \( L'(G) \): a sequence \( \{u_i\}_{i=1}^\infty \) which is bounded in \( L'(G) \) is precompact in \( L'(G) \) provided

(a) for every bounded \( G' \subset G \) the sequence \( \{u_i|G'\} \) is precompact in \( L'(G') \), and

(b) for each \( \varepsilon > 0 \) there exists \( R > 0 \) such that for all \( i, \|u_i\|_{0,r,G_\delta} < \varepsilon \).

Lemma 6 and condition (i) of the \((n-k)\)-tube property assures us that (b) is satisfied for any sequence \( \{u_i\} \) bounded in \( W^1_{0,p}(G) \). To establish (a) let \( G' \) be a bounded subset of \( G \). Then for some \( R, \, G' \subset K_R = \{x \in E_n : |x| \leq R\} \). Let \( W^1_{1,p}(G, R) \) denote the completion with respect to the norm \( \| \cdot \|_{1,p,G_\delta,K_R} \) of the space \( C_0(G) \). The imbedding \( W^1_{1,p}(K_R) \rightarrow L'(K_R) \) is known to be compact (Kondrašov’s Theorem) and since an element of \( W^1_{1,p}(G, R) \) can be extended to be zero outside its support so as to belong to \( W^1_{1,p}(K_R) \) it follows that \( W^1_{1,p}(G, R) \) is compactly imbedded in
L'(G \cap K_R). But \{u_k|_{K_R}\} is bounded in \(W^{1,p}(G, R)\) and hence precompact in \(L'(G \cap K_R)\) whence \(\{u_k|_{G'}\}\) is precompact in \(L'(G')\) as required.

By Sobolev's Imbedding Theorem \(W^{1,p}_0(G)\) is continuously imbedded in \(L^q(G)\) for any \(q\) satisfying \(p \leq q < \infty\) if \(p \geq n\) or \(p \leq q \leq np(n-p)^{-1}\) if \(p < n\). Select such a \(q\) and a sequence \(\{u_k\}\) bounded in \(W^{1,p}_0(G)\) so that, say, \(\|u_k\|_{0,q,G} \leq C\). We may assume, passing to a subsequence if necessary, that \(\{u_k\}\) converges in \(L^q(G)\). By Hölder's Inequality if \(p \leq r < q\)

\[
\|u_i - u_j\|_{r,G} \leq \|u_i - u_j\|_{0,p,G} \leq A \leq (2C)^{1-\lambda} \|u_i - u_j\|_{0,p,G}
\]

where \(\lambda = p(q-r)r^{-1}(q-p)^{-1} > 0\). Hence \(\{u_k\}\) converges in \(L'(G)\) and so the imbedding \(W^{1,p}_0(G) \to L'(G)\) is compact for \(1 \leq r < \infty\) if \(p \geq n\) and for \(1 \leq r < np(n-p)^{-1}\) if \(p < n\).

Finally, if \(\{u_k\}\) is bounded in \(W^{m+1,p}_0(G)\) then for any \(\alpha\) with \(0 \leq |\alpha| \leq m\), \(\{D^\alpha u_k\}\) is bounded in \(W^{1,p}_0(G)\) and so has a subsequence convergent to an element \(v_\alpha\) of \(L'(G)\). In particular (for a suitable subsequence) \(u_i \to v_0\) in \(L'(G)\) and so in the sense of distributions. Since \(D^\alpha u_i \to v_\alpha\) in \(L'(G)\) and \(D^\alpha v_i \to D^\alpha v_0\) in the sense of distributions it follows that \(v_\alpha = D^\alpha v_0\) and \(u_i \to v_0\) in \(W^{m,p}_0(G)\). This completes the proof.

This theorem affords for imbeddings of the sort \(W^{1,p}_0(G) \to L'(G)\) on domains \(G\) for which \((n-k)\)-quasiboundedness is equivalent to the \((n-k)\)-tube property, a complete converse to Theorem 1. For imbeddings \(W^{m,p}_0(G) \to L'(G)\), \(m \geq 2\), we do not fare quite so well.

**Theorem 3.** Let \(G\) be an unbounded, regular, flatly-bounded domain in \(E_n\) having the \((n-k)\)-tube property \((1 \leq k \leq n)\). Then the imbedding

\[
W^{m,p}_0(G) \to W^{1,q}_0(G), \quad 0 \leq j < m,
\]

is compact in any of the following cases:

(i) \(m=p=k=1\),

(ii) \(mp > n+p-np/k\), \(p \leq r < p^*\),

(iii) \(mp > n+(j+1)p-np/k\), \(1 \leq r < p^*\),

where \(p^* = np(n-mp+jp)^{-1}\) if \(n > mp-jp\) and \(p^* = \infty\) if \(n \leq mp-jp\).

**Proof.** The case \(m=1\) has already been proved. If \(m \geq 2\) the imbedding \(W^{m,p}_0(G) \to W^{1,q}_0(G)\) is continuous for \(p \leq q \leq np(n-mp+p)^{-1}\) if \(n > mp-p\) and \(p \leq r < \infty\) if \(n \leq mp-p\). By Theorem 2 the imbedding \(W^{1,q}_0(G) \to L'(G)\) is compact provided \(q > k\). Since \(mp > n+p-np/k\) is equivalent to \(np(n-mp+p)^{-1}\) if \(q > k\) can always be chosen and so the composed imbedding \(W^{m,p}_0(G) \to L'(G)\) is compact.

By a standard interpolation theorem for Sobolev spaces [5, Lemma 6] there exists a constant \(K\) such that for \(0 \leq j < m\), \(p \leq q < p^*\) we have

\[
\|u\|_{j,r,G} \leq K \|u\|_{m,p,G}^{\lambda} \|u\|_{0,p,G}^{1-\lambda}
\]
for all \( u \in W_0^{n,p}(G) \) where \( \lambda = (nr+jp-np)(mrp)^{-1} \). Note that \( 0 \leq \lambda < 1 \) for all relevant values of \( j, m, n, r \) and \( p \). If \( \{u_i\}_{i=1}^\infty \) is a bounded sequence in \( W_0^{n,p}(G) \) then it has a subsequence again denoted \( \{u_i\} \) which is convergent in \( L^p(G) \). Since

\[
\|u_i - u_k\|_{L^p(G)} \leq K \|u_i - u_k\|_{W_0^{n,p}(G)} \|u_i - u_k\|_{L^p(G)}^{1-\lambda} \\
\leq K(2 \sup_{u_i \in W_0^{n,p}(G)} \|u_i - u_k\|_{L^p(G)} \|u_i - u_k\|_{W_0^{n,p}(G)}),
\]

it follows that \( \{u_i\} \) is a Cauchy sequence and hence convergent in \( W_0^{l(G)} \) proving case (ii).

If \( mp > n + (j+1)p - np/k \) (and in particular if \( j=0 \) in case (ii)) we have by Sobolev's theorem and Theorem 2

\[
W_0^{n,p}(G) \rightarrow W_0^{l,1}, np(n + mp + jp + p)^{-1}(G) \rightarrow W_0^{l',r}(G)
\]

the second imbedding being compact since \( np(n + mp + jp + p)^{-1} > k \).

**Remark.** The condition \( mp > n + p - np/k \) implies \( mp > k \). The converse is, however, true for \( m \geq 2 \) only if \( k = n \) or \( k = p \). Thus even for domains \( G \) for which \( (n-k) \)-quasiboundedness is equivalent to the \( (n-k) \)-tube property imbeddings of \( W_0^{n,p}(G) \) corresponding to the cases \( 1 < k < m \leq n+p-np/k \) and \( 1=n-k \leq m \leq n + 1 - np/k \) fail to be covered either by Theorem 1 or Theorem 3.

4. Extensions to nonflatly-bounded domains. Let \( G, G' \) be open sets in \( \mathbb{R}^n \). A one-to-one transformation \( M \) from \( G \) onto \( G' \) is called an \( m \)-diffeomorphism of modulus \( C \) if all the components of \( M \) and \( M^{-1} \) have continuous partial derivatives of all orders up to and including \( m \), and these partials do not exceed \( C \) in modulus.

**Lemma 7.** Let \( G, G' \) be open in \( \mathbb{R}^n \) and let \( M \) be an \( m \)-diffeomorphism of modulus \( C \) from \( G \) onto \( G' \). Let

\[
A \mu(y) = \mu(M^{-1}y), \quad y \in G'.
\]

Then \( A \) is a homeomorphism from \( W^{m,p}(G) \) [respectively \( W_0^{m,p}(G) \)] onto \( W^{m,p}(G') \) [resp. \( W_0^{m,p}(G') \)] and there exist constants \( C_1 \) and \( C_2 \) depending only on \( n, p \) and \( C \) and not on \( G \) or \( G' \) such that for all \( u \in W^{m,p}(G) \)

\[
C_1 \|u\|_{m,p,G} \leq \|Au\|_{m,p,G'} \leq C_2 \|u\|_{m,p,G}.
\]

**Proof.** \( A \) is a homeomorphism from \( L^p(G) \) onto \( L^p(G') \) for if \( \partial M/\partial x \) represents the Jacobian determinant of \( M \) then

\[
\left\{ \sup_{y \in G'} \left| \frac{\partial M^{-1}(y)}{\partial y} \right|^{|1/p|} \right\}^{-1} \|u\|_{0,p,G} \leq \|Au\|_{0,p,G'} \leq \sup_{x \in G} \left| \frac{\partial M(x)}{\partial x} \right|^{|1/p|} \|u\|_{0,p,G}.
\]

By induction and formal applications of the chain rule it is easily verified that in the sense of distributions on \( G' \)

\[
D^q(Au) = \sum_{|\alpha| \leq q} M_{\alpha} A(D^q u)
\]
where $M_{a\beta}$ is a polynomial of degree $|\beta|$ in the derivatives of the components of $M^{-1}$ involving derivatives of orders not exceeding $|\alpha|$. It follows that

$$
\|Au\|_{n,p,c} = \sum_{|\alpha|=m} \sum_{|\beta|=|\alpha|} M_{a\beta} A(D^\alpha u)_{0,p,G}^p 
\leq \text{const} \sum_{|\beta|=m} \|A(D^\beta u)\|_{0,p,G}^p 
\leq \text{const} \|u\|_{n,p,g}.
$$

The reverse inequality follows in a similar manner.

**Lemma 8.** Under the hypotheses of Lemma 7 $W^{m,p}(G)$ is compactly imbedded in $W^{1,r}(G)$ if and only if $W^{m,p}(G')$ is compactly imbedded in $W^{1,r}(G')$. A similar statement holds for the spaces $W^n_0(G)$.

**Proof.** Suppose the imbedding $W^{m,p}(G) \rightarrow W^{1,r}(G)$ is compact. Let $\{u_\nu\}_{\nu=1}^\infty$ be a bounded sequence in $W^{m,p}(G')$. Then $\{A^{-1}u_\nu\}$ is bounded in $W^{m,p}(G)$ and so has a subsequence converging in $W^{1,r}(G)$. The corresponding subsequence of $\{u_\nu\}$ is convergent in $W^{1,r}(G')$ whence the imbedding $W^{m,p}(G') \rightarrow W^{1,r}(G')$ is compact. The other cases are proved similarly.

Of course Lemma 8 can be used to obtain immediately the conclusions of Theorems 1–3 for any domain $G$ which is $m$-diffeomorphic to an unbounded, regular, flatly-bounded domain $G'$ satisfying the conditions of the particular theorem. As most quasibounded domains do not have this property we obtain generalizations of these theorems with localized hypotheses.

**Theorem 4.** Let $G$ be a regular, unbounded domain in $\mathbb{R}^n$. Let $k$ be the largest integer ($1 \leq k \leq n$) for which for some constant $C$ there exist infinitely many mutually disjoint open sets $U$ in $\overline{G}$ each of which is $m$-diffeomorphic with modulus not greater than $C$ to the unit ball $B$ in $\mathbb{R}^n$ in such a way that $U \cap \partial G$ is mapped into a subset of the union of finitely many $(n-k)$-planes. (In particular $G$ is not $(n-k+1)$-quasibounded.) If either $mp \leq k$, $p > 1$ or $m < k$, $p = 1$ then no imbedding of the form $W^{m,p}_0(G) \rightarrow W^{1,r}_0(G)$ can be compact.

**Proof.** Let $\{U_\nu\}_{\nu=1}^\infty$ be a sequence of mutually disjoint open sets in $\overline{G}$ for which there correspond $m$-diffeomorphisms $M_\nu : U_\nu \rightarrow B$ having modulus $\leq C$ and such that $M_\nu(U_\nu \cap \partial G) \subset B \cap P_1$ where $P_1$ is the union of finitely many $(n-k)$-planes. Let $\varphi \in C_0^\infty(B)$ be such that

$$
\|\varphi\|_{0,r,B} = C_1 > 0, \quad \|\varphi\|_{m,p,B} = K_1 < \infty.
$$

By the method used in the proof of Theorem 1 we can construct functions $\gamma_\nu \in C_0^\infty(B - P_1)$ such that

$$
\|\gamma_\nu\|_{0,r,B} \geq C_2 > 0, \quad \|\gamma_\nu\|_{m,p,B} \leq K_2 < \infty,
$$

the constants $C_2$ and $K_2$ being independent of $\nu$. Denoting by $A_1$ the operator for
which $A_i u(y) = u(M_i^{-1} y)$, $y \in B$ we have by Lemma 7 that there exist constants $C_3$ and $K_3$ again independent of $i$ such that
\[\| A_i^{-1} \gamma_i \|_{0,r,G} \geq C_3 > 0, \quad \| A_i^{-1} \gamma_i \|_{p,m,G} \leq K_3 < \infty\]
and
\[A_i^{-1} \gamma_i \in C^0_\infty(U_i \cap G)\].

The noncompactness of the imbedding $W_0^{m,p}(G) \to W_0^{1,\gamma}(G)$ now follows as in Theorem 1.

As an analog of the compactness Theorems 2 and 3 we have

**Theorem 5.** Let $G$ be an unbounded open set in $E_n$ with the property that there exist constants $C$, $R_0$, and $K$ such that for each $R \geq R_0$ there exist positive numbers $d(R)$ and $\delta(R)$ with the following properties:

(i) $d(R) + \delta(R) \to 0$ as $R \to \infty$,
(ii) $d(R)/\delta(R) \leq K$, $R \geq R_0$,
(iii) for each $x \in G_R = \{ x \in G : |x| > R \}$ the ball $B_\delta(x)$ of radius $\delta(R)$ and center $x$ can be mapped by a $1$-diffeomorphism $M$ of modulus $\leq C$ onto a set $S$ in $E_n$ such that for some $(n-k)$-plane $H$ ($1 \leq k \leq n$) and some point $a \in H$ we have $S \subset T_{d(R), \delta(R)}(H, a)$ and $H \cap T = M(\partial G \cap B_\delta(x))$.

Then the imbedding $W_0^{m,p}(G) \to W_0^{1,\gamma}(G)$, $0 \leq j < m$ is compact in any of the following cases:

(a) $m = p = k = 1$,
(b) $mp > n + np/k$, $p \leq r < p^*$,
(c) $mp > n + (j+1)p - np/k$, $1 \leq r < p^*$,

where $p^* = np(n-mp+jp)^{-1}$ if $n > mp - jp$ and $p^* = \infty$ if $n \leq mp - jp$.

**Proof.** The conclusion is the same as that of Theorem 3 and the proof is identical if we reprove Lemma 6 (Poincaré's inequality) under the conditions of this theorem. Thus, let $p > k$ or $p = k = 1$ and let $1 \leq r \leq p$. Fix $R \geq R_0$ and let $d = d(R)$ and $\delta = \delta(R)$. Define the cubes $Q_a$ as in the proof of Lemma 6. If $x \in G_R$ then for some $a$, $x \in Q_a \subset B_\delta(x)$. There exists a $1$-diffeomorphism $M$ of $B = B_\delta(x)$ onto $S \subset E_n$ having modulus $\leq C$ and there exists an $(n-k)$-plane $H$ and a point $a \in H$ such that $S \subset T = T_{d, \delta}(H, a)$ and $H \cap T = M(\partial G \cap B_\delta(x))$. For any $\gamma \in C_\infty^0(G)$ we have that $A \gamma$ (defined by $A \gamma(y) = \gamma(M^{-1} y)$, $y \in S$) vanishes near $H \cap T$. Thus by the corollary of Lemma 5, Lemma 7 and the fact that $d \leq K \delta$
\[\| \gamma \|_{0,r,Q_a \cap G_R} \leq \| \gamma \|_{0,r,B} \leq \text{const} \| A \gamma \|_{0,r,S} \leq \text{const} d^{1+n/r-n/p} \| A \gamma \|_{1,p,T} \leq \text{const} \delta^{1+n/r-n/p} \| \gamma \|_{1,p,M^{-1}(T)} \leq \text{const} \delta^{1+n/r-n/p} \| \gamma \|_{1,p,Q_a^*}\]
where $Q_a^*$ is the union of all the cubes $Q_\delta$ which intersect $M^{-1}(T)$. Since the modulus of $M$ is bounded, $M^{-1}$ is Lipschitzian and there exists a constant $\lambda$ such that $M^{-1}(T) \subset B_{\lambda d}(x) \subset B_{\lambda R d}(x)$. Thus there is a constant $N$ independent of $R$ and $x$
such that any \( N+1 \) of the sets \( Q'_{\alpha} \) have empty intersection. Summing the above inequality over those \( \alpha \) for which \( Q_{\alpha} \) meets \( G_{\alpha} \) we obtain, as in Lemma 6, the required form of Poincaré’s inequality.

5. An application to differential operators. Let \( L \) be a linear partial differential operator of order \( 2m \) in \( G \) given by

\[
Lu(x) = \sum_{|\alpha| \leq 2m} a_{\alpha}(x) D^\alpha u(x)
\]

with coefficients \( a_{\alpha} \) infinitely differentiable, bounded, complex functions on \( G \). Suppose \( L \) is such that it satisfies the boundedness condition

\[
\left| \int_{G} L\varphi(x)\psi(x) \, dx \right| \leq c_0 \| \varphi \|_{m,2,G} \| \psi \|_{m,2,G}
\]

for all \( \varphi, \psi \in C^\infty_0(G) \), and also Garding’s inequality

\[
\text{Re} \left( \int_{G} L\varphi(x)\overline{\psi(x)} \, dx \right) \geq c_1 \| \varphi \|_{m,2,G}^2 - c_2 \| \varphi \|_{0,2,G}^2
\]

for all \( \varphi \in C^\infty_0(G) \), where \( c_0, c_1 > 0 \) and \( c_2 \) are constants. The realization of \( L \) in \( L^2(G) \) corresponding to null Dirichlet boundary data is an operator \( T \) in \( L^2(G) \) defined by

\[
\text{Dom}(T) = \{ f \in L^2(G) : Lf \in L^2(G) \} \quad T f = L f, \quad f \in \text{Dom}(T).
\]

Theorem 6. If \( G \) is open in \( E_n \) and satisfies the conditions of either Theorem 3 or Theorem 5 with \( 2m > n + 2 - 2n/k \) then \( T \) as defined above is a closed linear operator in \( L^2(G) \); the spectrum \( \sigma(T) \) is discrete and has no finite limit points; for \( \lambda \notin \sigma(T) \) the resolvent operator \( R_{\lambda}(T) = (\lambda I - T)^{-1} \) is completely continuous.

The proof is identical to that of the standard theorem of this type. A sketch can be found in [6].

References


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