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A CLASS OF DECOMPOSITIONS OF E^n WHICH ARE FACTORS OF E^{n+1}(1)

BY
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R. H. Bing has given an example of a space which fails to be a 3-manifold at each of a Cantor set of points, but such that the cartesian product of the space and E^1 is homeomorphic to E^4 [4]. This result has led to the study of other nonmanifold factors of E^4; a summary of results in this area may be found in Armentrout's paper in [5].

Bing's example arises from a decomposition of E^3 into points and arcs; the defining sequence for the arcs consists of collections of double tori resembling bones, hence the example is generally called the dogbone space. In this paper we generalize Bing's result with the following theorem.

THEOREM 1. *For integers $n>1$ let G be a monotone upper semicontinuous decomposition of E^n into points and arcs. Let A denote the set of nondegenerate elements of G. If there is a homeomorphism h of A onto the product of a Cantor set and* $[0, 1]$ *then* $E^n/G\times E^1=E^{n+1}$.

Standard notation is used with the following adaptations. We use $S(A, \varepsilon)$ to denote an ε neighborhood of the set A, Cl denotes closure, and equality of topological spaces indicates only that the spaces are homeomorphic. As a convention we use C' to denote the "middle thirds" Cantor set on $[0, 1]$. We let $C=C'\times[0, 1]$ and call any set homeomorphic to C a Cantor set of arcs.

The material in this paper was contained in a dissertation [2] written at the University of Tennessee under the direction of Ralph J. Bean to whom the author wishes to express his gratitude.

To prove Theorem 1 we generalize the technique used by Bing in [4]. We define a decomposition B of E^{n+1} by $B=\{(g, w)\in E^n\times E^1 : g\in G,\ w\in E^1\}$. The identity map is a homeomorphism of $E^n/G\times E^1$ onto E^{n+1}/B. A pseudo-isotopy f of $E^{n+1}\times[0, 1]$ onto E^{n+1} is then constructed in such a way that $f(x, 0)$ is the identity map and $f(x, 1)$ takes each element of B onto a distinct point of E^{n+1}. This establishes that $E^{n+1}/B=E^{n+1}$ [4], [8].

For the case $n=2$, Moore's Theorem establishes that $E^2/G=E^2$ and Theorem 1 is immediate. The remainder of this paper is the proof of Theorem 1 for the case

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$n=3$. Higher dimensional cases may be proved analogously, but since the geometry remains at least partially visible for $n=3$ we do the proof for that specific case.

In the following sequence of theorems we are working to reduce the problem to a point where we can invoke the proof in [4] to draw our conclusion. We must first define a generalized defining sequence for the nondegenerate elements of G.

LEMMA 1. *Suppose A is a Cantor set of arcs in E^3. There is a collection of sets $\{X_i''\}_0^\infty$ in $E^3\times E^1$ such that each X_i'' is the union of a collection of mutually disjoint 4-cells; for each i, $X_{i+1}''\subseteq \operatorname{int} X_i''$; and $\bigcap_{i=0}^\infty X_i''=A\times\{0\}$.*

Proof of Lemma 1. The procedure defined below is illustrated in Figure 1. Let h be the homeomorphism of A onto C. We may consider C to be embedded in $I^2=[0,1]\times[0,1]$ as $C'\times[0,1]$. The existence of a continuous function h' of E^3 into I^2 which agrees with h when restricted to A is given by Tietze's Extension Theorem.

Let π be the projection map of I^2 onto I defined by $\pi(x,y)=x$. If $\boldsymbol{a}$ is an arc of A, define p_a to be $\pi(h(\boldsymbol{a}))$. Notice that $p_a\in C'$; we will call p_a the *index* of $\boldsymbol{a}$ in C'.

Define a map g of $E^3\times E^1$ onto $E^3\times E^1$ by

$$g(x,w) = (x, w+\pi(h'(x))).$$

Continuity of h' makes g a homeomorphism. Notice that if $\boldsymbol{a}$ is an arc of A, then

$$g(\boldsymbol{a},w) = (\boldsymbol{a}, p_a+w).$$

The effect of g is to lift each arc of $A\times\{w_0\}$ in the w direction by the index of that arc in C'. Each arc $\boldsymbol{a}$ of $g(A\times\{0\})$ will lie in $E^3\times\{p_a\}$. Let $A'=g(A\times\{0\})$. Because each arc $\boldsymbol{a}'$ of A' lies in a 3-plane in E^4 there is a homeomorphism g_a of E^4 onto E^4 such that $g_a(\boldsymbol{a}')=I$, the unit interval on the x-axis in E^4. This follows from a theorem of Klee [7].

We now construct the sets $\{X_i''\}_0^\infty$. This is done in a recursive manner. We first choose a set of numbers $\{\delta_0^a\}_{a\in A}$ and then use these numbers to form sets X_0' and X_0''. We use X_0' to choose the numbers $\{\delta_1^a\}_{a\in A}$ and continue in this manner to form $\{X_i'\}_0^\infty$, $\{X_i''\}_0^\infty$, and $\{\{\delta_i^a\}_{a\in A}\}_0^\infty$.

For each $\boldsymbol{a}$ we choose δ_0^a so that $g_a^{-1}(S(I,\delta_0^a))\subset S(g(\boldsymbol{a}),1)$. We now describe the inductive steps used in our construction. First we assume we have constructed the sets $\{X_k'\}_0^{i-1}$, $\{X_k''\}_0^{i-1}$ and $\{\{\delta_k^a\}_{a\in A}\}_0^i$ and show how to construct X_i' and X_i''. We will make use of the homeomorphism which lifted the arcs of $A\times\{0\}$ onto distinct 3-planes.

Let $T_i^a=g_a^{-1}(\operatorname{Cl} S(I,\delta_i^a))$. Each T_i is a 4-cell. We will adjust these cells and then use a finite number of them to form X_i'; X_i' will then be used to form X_i''.

Let

$$l_i^a = \inf\{p\in[-1,2] : \forall q\in[p,p_a]\cap C',\ gh^{-1}(q\times I)\subset \operatorname{int} T_i^a\},$$
$$m_i^a = \sup\{p\in[-1,2] : \forall q\in[p_a,p]\cap C',\ gh^{-1}(q\times I)\subset \operatorname{int} T^a\}.$$

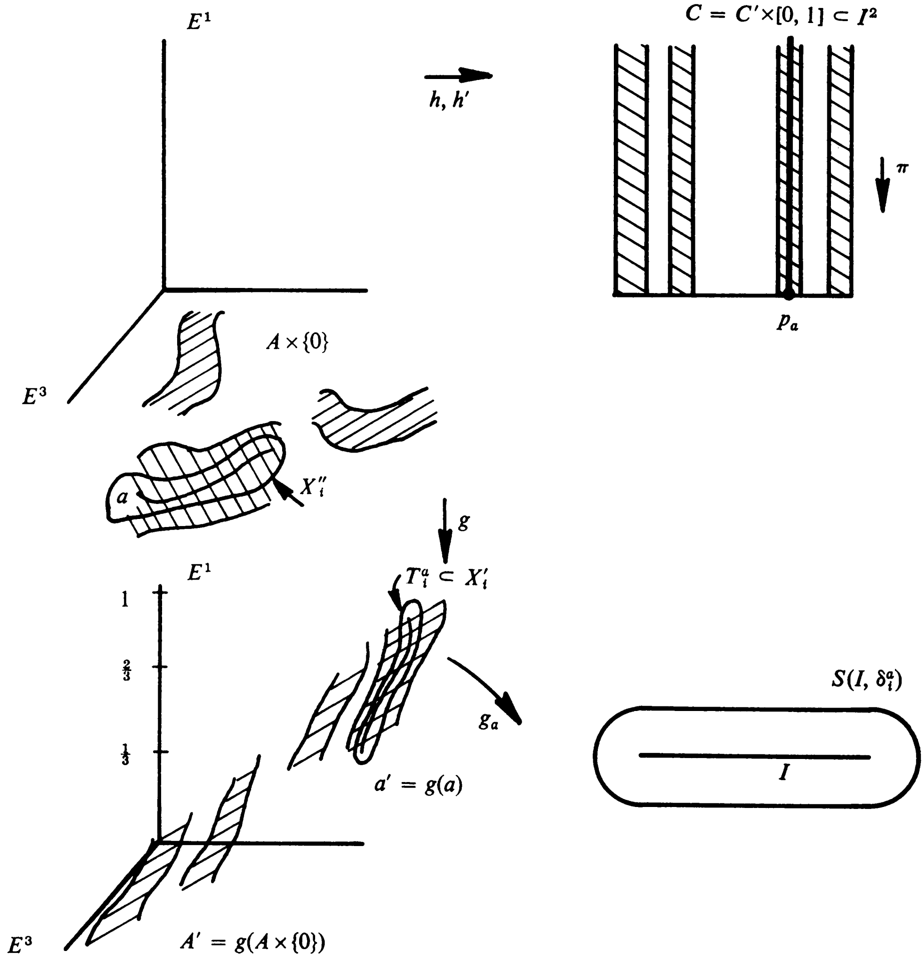

FIGURE 1. Constructing a defining sequence

Because $a' \subset \operatorname{int} T_i^a$, it follows that $l_i^a < p_a < m_i^a$. Choose intervals $[l_1, l_2]$ and $[m_1, m_2]$ lying in $[-1, 2] - C'$ such that

$$l_i^a < l_1 < l_2 < p_a < m_1 < m_2 < m_i^a.$$

Define a homeomorphism f_a of $E^3 \times E^1$ onto $E^3 \times E^1$ so that f_a is the identity map on $E^3 \times [l_2, m_1]$ and f_a shrinks rays in the w direction in such a way that

$$f_a(T_i^a \cap (E^3 \times (-\infty, l_2])) \subset E^3 \times [l_1, l_2]$$

and

$$f_a(T_i^a \cap (E^3 \times [m_1, \infty))) \subset E^3 \times [m_1, m_2].$$

The action of f_a is illustrated in Figure 2.

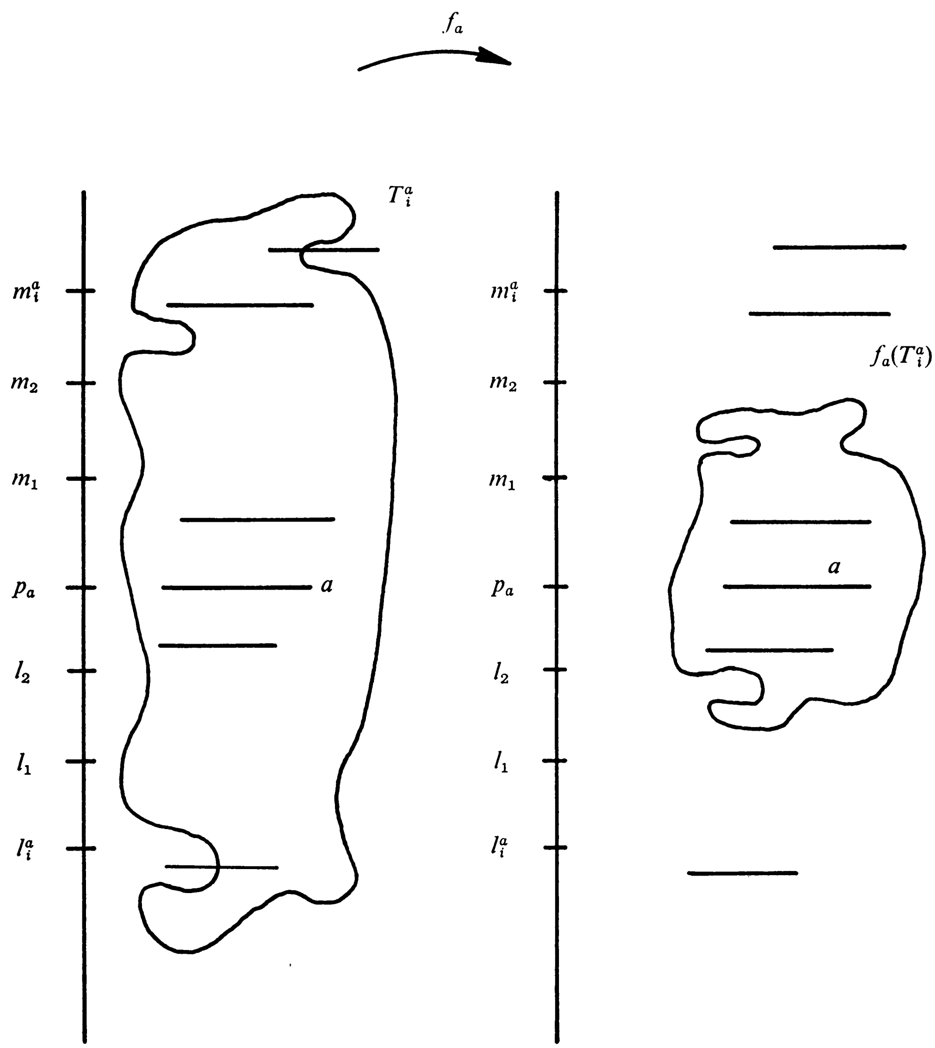

FIGURE 2. Adjusting 4-cells

Let $(T_i^a)' = f_a(T_i^a)$, then $A' \subset \bigcup_{a' \in A'} \operatorname{int}(T_i^a)'$; for each i we may use compactness of A' to find a finite collection of arcs $\{a(j)\}_{j=0}^{n_i}$ such that $A' \subset \bigcup_{j=0}^{n_i} \operatorname{int}(T_i^{a(j)})'$. The effect of f_a is to assure that if any arc of A' intersects $(T_i^{a(j)})'$ then it lies in the interior of $(T_i^{a(j)})'$. We may apply homeomorphisms similar to f_a to certain of the $(T_i^{a(j)})'$ to get a mutually disjoint collection $\{(T_i^{a(j)})''\}_{j=0}^{n_i}$ with elements which still have the above property and so that for each j, $(T_i^{a(j)})'' \subset T_i^{a(j)}$. For convenience of notation we assume that $\{T_i^{a(j)}\}_{j=0}^{n_i}$ is already a mutually disjoint collection of 4-cells with these properties. Let $f^i_{a(j)}$ denote the complete homeomorphism so that

$$T_i^{a(j)} = f^i_{a(j)} g^{-1}_{a(j)}(\mathrm{Cl}\,(S(I, \delta_i^{a(j)}))).$$

Let $X_i' = \bigcup_{j=0}^{n_i} T_i^{a(j)}$. Each X_i' is a collection of mutually disjoint 4-cells and

$A' \subset \text{int } X_i' \subset X_i' \subset \text{int } X_{i-1}'$. Let $X_i'' = g^{-1}(X_i')$. We will choose the numbers $\{\{\delta_i^a\}_A\}_0^\infty$ in such a manner that $\bigcap_0^\infty X_i'' = A \times \{0\}$. It will then follow from the above that the sequence $\{X_i''\}$ satisfies the conditions of Lemma 1. Notice that it was necessary to have the arcs of A' on distinct 3-planes in order to construct sets X_i' with all the desired properties.

We now define the inductive step for picking $\{\delta_k^a\}_{a \in A}$ for $k \geqq 1$. We may assume we know X_i' and $\{\delta_i^a\}_A$ for $i \leqq k-1$. To determine δ_k^a first choose an integer m so that if I is divided into m consecutive subintervals $\{I_l\}_1^m$ each of length $1/m$ then for each l the diameter of $g^{-1}g_a^{-1}(I_l)$ is less than $1/16k$. Existence of m follows from continuity of g and g_a.

Let P_i be a 3-plane perpendicular to I at point i/m, $1 \leqq i \leqq m-1$. Let C_l be the closure of the section of E^4 between P_{l-1} and P_l; C_1 and C_m are closed half 4-planes respectively to the left of P_1 and to the right of P_{m-1}. These are shown in Figure 3. Choose a positive number δ_k^a so small that if a' is an arc of A' and

$$g_a(a') \subset S(I, \delta_k^a)$$

then the following conditions are satisfied:

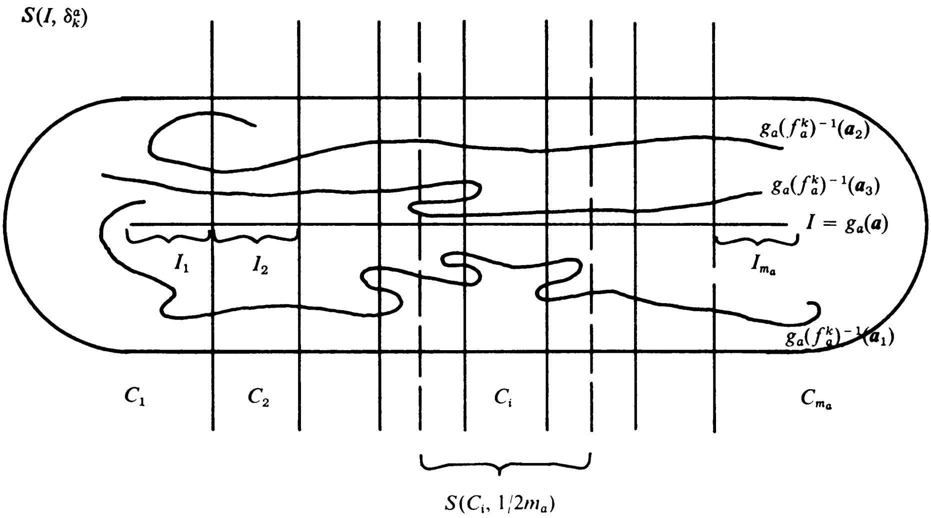

FIGURE 3. Position of arcs

1. One endpoint of $g_a(a')$ lies in C_1, the other lies in C_m.

2. If p_1 and p_2 are points of $g_a(a')$ and $p_1 < p_2$ in the natural order on $g_a(a')$, let $[p_1, p_2]$ denote the corresponding interval on $g_a(a')$. Choose δ_k^a so that if p_1 and p_2 belong to C_i then $[p_1, p_2] \subset S(C_i, 1/2m)$.

3. The diameter of $g^{-1}g_a^{-1}(S(I, \delta_k^a) \cap C_l)$ is less than $1/8k$ for each l, $1 \leqq l \leqq m$.

4. The set $g_a^{-1}(S(I, \delta_k^a))$ is a subset of $S(g(a), 1/k)$.

5. The arc $g(a)$ will be contained in some component $T_{k-1}^{a(j_0)}$ of X_{k-1}'. Let η be so small that if E is an η subset of $T_{k-1}^{a(j_0)}$ then $g_{a(j_0)} f_{a(j_0)}^{-1}(E)$ has diameter less than $1/3m$.

Choose δ_k^a so small that if D is a $2\delta_k^a$ subset of $S(I, \delta_k^a)$ then the diameter of $g_a^{-1}(D)$ is less than η.

6. $\delta_k^a < \delta_{k-1}^a$.

The existence of a δ_k^a which will satisfy conditions 1 and 2 follows from the hypothesis that A is a Cantor set of arcs. The existence of a δ_k^a satisfying conditions 3, 4, and 5 is assured by the continuity of g, g_a, and $g_{a(j_0)}$ and the fact that m was chosen so that $g^{-1}g_a^{-1}(I_l)$ has diameter less than $1/16k$ for each l, $1 \leqq l \leqq m$.

Condition 2 may be be interpreted to say that no arc of $g_a(A')$ contained in $S(I, \delta_k^a)$ can double back more than halfway through C_{l-1} after it has intersected C_l (in the natural order on the image of the arc from the end lying in C_1 to the end lying in C_m). Consider the arcs shown in Figure 3, the choice of δ_k^a does allow arcs like $g_a(\boldsymbol{a}_1)$ to lie in $S(I, \delta_k^a)$; it does not allow arcs like $g_a(\boldsymbol{a}_2)$ and $g_a(\boldsymbol{a}_3)$ to lie in $S(I, \delta_k^a)$. Condition 4 will assure that $\bigcap_{k=0}^{\infty} X_k' = A'$.

The restrictions on the choice of δ_k^a are, of course, not all necessary to satisfy Lemma 1; however, these conditions are used later when we define a shrinking homeomorphism.

Let $X_i = X_i'' \cap (E^3 \times \{0\})$. For each i,

$$A \times \{0\} \subset \operatorname{int} X_{i+1} \subset X_{i+1} \subset \operatorname{int} X_i$$

with interior taken in the E^3 sense, and $\bigcap_0^{\infty} X_i = A \times \{0\}$. Each X_i is contained in $E^3 \times \{0\}$; to simplify notation in the remainder of this paper we will regard each X_i as a subset of E^3 without regard to its embedding in E^4. Thus we now have $A = \bigcap_0^{\infty} X_i$. The sequence $\{X_i\}_0^{\infty}$ is a 3-dimensional defining sequence for A.

This sequence, $\{X_i\}_0^{\infty}$, will now be used in much the same way that Bing used the defining sequence of cells with handles for the nondegenerate elements of the dogbone space in [4]. Due to the generality of the construction of $\{X_i\}_0^{\infty}$, 4-cells must be obtained by a more general method than the one Bing used and the shrinking must be done differently.

We call T a *pseudo-component* of X_i if for some component T'' of X_i'', $T \times \{0\} = T'' \cap (E^3 \times \{0\})$.

THEOREM 2. *For each nonnegative integer i, each interval $[a, b]$, each pseudo-component T of X_i, and each positive number ε there is an integer $n_i > i$ and a set K_i, the union of mutually disjoint 4-cells, such that $(T \cap X_{n_i}) \times [a, b] \subset \operatorname{int} K_i \subset K_i \subset T \times [a-\varepsilon, b+\varepsilon]$.*

Proof of Theorem 2. Let $A^* = T \cap A$. Define a homeomorphism

$$f\colon E^3 \times E^1 \to E^3 \times E^1$$

by $f(x, w) = (x, w+a)$. For each $j \geqq i$ let $Y_j'' = f(X_j'' \cap T'')$. Then $A^* \times \{a\} = \bigcap_{j=i}^{\infty} Y_j''$. ($Y_j''$ is just $T'' \cap X_j''$ translated a units in the w direction.) It follows that there is an integer $n > i$ such that $Y_{n-1}'' \subset \operatorname{int}(T \times [a-\varepsilon, b+\varepsilon])$. See Figure 4.

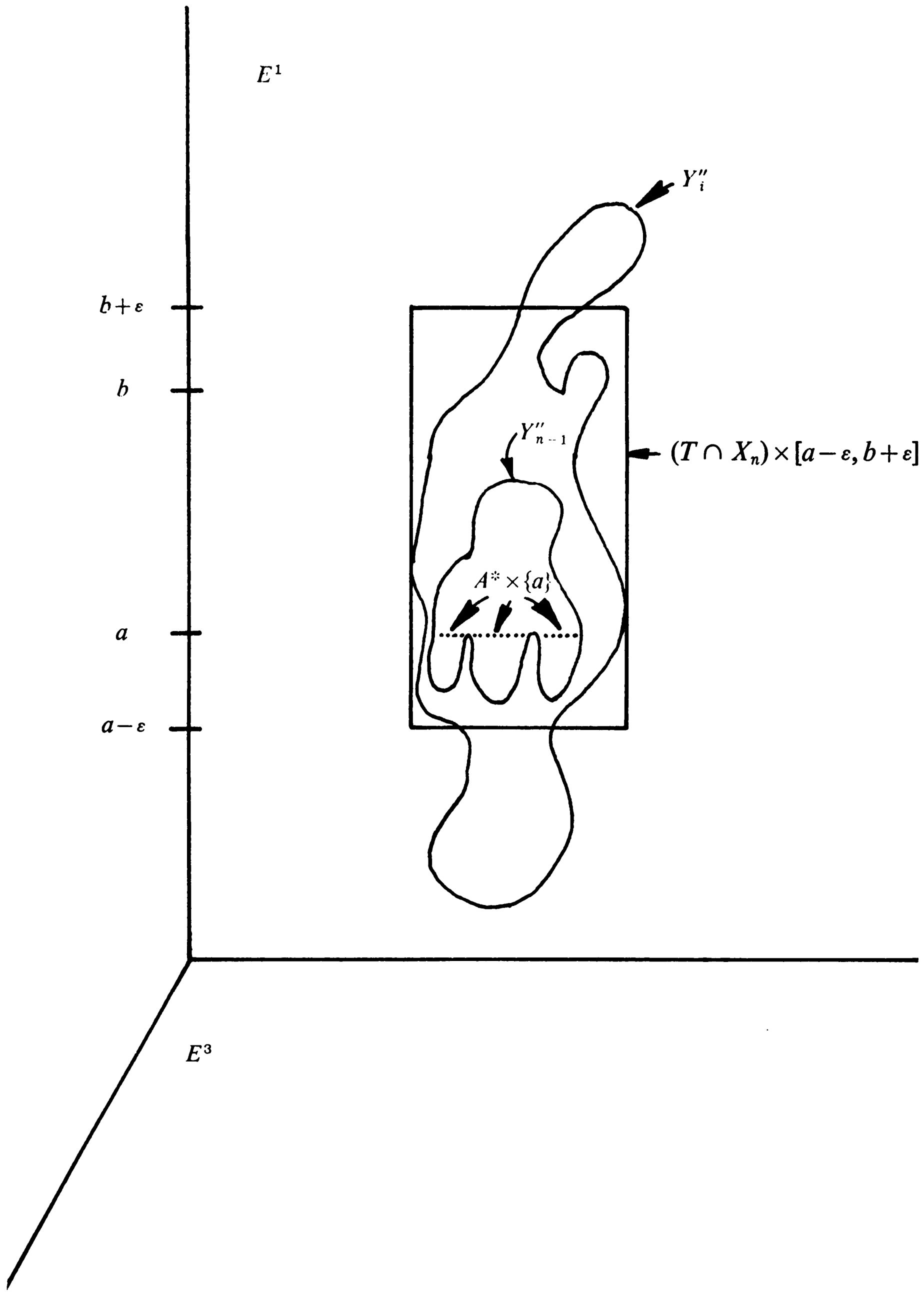

FIGURE 4. 4-cells containing arcs

Since $Y''_n \subset \operatorname{int} Y''_{n-1}$, it follows that $(T \cap X_n) \times \{a\} \subset \operatorname{int} Y''_{n-1}$; so there is a $\delta > 0$ such that $(T \cap X_n) \times [a, a+\delta] \subset \operatorname{int} Y''_{n-1}$.

Define $g_{ab}\colon E^3\times E^1 \to E^3\times E^1$ by

$$\begin{aligned} g_{ab}(x, w) &= (x, (w-a)(b-a)/\delta+a), \qquad w\in[a, a+\delta] \\ &= (x, b+\varepsilon-((b+\varepsilon-w)/(b+\varepsilon-a-\delta))\varepsilon), \qquad w\in[a+\delta, b+\varepsilon] \\ &= (x, w), \qquad w\in E^1-[a, b+\varepsilon]; \end{aligned}$$

g_{ab} linearly expands the interval $[a, a+\delta]$ on the w axis into the interval $[a, b]$ while shrinking $[a+\delta, b+\varepsilon]$ into $[b, b+\varepsilon]$ as shown in Figure 5. Let $K_i = g_{ab}(Y''_{n-1})$. Since g_{ab} is a homeomorphism, K_i is the union of a collection of mutually disjoint 4-cells.

Because $(T\cap X_n)\times[a, a+\delta]\subset \operatorname{int} Y''_{n-1}$, it follows that

$$(T\cap X_n)\times[a, b] \subset \operatorname{int} g_{ab}(Y''_{n-1}) = \operatorname{int} K_i;$$

and because

$$Y''_{n-1} \subset \operatorname{int}(T\times[a-\varepsilon, b+\varepsilon]),$$

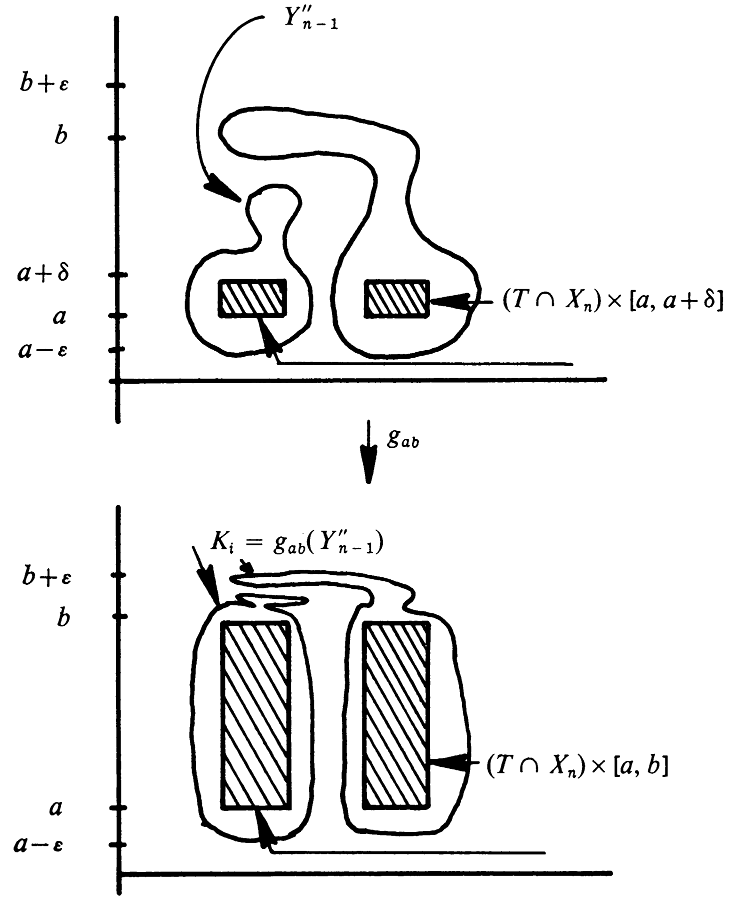

FIGURE 5. Expanding 4-cells

we have

$$K_i = g_{ab}(Y''_{n-1}) \subset \operatorname{int} g_{ab}(T\times[a-\varepsilon, b+\varepsilon]) = \operatorname{int}(T\times[a-\varepsilon, b+\varepsilon]).$$

Therefore K_i satisfies the required conditions. We take n to be the number n_i in the statement of the theorem.

COROLLARY 1. *Let m and k be positive integers and let T be a pseudo-component of X_k. There is a sequence $K_1, K_2, \ldots, K_{m-2}$, with each K_i the union of mutually disjoint 4-cells, and a sequence of positive integers $n_1, n_2, \ldots, n_{m-2}$ such that*

$$\begin{aligned} T\times[0, 2m-3] \supset K_1 \supset \operatorname{int} K_1 \supset (T\cap X_{n_1})\times[1, 2m-4] \\ \supset K_2 \supset \operatorname{int} K_2 \supset (T\cap X_{n_2})\times[2, 2m-5] \supset K_3 \supset \cdots \\ \supset K_{m-2} \supset \operatorname{int} K_{m-2} \supset (T\cap X_{n_{m-2}})\times[m-2, m-1]. \end{aligned}$$

Proof of Corollary 1. The proof follows immediately from a series of applications of Theorem 2 with varying choices of a and b; $\varepsilon=1$ in each application.

The following theorem is almost like Theorem 2 of [4]. It may be used to prove Theorem 1 just as Bing uses his Theorem 2 in the proof of Theorem 4 of [4].

THEOREM 3. *For each nonnegatve integer k and each $\varepsilon>0$, if T^* is a pseudo-component of X_k, there is an integer $n>k$ and a uniformly continuous homeomorphism f of E^4 onto E^4 such that:*

(a) *f is isotopic to the identity map,*

(b) *f is the identity map on $E^4-(T^*\times E^1)$,*

(c) *f moves no point more than ε in the w direction, and*

(d) *if T' is a pseudo-component of $T^*\cap X_n$, the diameter of $f(T'\times\{w\})$ is less than ε for each $w\in E^1$.*

Choose an integer $q\geqq k$ so that $1/q<\varepsilon$. A sequence of arcs $\{a(j)\}_0^n$ was used to form $X_q\cap T^*$ by the method given in the proof of Lemma 1. We now must define chambers similar to those used in [4]. Let m_j be the integer chosen in defining $\delta_q^{a(j)}$ in Lemma 1 and for each l, $1\leqq l\leqq m_j$, let

$$\begin{aligned} C^*_{jl} &= C_l \cap \operatorname{Cl} S(I, \delta_q^{a(j)}), \\ C'_{jl} &= f^q_{a(j)} g^{-1}_{a(j)}(C^*_{jl}), \\ C''_{jl} &= g^{-1}(C'_{jl}), \\ C_{jl}\times\{0\} &= C''_{jl}\cap(E^3\times\{0\}). \end{aligned}$$

Notice that $C'_{jl}\subset X'_q$, $C''_{jl}\subset X''_q$, $C_{jl}\subset X_q$; so by condition 3 on the choice of $\delta_q^{a(j)}$, and the observation that

$$f^q_{a(j)} g^{-1}_{a(j)}(S(I, \delta_q^{a(j)})) \subset g^{-1}_{a(j)}(S(I, \delta_q^{a(j)})),$$

the diameter of each C_{jl} is less than $\varepsilon/8$. To complete the proof of Theorem 3 we must prove two lemmas.

LEMMA 2. *Let T be a component of X'_q; say $T=T_q^{a(0)}$. Let m be the integer used in defining $\delta_q^{a(0)}$ in Lemma* 1. *Let $\{K_i\}_1^{m-2}$ and $\{n_i\}_1^{m-2}$ be sequences obtained by applying Corollary* 1 *to the pseudo-component $g^{-1}(T)\cap(E^3\times\{0\})$. There is a sequence of homeomorphisms of E^4 onto E^4, $\{f_i\}_1^{m-2}$, each isotopic to the identity map, such that for each i:*

(a) *The function f_i is the identity on $E^4-(X'_{n_i-1}\cap T)$ and on $\bigcup_{l=i+2}^{m} C'_{0l}$.*

(b) *The set $f_i(X'_{n_i}\cap T)$ is a subset of $\bigcup_{l=i+1}^{m} C'_{0l}$.*

Proof of Lemma 2. We define f_i to be the identity map on $E^4-(X'_{n_i-1}\cap T)$. Let $T^a_{n_i-1}$ be a component of $X'_{n_i-1}\cap T$ formed by using an arc $\boldsymbol{a}$ of A; let $\boldsymbol{a}'=g(\boldsymbol{a})$. By condition 2 on the choice of $\delta_q^{a(0)}$ we may find a point p of

$$g_{a(0)}(\boldsymbol{a}')\cap(C^*_{0(i+1)}-S(C^*_{0i},1/3m)-S(C^*_{0(i+2)},1/3m))$$

such that the component of $g_{a(0)}(\boldsymbol{a}'-\{p\})$ with endpoint in C^*_{01} lies entirely in $\bigcup_{l=1}^{i+1} C_{0l}$. Since $\delta^a_{n_i-1}\leqq\delta^a_{q+1}$, the choice of δ^a_{q+1} implies that there is a round disk D perpendicular to I at $g_a(f_a^{n_i-1})^{-1}f^q_{a(0)}g^{-1}_{a(0)}(p)$ separating Cl $S(I,\delta^a_{n_i-1})$ such that

$$g_{a(0)}(f^q_{a(0)})^{-1}f_a^{n_i-1}g_a^{-1}(D)\subset C^*_{0(i+1)}.$$

To see this first observe that D is a $2\delta^a_{n_i-1}$ subset of $S(I,\delta^a_{n_i-1})$ so by condition 5

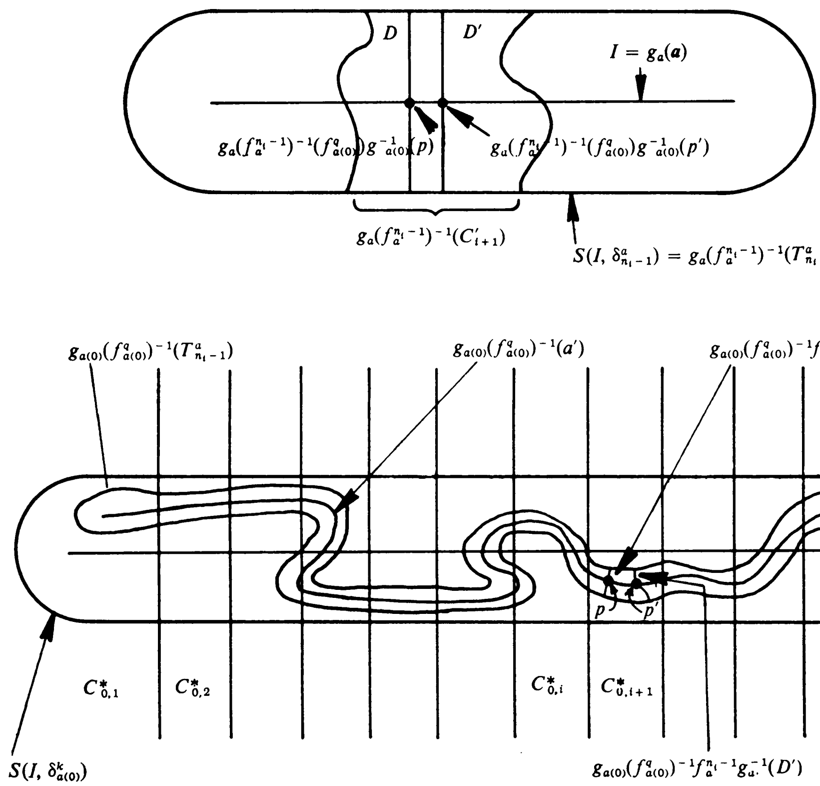

FIGURE 6. Position of disks

on the choice of δ_{n_i-1} the diameter of $g_a^{-1}(D)$ is less than the number η mentioned in that condition. Because the action of $f_a^{n_i-1}$ is to shrink $g_a^{-1}(D)$ it will follow that the diameter of $g_{a(0)}(f_{a(0)}^q)^{-1}f_a^{n_i-1}g_a^{-1}(D)$ is less than $1/3m$. The point p of $g_{a(0)}(f_{a(0)}^q)^{-1}f_a^{n_i-1}g_a^{-1}(D)$ is more than $1/3m$ units away from $E^4-C^*_{0(i+1)}$ so this choice of D will satisfy the above condition.

Now choose a point p' and a disk D' very near p and D respectively, having the same properties that p and D have. Figure 6 illustrates the positition of these points and disks. Each component of X'_{n_i} which intersects $T^a_{n_i-1}$ lies in the interior of $T^a_{n_i-1}$, so there is a $\delta'<\delta^a_{n_i-1}$ such that the image of each component of X_{n_i} under $g_a(f_a^{n_i-1})^{-1}$ lies in $S(I,\delta')$. Cl $S(I,\delta^a_{n_i-1})-S(I,\delta')$ will form a collar, as illustrated in Figure 7. There is a homeomorphism f_i' which will expand the collar to fill the area to the left of the disk D while shrinking all of $S(I,\delta')$ to the left of D into the area between D and D'. See Figure 7.

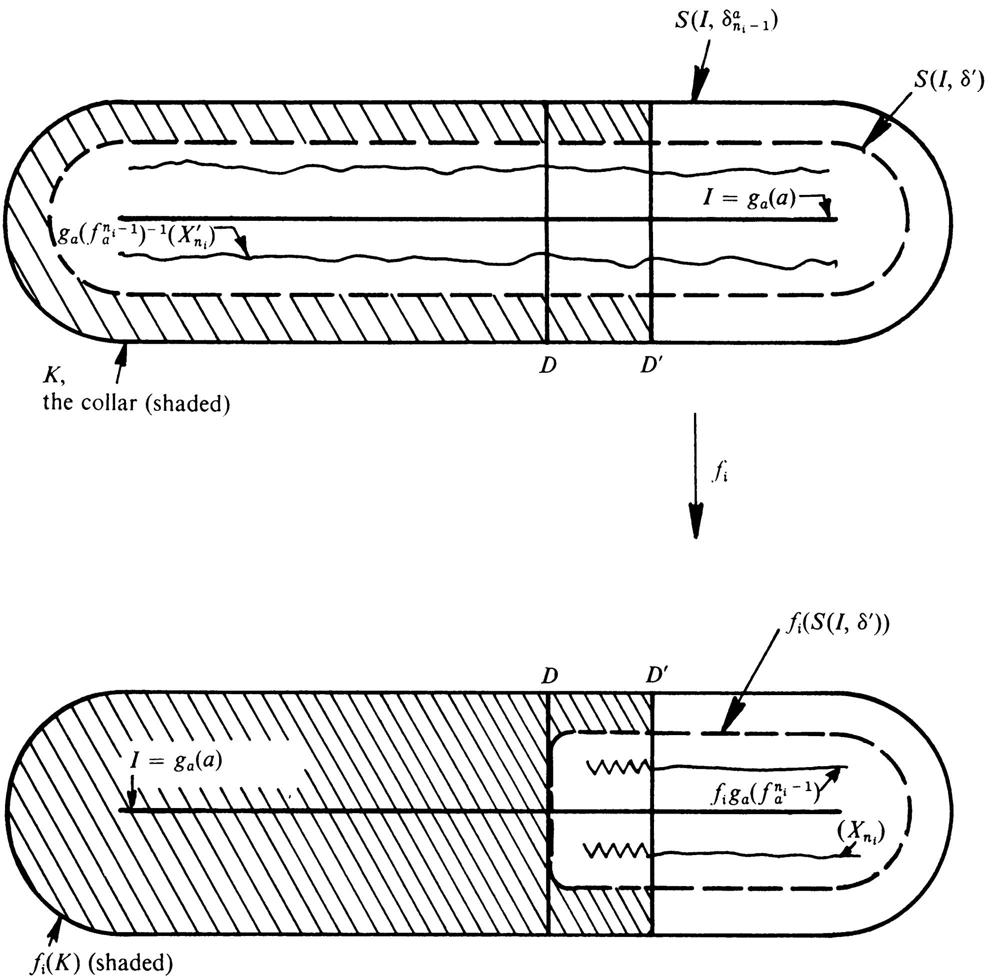

FIGURE 7. Shrinking inside a 4-cell

The homeomorphism f_i' is isotopic to the identity map and is equal to the identity to the right of D' and outside $S(I, \delta^a_{n_i-1})$. Let

$$f_i = f_a^{n_i-1} g_a^{-1} f_i' g_a (f_a^{n_i-1})^{-1}$$

on $T^a_{n_i-1}$; define f_i similarly on each component of $T \cap X'_{n_i-1}$ and let f_i be the identity on $E^4-(T \cap X'_{n_i-1})$. Because f_i will be the identity map on the boundary of each component of X'_{n_i-1}, it follows that f_i is a homeomorphism isotopic to the identity map.

Suppose x is a point of $X'_{n_i} \cap T$; x will belong to some component $T^a_{n_i-1}$ of X'_{n_i-1}. The point $f_i(x)$ will lie in the section to the right of the disk $f_a^{n_i-1}g_a^{-1}(D)$ in $T^a_{n_i-1}$. By the choice of the positions of D and D', f_i will be the identity on $\bigcup_{l=i+2}^m C'_{0l}$ and $g_a(f_a^{n_i-1})^{-1}f_i(x)$ will lie in $\bigcup_{l=i+1}^m C^*_{0l}$. It follows that $f_i(x) \in \bigcup_{l=i+1}^m C'_{0l}$. This concludes the proof of Lemma 2.

LEMMA 3. *Let T be a pseudo-component of X_q; $T=T'' \cap E^3 \times \{0\}$, where T'' is a component of X''_q. Let $\{C_l\}_1^m$ be the chambers of T and $\{n_i\}_1^{m-2}$ be integers determined by the corollary to Theorem* 2. *There is a homeomorphism f, isotopic to the identity, of E^4 onto E^4 such that*:

(1) *f is the identity map on*

(a) $$E^4-(T\times [0, 2m-3]),$$

and on

(b) $$\begin{gathered}(C_3+C_4+\cdots+C_m)\times([0, 1]+[2m-4, 2m-3]),\\ (C_4+C_5+\cdots+C_m)\times([1, 2]+[2m-5, 2m-4]),\end{gathered}$$

and so on through

$$C_m\times[m-3, m];$$

and

(2) $f((T\cap(C_1+C_2))\times([0, 1]+[2m-4, 2m-3]))\subset(C_1+C_2)\times[0, 2m-3]$,

$$f((X_{n_1}\cap T\cap(C_1+C_2+C_3))\times([1, 2]+[2m-5, 2m-4])) \subset (C_2+C_3)\times[0, 2m-3],$$

$$f((X_{n_2}\cap T\cap(C_1+C_2+C_3+C_4))\times([2, 3]+[2m-6, 2m-5])) \subset (C_3+C_4)\times[0, 2m-3],$$

and so on through

$$f((X_{n_{m-2}}\cap T)\cap(C_1+C_2+\cdots+C_m))\times[m-2, m-1] \subset (C_{m-1}+C_m)\times[0, 2m-3].$$

Interpretation of Lemma 3. See Figure 8. A similar figure appears in [4]. The diagram is misleading in that we have no idea of the exact structure of each C_i, this is determined by the homeomorphisms g, $g_{a(0)}$, and $f^q_{a(0)}$; but the diagram should suffice as a schematic. The columns represent the chambers, $\{C_i\}_1^m$, into which X_k has been divided. The y axis of the diagram represents the w coordinate of points of $E^3\times E^1$. For example, the point x in the diagram lies in $C_2\times[2, 3]$.

The region R is bounded on the left by the vertical line from 0 to $2m-3$ and by

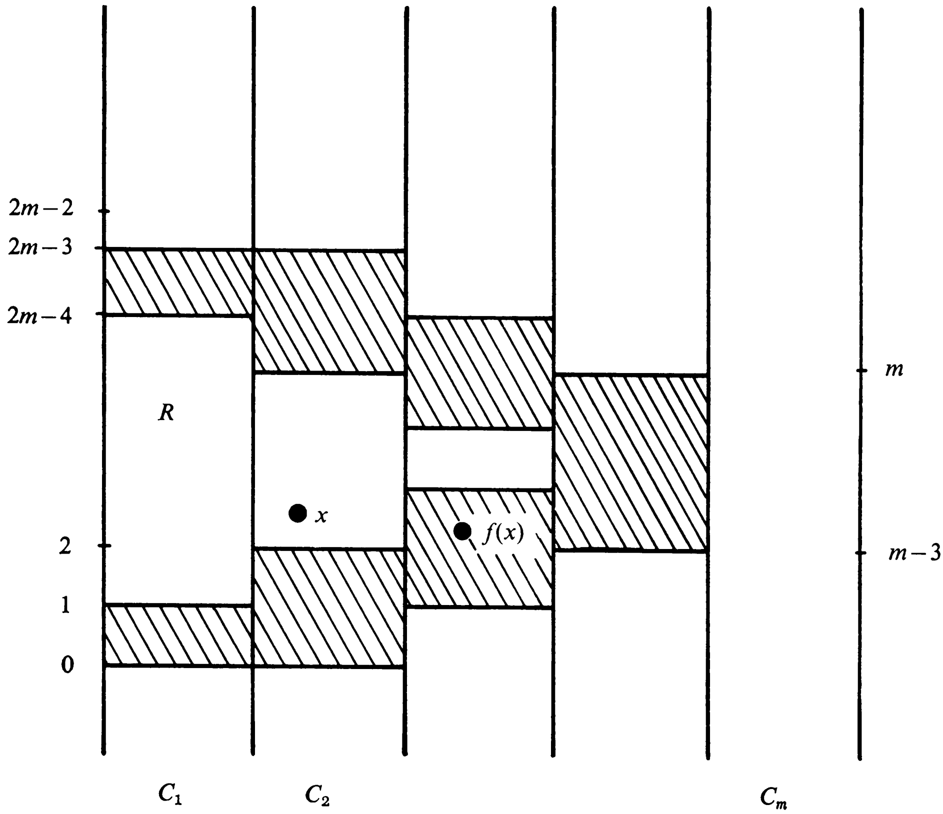

FIGURE 8. Sectioning $T\times[0, 2m-3]$

the heavy broken line to the right of this line. The shaded region is contained in R; f does not move any points outside of the region represented by R. This is implied in condition 1 of the statement of the lemma.

If x lies in $R\cap((X_{n_{m-2}}\cap T)\times[0, 2m-3])$ consider the horizontal line through x and the shaded rectangles which intersect this line. The first coordinate of $f(x)\in E^3\times E^1$ belongs to a C_i corresponding to one of the shaded rectangles. For the x shown, $f(x)$ would belong to $(C_3+C_4)\times[0, 2m-3]$. The diagram does not show anything about where the w coordinate of $f(x)$ will lie; but if x belongs to R, the w coordinate of $f(x)$ will lie inside R and therefore in the interval $[0, 2m-3]$. Roughly speaking, f moves certain points to the right but it does not move any point vertically by more than $2m-3$ units.

Proof of Lemma 3. Let $T'=g(T'')$. Use $\{f_i\}_1^{m-2}$ as defined in Lemma 2 on each component of $T'\cap X'_{n_i}$; each function is the identity map on $E^4-(T'\cap X'_{n_i})$. Let $g_i=g_{i,2m-3-i}$ with $g_{a,b}$ as defined in the proof of Theorem 2.

Let $h_i=g_ig^{-1}f_igg_i^{-1}$ for each integer $1\leqq i\leqq m-2$. Let $f=h_1h_2\cdots h_{m-2}$. Each f_i is a homeomorphism isotopic to the identity so each h_i is also, and it follows that f is a homeomorphism isotopic to the identity.

The verification that f satisfies the conditions in the statement of the lemma is quite laborious. A detailed proof is included in the author's dissertation [2]. We now give only a short indication of how this is done.

To check condition 1 notice that each h_i is the identity on $E^4 - K_1$ and thus on $E^4-(T\times[0, 2m-3])$. Each h_i is the identity on

$$(E^4-K_i)\cup\left(\left(\bigcup_{i+2}^{m} C_l\right)\times[0, 2m-3]\right).$$

We may conclude from this that f is the identity on

$$(C_3+C_4+\cdots+C_m)\times([0, 1]+[2m-4, 2m-3]),$$

and use it to check the other parts of condition 1(b).

To check condition 2 notice that

$$((T\cap X_{n_i})\cap(C_1+C_2+\cdots+C_{i+2}))\\ \times([i, i+1]+[2m-4-i, 2m-3-i])\subset \mathrm{Cl}\,(E^4-K_{i+2})$$

so $f=h_1h_2\cdots h_{i+1}$ on this set; by definition of the functions $\{h_i\}$ the image of this set under f will be contained in $(C_{i+1}+C_{i+2})\times[0, 2m-3]$.

Lemma 3 may be used to prove Theorem 3 in the same manner that Lemma 1 of [4] was used to prove Theorem 2 of that paper. The function f of Lemma 3 is extended to satisfy Theorem 3. Theorem 3 may then be used to prove Theorem 1 for the case $n=3$ in the same manner that Bing used Theorem 2 of [4] to prove that the dogbone space is a factor of E^4.

For $n>3$ the same proof applies if we replace E^3 by E^n, E^4 by E^{n+1}, and 4-cells by $n+1$ cells in the appropriate places.

The following corollaries to Theorem 1 follow because in each case the set of nondegenerate elements of the decomposition is a Cantor set of arcs.

COROLLARY 2. *If G is the "unused example" decomposition defined by Bing in* [6] *then $E^3/G\times E^1=E^4$.*

COROLLARY 3. *If G is the toroidal decomposition described by Armentrout and Bing in* [1] *then $E^3/G\times E^1=E^4$.*

COROLLARY 4. *If G is the decomposition described by Bing on pp.* 14–17 *of* [6] *then $E^3/G\times E^1=E^4$.*

The methods used to prove Theorem 1 may be applied to prove the following theorems. In each case G is a monotone decomposition of E^n into points and a Cantor set of arcs.

THEOREM 4. *The cartesian product of the one point compactification of E^n/G with E^1 is homeomorphic to $S^n\times E^1$.*

THEOREM 5. $E^n/G\times S^1=E^n\times S^1$.

THEOREM 6. *The cartesian product of the one point compactification of E^n/G with S^1 is homeomorphic to $S^n\times S^1$.*

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