

CORRECTIONS TO "ON SEQUENTIAL CONVERGENCE"

BY

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In my paper [2], Theorems 8.2 and 8.3 are false. I am indebted to Dennis Sentilles who sent me a counterexample to Theorem 8.2.

The wrong statements concern "LS-spaces." These are linear spaces S on which there is a metric ρ and a function f such that $x_n \rightarrow x$ in S if and only if both $\rho(x_n, x) \rightarrow 0$ and $f(x_n)$ remains bounded. Without restating the further conditions imposed on ρ and f , we single out for the present two tractable subclasses of the class of LS-spaces:

(I) LF -spaces, i.e. strict inductive limits of Fréchet spaces ([1], [2, Theorem 7.7]).

(II) If B is a separable Banach space, and S is its dual B^* with weak-star convergence of sequences, then S is an LS-space [2, Theorem 7.1]. (Actually the same is true with "Fréchet" in place of "Banach": see [1, Théorème 5, p. 84].)

In case (I), Theorems 8.2 and 8.3 are true. The Banach-Steinhaus theorem for LF -spaces is well known [1, Théorème 2, p. 73]. But in case (II), if B is infinite-dimensional, the elements of its unit ball give pointwise bounded continuous linear forms on S which are not equicontinuous for the LS-topology. The proof of Theorem 8.2 in [2] errs in assuming that multiplication by a positive scalar is an open mapping in the relative topology of a (convex, symmetric, closed, metrizable) set.

The proof of Theorem 8.3 is invalid since it rests on Theorem 8.2. Disproving the statement of 8.3 requires some further work. Here is one counterexample.

Let \mathcal{C} be the space of continuous real functions on $[0, 1]$ with supremum norm. Then its dual \mathcal{C}^* is the space of finite signed measures on $[0, 1]$. \mathcal{C}^* with weak* sequential convergence is an LS-space. It has a countable dense set and is complete as an LS-space. The LS-topology \mathcal{T} on \mathcal{C}^* is the topology of uniform convergence on sequences $\{f_n\}$ in \mathcal{C} with $\|f_n\| \rightarrow 0$ [2, around Theorem 7.8]. By the Mackey-Arens theorem, the dual space of $(\mathcal{C}^*, \mathcal{T})$ is \mathcal{C} (see also [1, Théorème 6, p. 85]).

A sequence in \mathcal{C} is weakly convergent if and only if it is uniformly bounded and converges pointwise. Thus the following fact contradicts Theorem 8.3:

PROPOSITION. *Bounded pointwise convergence in \mathcal{C} is not countably quasi-metric.*

Proof. Suppose there is a metric d and a countable set $\{G_m\}_{m=1}^\infty$ of functions on \mathcal{C} such that if $\|f_n\| \leq 1$, then $f_n \rightarrow 0$ pointwise if and only if both:

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- (a) $d(f_n, 0) \rightarrow 0$ and
 (b) for each m , $\sup_n G_m(f_n) < \infty$.

For some intervals $I_m = [a_m, b_m]$, $0 = a_0 < a_1 < \dots < a_m < \dots < b_m < \dots < b_1 < b_0 = 1$, let $\mathcal{C}_m = \{f \in \mathcal{C} : \|f\| \leq 1, f(x) = 0 \text{ for all } x \notin I_m\}$. We can define a_m and b_m inductively so that for each m ,

- (c) $\sup \{G_m(f) : f \in \mathcal{C}_m\} < \infty$ and
 (d) $\sup \{d(0, f) : f \in \mathcal{C}_m\} < 1/m$.

To do this, we can let $a_{m+1} = (a_m + b_m)/2$ and $b_{m+1} = a_{m+1} + 1/r$ for r large enough (if $g_r \in \mathcal{C}$, $\|g_r\| \leq 1$, and $g_r = 0$ outside $[a_{m+1}, a_{m+1} + 1/r]$, then $g_r \rightarrow 0$ pointwise).

Now there is a c with $a_m < c < b_m$ for all m , and there are f_m in \mathcal{C}_m with $f_m(c) = 1$ for all m . But (a) through (d) imply $f_m(c) \rightarrow 0$, a contradiction. Q.E.D.

Another correction: on p. 506 in [2], it is wrongly stated that topologies $T(C)$ on distribution spaces \mathcal{E}' and \mathcal{S}' are not locally convex (see [3]).

REFERENCES

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