TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 149, May 1970

ON THE MANN ITERATIVE PROCESS

BY
W. G. DOTSON, JR.

1. **Introduction.** A self-mapping T of a Banach space E is said to be *nonexpansive* provided $\|Tx-Ty\| \leqq \|x-y\|$ for all $x, y \in E$, and is said to be *quasi-nonexpansive* provided that if $Tp=p$ then $\|Tx-p\| \leqq \|x-p\|$ for all $x \in E$ (i.e., T is nonexpansive about each of its fixed points). Nonexpansive mappings are clearly quasi-nonexpansive, and linear quasi-nonexpansive mappings are nonexpansive; but it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive, e.g. $Tx=(x/2)\sin(1/x)$, $T(0)=0$, on E^1. The concept of quasi-nonexpansiveness is closely related to some ideas which have been investigated recently by J. B. Diaz and F. T. Metcalf [2]. A mapping T is said to be quasi-nonexpansive on a subset C of E provided T maps C into C, and if $p \in C$ and $Tp=p$ then $\|Tx-p\| \leqq \|x-p\|$ holds for all $x \in C$.

In this paper, an iterative process introduced by W. R. Mann [7] is applied to the approximation of fixed points of quasi-nonexpansive mappings in Hilbert space and in uniformly convex and strictly convex Banach spaces. As corollaries, we obtain some results of M. A. Krasnosel'skiĭ [6], H. Schaefer [12], F. E. Browder and W. V. Petryshyn [1], and M. Edelstein [5]. An affirmative answer is obtained for a recent conjecture of C. L. Outlaw and C. W. Groetsch [10], and a partial affirmative answer is obtained for a conjecture of H. Schaefer (for which Z. Opial [9] has recently obtained another partial affirmation).

2. **The Mann iterative process.** Suppose $A=[a_{nj}]$ is an infinite real matrix satisfying (A1) $a_{nj} \geqq 0$ for all n, j, and $a_{nj}=0$ for $j>n$; (A2) $\sum_{j=1}^{n} a_{nj}=1$ for all n; (A3) $\lim_n a_{nj}=0$ for all j. Suppose E is a linear space, C is a convex subset of E, T is a mapping of C into C, and $x_1 \in C$. Then the *Mann iterative process* $M(x_1, A, T)$ is defined by $v_n=\sum_{j=1}^{n} a_{nj}x_j$, $x_{n+1}=Tv_n$, $n=1, 2, 3, \ldots$. W. R. Mann [7] introduced this process and proved that in case E is a Banach space, and C is closed, and T is continuous, then the convergence of either $\{x_n\}$ or $\{v_n\}$ to a point y implies the convergence of the other to y, and also implies $Ty=y$. Mann's proof is easily extended to a locally convex Hausdorff linear topological space E, by using the regularity of the matrix A together with properties of the continuous pseudo-norms which generate the topology of E. We state this as our first result.

THEOREM 1. *Suppose E is a locally convex Hausdorff linear topological space, C is a closed convex subset of E, $T\colon C \to C$ is continuous, $x_1 \in C$, and $A=[a_{nj}]$ satisfies*

Received by the editors August 8, 1968.

(A1), (A2), (A3). *If either of the sequences* $\{x_n\}$, $\{v_n\}$ *in the process* $M(x_1, A, T)$ *converges to a point* y, *then the other sequence also converges to* y, *and* $Ty=y$.

DEFINITION. A Mann process $M(x_1, A, T)$ is said to be *normal* provided $A=[a_{nj}]$ satisfies (A1), (A2), (A3), and (A4) $a_{n+1,j}=(1-a_{n+1,n+1})a_{nj}$, $j=1, 2, \ldots, n$; $n=1, 2, 3, \ldots$, and (A5) either $a_{nn}=1$ for all n, or $a_{nn}<1$ for all $n>1$.

THEOREM 2. *The following are true*: (a) *In order that* $M(x_1, A, T)$ *be a normal process, it is necessary and sufficient that* $A=[a_{nj}]$ *satisfy* (A1), (A2), (A4), (A5), *and* (A3′) $\sum_{n=1}^{\infty} a_{nn}$ *diverges.* (b) *The matrices* $A=[a_{nj}]$ (*other than the infinite identity matrix*) *in all normal Mann processes* $M(x_1, A, T)$ *are constructed as follows*: *Choose* $\{t_n\}$ *such that* $0 \leqq t_n < 1$ *for all n and* $\sum_{n=1}^{\infty} t_n$ *diverges, and define* $A=[a_{nj}]$ *by*: $a_{11}=1$, $a_{1j}=0$ *for* $j>1$; $a_{n+1,n+1}=t_n$, $n=1, 2, 3, \ldots$; $a_{n+1,j}=a_{jj}\prod_{i=j}^{n}(1-t_i)$ *for* $j=1, 2, \ldots, n$, *and* $a_{n+1,j}=0$ *for* $j>n+1$, $n=1, 2, 3, \ldots$. (c) *The sequence* $\{v_n\}$ *in a normal Mann process* $M(x_1, A, T)$ *satisfies*

$$v_{n+1} = (1-t_n)v_n + t_n T v_n$$

for all $n=1, 2, 3, \ldots$, *where* $t_n=a_{n+1,n+1}$.

Indication of proof. Statement (a) follows from a standard result in the theory of infinite products, viz. if $0 \leqq t_n < 1$ for each n, then $\lim_n \prod_{k=1}^{n}(1-t_k)=0$ if and only if the series $\sum_{k=1}^{\infty} t_k$ diverges. Intuitively, the divergence of the diagonal series is precisely what is required (in the presence of the other conditions on the matrix A) to keep enough weight on the diagonal so that all the column limits will be zero. The proofs of statements (b) and (c) consist of fairly simple algebraic computations. For example, one verifies that the construction in (b) yields a matrix A having the five properties (A1)–(A5), and, conversely, that any matrix A having the properties (A1)–(A5) can be obtained via the construction in (b), starting with t_n defined to be $a_{n+1,n+1}$ for all n. To obtain (c), one uses condition (A4) together with the original definition of the sequences $\{v_n\}$, $\{x_n\}$ in a Mann process $M(x_1, A, T)$.

Important examples of normal Mann processes can be obtained as follows. For each λ, $0 \leqq \lambda < 1$, we define an infinite matrix $A_\lambda=[a_{nj}]$ by setting $a_{n1}=\lambda^{n-1}$, $a_{nj}=\lambda^{n-j}(1-\lambda)$ for $j=2, 3, \ldots, n$, and $a_{nj}=0$ for $j>n$, $n=1, 2, 3, \ldots$. In case $\lambda=0$, we agree that $a_{nn}=1$ for all n, so that A_0 is the infinite identity matrix. It is easily checked that for each λ, $0 \leqq \lambda < 1$, $M(x_1, A_\lambda, T)$ is a normal Mann process. Since the diagonal sequence for A_λ is given by $t_n=a_{n+1,n+1}=1-\lambda$ for all $n=1, 2, 3, \ldots$, part (c) of Theorem 2 gives us that the sequence $\{v_n\}$ in the normal process $M(x_1, A_\lambda, T)$ satisfies $v_{n+1}=\lambda v_n+(1-\lambda)Tv_n$. Defining a mapping S_λ by $S_\lambda=\lambda I+(1-\lambda)T$ (where I is the identity mapping), we see that in the process $M(x_1, A_\lambda, T)$ the sequence $\{v_n\}$ satisfies $v_{n+1}=S_\lambda v_n=S_\lambda^n v_1=S_\lambda^n x_1$. Of course $S_0=T$, and so we have the ordinary Picard iterates $\{T^n x_1\}$ of T as the sequence $\{v_n\}$ in the normal Mann process $M(x_1, A_0, T)$. The sequence $\{S_{1/2}^n x_1\}$ of Picard iterates of $S_{1/2}=\frac{1}{2}(I+T)$ has been studied by Krasnosel'skiĭ [6] and Edelstein [5], and the sequence $\{S_\lambda^n x_1\}$ of Picard iterates of S_λ $(0<\lambda<1)$ has been studied by Schaefer [12], Browder and Petryshyn [1], and Opial [9]. Another interesting normal Mann process is $M(x_1, A_c, T)$ where A_c is the Cesàro matrix (defined by $a_{nj}=1/n$ for $1\leqq j\leqq n$, $a_{nj}=0$ for $j>n$); it has been studied by Mann [7].

3. **Quasi-nonexpansive mappings in strictly convex spaces.** A Banach space is said to be *strictly convex* provided that if $\|x+y\|=\|x\|+\|y\|$ and $x\neq 0$, $y\neq 0$, then $y=cx$, $c>0$. The following lemma is an immediate consequence of strict convexity.

LEMMA 1. *If E is a strictly convex Banach space, and u, $v\in E$, and $\|v\|\leqq\|u\|$, and $0<t<1$, and $\|(1-t)u+tv\|=\|u\|$, then $u=v$.*

We shall also have need of the following lemma which holds in any Banach space.

LEMMA 2. *Suppose C is a closed convex subset of a Banach space E, $T\colon C\to C$ is quasi-nonexpansive on C, $p\in C$ is a fixed point of T, and $x_1\in C$. Suppose $M(x_1, A, T)$ is any normal Mann process (with sequences $\{x_n\}$, $\{v_n\}$). Then the following are true*:

(1) $\|v_{n+1}-p\|\leqq\|v_n-p\|$ *for each* $n=1, 2, 3, \ldots$.

(2) *If $\{v_n\}$ clusters at p, then $\{v_n\}$ converges to p.*

(3) *If $\{v_n\}$ clusters at y and z, then $\|y-p\|=\|z-p\|$.*

Proof. From part (c) of Theorem 2 we have

$$v_{n+1}-p = (1-t_n)(v_n-p)+t_n(Tv_n-p)$$

where $t_n=a_{n+1,n+1}$. Since $\|Tv_n-p\|\leqq\|v_n-p\|$ we get

$$\|v_{n+1}-p\| \leqq (1-t_n)\|v_n-p\|+t_n\|v_n-p\| = \|v_n-p\|.$$

This proves (1). Statements (2) and (3) follow immediately from (1). Q.E.D.

THEOREM 3. *Suppose E is a strictly convex Banach space, C is a closed convex subset of E, $T\colon C\to C$ is continuous and quasi-nonexpansive on C, and $T(C)\subset K\subset C$ where K is compact. Suppose $x_1\in C$ and $M(x_1, A, T)$ is a normal Mann process such that $\{t_n\}=\{a_{n+1,n+1}\}$ clusters at some $t\in(0, 1)$. Then the sequences $\{x_n\}$, $\{v_n\}$ in the process $M(x_1, A, T)$ converge (strongly) to a fixed point of T.*

Proof. The closed convex hull of a set D is denoted by co D. Since co $K\subset C$, we have $T(\text{co } K)\subset T(C)\subset K\subset \text{co } K$, and by a theorem of Mazur [8] co K is compact. Since T is continuous, the fixed point theorem of Schauder [13] gives us that there exists $p\in\text{co } K$ such that $Tp=p$. There is a subsequence $\{t_m\}$ of $\{t_n\}$ such that $t_m\to t$. The corresponding subsequence $\{v_m\}$ of $\{v_n\}$ is contained in co $(K\cup\{x_1\})$ which is compact by Mazur's theorem. Hence, there is a subsequence $\{v_k\}$ of $\{v_m\}$ which converges to some $y\in C$. Of course $t_k\to t$. From part (c) of Theorem 2, and the continuity of T, we have

$$v_{k+1} = (1-t_k)v_k+t_kTv_k\to(1-t)y+tTy.$$

Since $\{v_n\}$ clusters at y and $(1-t)y+tTy$, and $Tp=p$, it follows from part (3) of Lemma 2 that

$$\|y-p\| = \|[(1-t)y+tTy]-p\|,$$

which can also be written

$$\|(1-t)(y-p)+t(Ty-p)\| = \|y-p\|.$$

Since $\|Ty-p\| \leqq \|y-p\|$ and $0<t<1$, we now have from Lemma 1 that $y-p = Ty-p$. So $Ty=y$, and since $\{v_n\}$ clusters at y it now follows from part (2) of Lemma 2 that $v_n \to y$. Theorem 1 gives us $x_n \to y$ also. Q.E.D.

REMARK. If T is nonexpansive and the normal Mann process used is $M(x_1, A_{1/2}, T)$, then Theorem 3 specializes to yield a result of Edelstein [5] which is in turn a generalization of a result of Krasnosel'skiĭ [6]. If T is nonexpansive and one uses $M(x_1, A_\lambda, T)$, $0<\lambda<1$, then Theorem 3 specializes to yield a generalization of a result of Schaefer [12] (i.e., his hypothesis of uniform convexity is replaced by our weaker hypothesis of strict convexity).

4. Quasi-nonexpansive mappings in uniformly convex spaces.

A Banach space is said to be *uniformly convex* provided that for each $\varepsilon>0$ there exists a $\delta>0$ such that if $\|x\|\leqq 1$, $\|y\|\leqq 1$, and $\|x-y\|\geqq\varepsilon$, then $\|\frac{1}{2}(x+y)\|\leqq 1-\delta$. By the Pettis-Milman theorem (e.g., see Pettis [11]) uniformly convex Banach spaces are reflexive, and so closed bounded convex sets are weakly compact in these spaces. It is easily seen that uniform convexity implies strict convexity, and so Theorem 3 holds in all uniformly convex Banach spaces. The following lemma is an easy consequence of uniform convexity.

LEMMA 3. *Suppose E is a uniformly convex Banach space. Suppose $0<a<b<1$, and $\{t_n\}$ is a sequence in $[a, b]$. Suppose $\{w_n\}$, $\{y_n\}$ are sequences in E such that $\|w_n\|\leqq 1$, $\|y_n\|\leqq 1$ for all n. Define $\{z_n\}$ in E by $z_n=(1-t_n)w_n+t_ny_n$. If $\lim \|z_n\|=1$, then $\lim \|w_n-y_n\|=0$.*

THEOREM 4. *Suppose E is a uniformly convex Banach space, C is a closed convex subset of E, T: $C \to C$ is quasi-nonexpansive on C and has at least one fixed point $p\in C$. Suppose $x_1\in C$ and $M(x_1, A, T)$ is a normal Mann process such that $\{t_n\} =\{a_{n+1,n+1}\}$ is bounded away from 0 and 1. Then each of the sequences $\{v_{n+1}-v_n\}$ and $\{Tv_n-v_n\}$ converges (strongly) to $0\in E$.*

Proof. From part (c) of Theorem 2 we have $\|v_{n+1}-v_n\|=t_n\|Tv_n-v_n\|$, and since we also have $0<a\leqq t_n\leqq b<1$, it follows that if either of the sequences in question converges to 0 then the other does also. If $\lim \|v_n-p\|=0$, then clearly $\lim \|v_{n+1}-v_n\|=0$. Otherwise, since by Lemma 2 $\{\|v_n-p\|\}$ is nonincreasing, we have $\lim \|v_n-p\|=d>0$. We define sequences $\{w_n\}$, $\{y_n\}$, $\{z_n\}$ by

$$w_n = (v_n-p)/\|v_n-p\|, \qquad y_n = (Tv_n-p)/\|v_n-p\|, \qquad z_n = (v_{n+1}-p)/\|v_n-p\|.$$

Then the equation in the proof of Lemma 2 can be divided through by $\|v_n-p\|$

to give $z_n=(1-t_n)w_n+t_n y_n$. Since $\|w_n\|=1$, $\|y_n\|\leqq 1$, and $\|z_n\|\to d/d=1$, it now follows from Lemma 3 that $\lim \|w_n-y_n\|=0$; and this immediately gives

$$\lim \|Tv_n-v_n\| = 0. \qquad \text{Q.E.D.}$$

The following corollary is a result of Browder and Petryshyn [1].

COROLLARY 1. *Suppose E is a uniformly convex Banach space, and T: $E\to E$ is nonexpansive and has at least one fixed point. For any λ, $0<\lambda<1$, the mapping $S_\lambda=\lambda I+(1-\lambda)T$ is nonexpansive, has the same fixed points as T, and is asymptotically regular, i.e., $\{S_\lambda^{n+1}x-S_\lambda^n x\}$ converges (strongly) to 0 for each $x\in E$.*

Proof. S_λ obviously is nonexpansive and has the same fixed points as T. Using the normal Mann process $M(x_1, A_\lambda, T)$ in Theorem 4, we get that for each $x_1\in E$ $\{v_{n+1}-v_n\}=\{S_\lambda^n x_1-S_\lambda^{n-1}x_1\}\to 0$. Q.E.D.

The following two theorems are further generalizations of results of Browder and Petryshyn [1].

THEOREM 5. *Suppose E is a uniformly convex Banach space, C is a closed convex subset of E, T: $C\to C$ is quasi-nonexpansive on C and has at least one fixed point $p\in C$, and $I-T$ is closed. Suppose $x_1\in C$ and $M(x_1, A, T)$ is a normal Mann process such that $\{t_n\}=\{a_{n+1,n+1}\}$ is bounded away from 0 and 1. If $\{v_n\}$ clusters (strongly) at some $y\in C$, then $Ty=y$ and the sequences $\{x_n\}$, $\{v_n\}$ converge (strongly) to y.*

Proof. There is a subsequence $\{v_m\}$ of $\{v_n\}$ such that $v_m\to y$. It follows from Theorem 4 that $(I-T)v_n\to 0$, and so $(I-T)v_m\to 0$. Since $I-T$ is closed, we have $(I-T)y=0$. Since $Ty=y$ and $\{v_n\}$ clusters at y, it now follows from Lemma 2 that $v_n\to y$. Since $v_n-x_{n+1}=v_n-Tv_n\to 0$, we get $x_n\to y$ also. Q.E.D.

REMARK. We note that if T is continuous on C (as would be the case if T were nonexpansive, for example), then $I-T$ is also continuous on C, and hence closed.

A mapping S: $C\to E$ is said to be *demiclosed* (Browder and Petryshyn [1], Opial [9]) provided that if $\{u_n\}$ is a sequence in C which converges weakly to $u\in C$, and if $\{Su_n\}$ converges strongly to $v\in E$, then $Su=v$. For C closed and convex (and so weakly closed) every weakly continuous self-mapping of C is weakly closed, and every weakly closed self-mapping is demiclosed. We continue to use $\to$ to denote strong convergence, and we use $\rightharpoonup$ to denote weak convergence.

THEOREM 6. *Suppose E is a uniformly convex Banach space, C is a closed convex subset of E, T: $C\to C$ is quasi-nonexpansive on C and has at least one fixed point $p\in C$. Suppose $x_1\in C$ and $M(x_1, A, T)$ is a normal Mann process such that $\{t_n\}=\{a_{n+1,n+1}\}$ is bounded away from 0 and 1. Then the following are true:* (a) *There is a subsequence of $\{v_n\}$ which converges weakly to some $y\in C$, and if $I-T$ is demiclosed then each weak subsequential limit point of $\{v_n\}$ is a fixed point of T.* (b) *If $I-T$ is demiclosed and T has only one fixed point $p\in C$, then the sequences $\{x_n\}$, $\{v_n\}$ converge weakly to p.* (c) *If $I-T$ is weakly closed, then each weak cluster point of $\{v_n\}$ is a fixed point of T.*

Proof. (a) If $x_1=v_1=p$ then $v_n \to p$ and we are through. Suppose $\|v_1-p\| = r>0$, and let $S_r(p)=\{x : \|x-p\| \leq r\}$, and let $D=C \cap S_r(p)$. Then D is weakly compact (since closed, bounded, and convex), and from the Eberlein-Šmulian theorem (e.g., see Dunford-Schwartz [3, p. 430]) it follows that D is weakly sequentially compact. Since by part (1) of Lemma 2 $\{v_n\} \subset D$, there is a subsequence of $\{v_n\}$ which converges weakly to some $y \in D \subset C$. Now suppose $\{v_m\}$ is a subsequence of $\{v_n\}$ such that $v_m \rightharpoonup z$. Then $z \in D$ since D is weakly closed. By Theorem 4 we have $(I-T)v_m \to 0$, and so if $I-T$ is demiclosed then $(I-T)z=0$. (b) If $\{v_n\}$ does not converge weakly to p then there is a weakly open set U with $p \in U$ such that $v_n \in U^c \cap D$ for infinitely many n. So there is a subsequence $\{v_k\}$ of $\{v_n\}$ which is contained in the weakly compact set $U^c \cap D$; and there is a subsequence $\{v_m\}$ of $\{v_k\}$ such that $v_m \rightharpoonup z \in U^c \cap D$. But from (a) it now follows that $Tz=z$, which is impossible since $z \neq p$. Hence $v_n \rightharpoonup p$; and since $x_{n+1}-v_n=Tv_n-v_n \to 0$ we have $x_{n+1}=(x_{n+1}-v_n)+v_n \rightharpoonup 0+p=p$. (c) Suppose y is a weak cluster point of $\{v_n\}$; then $y \in D$ since D is weakly closed. There is a subnet $\{v_{n(\alpha)}\}$ of $\{v_n\}$ such that $v_{n(\alpha)} \rightharpoonup y$. Since $\{(I-T)v_{n(\alpha)}\}$ is a subnet of $\{(I-T)v_n\}$, it follows from Theorem 4 that $(I-T)v_{n(\alpha)} \to 0$, and so $(I-T)v_{n(\alpha)} \rightharpoonup 0$. With $I-T$ weakly closed, it now follows that $(I-T)y=0$. Q.E.D.

REMARK 1. If C is bounded (as well as closed and convex) then: (i) If T is weakly continuous then T has at least one fixed point by the Tychonoff fixed point theorem (e.g., see [3, p. 456]); and (ii) If T is nonexpansive, then T has at least one fixed point by the Browder-Kirk fixed point theorem (e.g., see [1, p. 572]).

REMARK 2. In case T is nonexpansive and the uniformly convex Banach space E has a weakly continuous duality mapping, it has been shown by Opial [9] that $I-T$ is necessarily demiclosed. A conjecture of H. Schaefer (as recently stated by Opial [9]) is that if C is a closed convex subset of a uniformly convex Banach space E, and T: $C \to C$ is nonexpansive and weakly continuous and has at least one fixed point $p \in C$, and $0<\lambda<1$, and $x_1 \in C$, then $\{S_\lambda^n x_1\}$ converges weakly to a fixed point of T. Opial [9] showed that this conjecture is true even without the assumption of weak continuity provided the space E has a weakly continuous duality mapping. But, as shown by Opial [9], there are some uniformly convex Banach spaces that do not have weakly continuous duality mappings (e.g., L^p, $1<p<\infty$, $p \neq 2$). Using $M(x_1, A_\lambda, T)$ in Theorem 6, and noting that T weakly continuous implies $I-T$ demiclosed, we find in part (b) that if T has only one fixed point $p \in C$ then $v_{n+1}=S_\lambda^n x_1 \rightharpoonup p$; and this is valid in any uniformly convex Banach space.

A closely related conjecture has recently been proposed by Outlaw and Groetsch [10], viz. if E is a uniformly convex Banach space and T: $E \to E$ is nonexpansive and linear (and therefore weakly continuous), and $x_1 \in E$, then $\{S_{1/2}^n x_1\}$ converges (strongly) to a fixed point of T. Setting $\lambda=\frac{1}{2}$ in the following theorem provides an affirmative answer to this conjecture.

THEOREM 7. *Suppose E is a uniformly convex Banach space, T: $E \to E$ is non-*

expansive and linear, and $0<\lambda<1$, *and* $x_1 \in E$. *Then* $\{S_\lambda^n x_1\}$ *converges (strongly) to a fixed point of* T.

Proof. The proof consists of showing that the sequence $\{S_\lambda^n\}$ of linear operators is a system of almost invariant integrals for the semigroup $\{T^m : m=0, 1, 2, \ldots\}$, and then applying the mean ergodic theorem of Eberlein [4]. T has at least one fixed point, viz. $p=0$; and, of course, for any $x \in E$ the sequence $\{S_\lambda^n x\}$ is the sequence $\{v_{n+1}\}$ in the normal Mann process $M(x, A_\lambda, T)$. Hence from Theorem 4 we have $TS_\lambda^n x - S_\lambda^n x \to 0$ for all $x \in E$. Since T is linear and continuous, we have

$$T^2 S_\lambda^n x - S_\lambda^n x = T(TS_\lambda^n x - S_\lambda^n x) + (TS_\lambda^n x - S_\lambda^n x) \to T(0)+0 = 0,$$

and by induction we get $T^m S_\lambda^n x - S_\lambda^n x \to 0$ as $n \to \infty$ for all $x \in E$ and all $m=0, 1, 2, \ldots$. Since T is linear, we have

$$S_\lambda^n = [\lambda I + (1-\lambda)T]^n = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j T^j,$$

so that S_λ^n is a convex combination of $I, T, T^2, \ldots, T^n$. Thus, for each n and for each $x \in E$, $S_\lambda^n x$ is in the convex hull of $x, Tx, T^2x, \ldots, T^n x$. Also we get

$$\|S_\lambda^n\| \leqq \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j \|T\|^j \leqq 1$$

for all n. Finally, since S_λ^n is a polynomial in T, we have $T^m S_\lambda^n = S_\lambda^n T^m$, and so $S_\lambda^n T^m x - S_\lambda^n x \to 0$ as $n \to \infty$ for all $x \in E$ and all $m=0, 1, 2, \ldots$. This completes the proof that the sequence $\{S_\lambda^n\}$ of linear operators forms a system of almost invariant integrals for the semigroup $\{T^m : m=0, 1, 2, \ldots\}$. Since by Lemma 2 (part (1)) $\{S_\lambda^n x_1\}$ lies in the weakly compact set $\{x : \|x\| \leqq \|x_1\|\}$, $\{S_\lambda^n x_1\}$ has a weak cluster point y. From Eberlein's mean ergodic theorem [4], it now follows that $S_\lambda^n x_1 \to y$ and $T^m y = y$, $m=0, 1, 2, \ldots$. Q.E.D.

EXAMPLE. The following example shows the importance of uniform convexity in several of the above results. In the (not uniformly convex) space l^1, define $T(u_1, u_2, \ldots) = (0, u_1, u_2, \ldots)$. Then T is nonexpansive and linear, and has $(0, 0, 0, \ldots)$ as its unique fixed point. But for $x_1 = (1, 0, 0, \ldots)$ it is easily seen that $\|S_{1/2}^n x_1\| = 1$ for all n. So $\{S_{1/2}^n x_1\}$ does not converge strongly to $(0, 0, 0, \ldots)$. Since weak convergence is the same as strong convergence in l^1, we get that $\{S_{1/2}^n x_1\}$ does not converge weakly to $(0, 0, 0, \ldots)$.

5. **Quasi-nonexpansive mappings in Hilbert space.** Since Hilbert space is uniformly convex (an easy consequence of the parallelogram law), all of the above results hold in Hilbert space. The following theorem generalizes a result of Schaefer [12].

THEOREM 8. *Suppose* E *is a real Hilbert space,* C *is a closed convex subset of* E, $T: C \to C$ *is quasi-nonexpansive on* C *and has at least one fixed point* $p \in C$, *and* $I-T$ *is demiclosed. Suppose* $x_1 \in C$ *and* $M(x_1, A, T)$ *is a normal Mann process* *such that* $\{t_n\}=\{a_{n+1,n+1}\}$ *is bounded away from* 0 *and* 1. *Then the sequences* $\{x_n\}$, $\{v_n\}$ *converge weakly to a fixed point of* T.

Proof. From Theorem 6 (part (a)) we have that there exists $p_1 \in C$ such that $Tp_1=p_1$ and p_1 is a weak subsequential limit point of $\{v_n\}$. If $\{v_n\}$ does not converge weakly to p_1 then, as in the proof of part (b) of Theorem 6, $\{v_n\}$ must have at least one other weak subsequential limit point $p_2 \in C$ such that $Tp_2=p_2$ and $p_2 \neq p_1$. We proceed below to obtain the contradiction that $p_1=p_2$, thereby proving that $\{v_n\}$ does converge weakly to p_1. It will then follow exactly as in the proof of Theorem 6 (part (b)) that $\{x_n\}$ also converges weakly to p_1. Since by Lemma 2 (part (1)) each of the sequences $\{\|v_n-p_1\|\}$, $\{\|v_n-p_2\|\}$ is nonincreasing, we have $\lim \|v_n-p_1\|=\inf \|v_n-p_1\|=r_1>0$, and $\lim \|v_n-p_2\|=\inf \|v_n-p_2\|=r_2>0$ (of course, if either r_1 or r_2 is 0 then $\{v_n\}$ is strongly convergent to a fixed point of T). For each positive integer k, let

$$C_k = \{x : r_1 \leqq \|x-p_1\| \leqq r_1+(1/k) \text{ and } r_2 \leqq \|x-p_2\| \leqq r_2+(1/k)\}.$$

Then for each k there is a positive integer N_k such that for all $n \geqq N_k$ $v_n \in C_k \subset \text{co } C_k$ = a weakly closed set (since closed and convex). Hence, for each k, co C_k contains all weak cluster points of $\{v_n\}$; and so p_1 and p_2 are both in $\bigcap_{k=1}^{\infty} \text{co } C_k$. Squaring the inequalities in the definition of C_k, using $\|x-p_i\|^2=\|x\|^2-2(x,p_i)+\|p_i\|^2$, $i=1, 2$, and, finally, multiplying the second inequality through by -1 and adding to the first inequality, we find that, for each $x \in C_k$,

$$m_k \leqq (x, p_2-p_1) \leqq M_k$$

where $m_k=m-r_2/k-1/2k^2$, $M_k=m+r_1/k+1/2k^2$, and

$$m = \tfrac{1}{2}(r_1^2-r_2^2-\|p_1\|^2+\|p_2\|^2).$$

A continuous linear functional g on E is defined by $g(x)=(x, p_2-p_1)$, and so the sets $s_k=\{x : g(x) \geqq m_k\}$, $S_k=\{x : g(x) \leqq M_k\}$ are closed convex half-spaces. For each k we have $C_k \subset s_k \cap S_k$; whence co $C_k \subset s_k \cap S_k$ also. Thus

$$\bigcap_{k=1}^{\infty} \text{co } C_k \subset \bigcap_{k=1}^{\infty} (s_k \cap S_k) = \{x : g(x) = m\},$$

since $m_k \to m$ (monotonically increasing) and $M_k \to m$ (monotonically decreasing) as $k \to \infty$. So p_1 and p_2 are both in the hyperplane $\{x : g(x)=m\}$; and therefore $\|p_2-p_1\|^2=(p_2-p_1, p_2-p_1)=g(p_2-p_1)=g(p_2)-g(p_1)=m-m=0$, so that $p_1=p_2$. As indicated above, this completes the proof. Q.E.D.

REMARK. If T is a weakly continuous nonexpansive mapping, and the normal Mann process used is $M(x_1, A_\lambda, T)$, $0<\lambda<1$, then Theorem 8 specializes to give a result of Schaefer [12]. Our method of proof seems to be considerably simpler than the method used by Schaefer and subsequently refined by Opial [9].

ACKNOWLEDGMENT. The author would like to thank Professor W. Robert Mann for his guidance, encouragement, and assistance in this work.

Bibliography

1. F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc. **72** (1966), 571–575. MR **32** #8155b.

2. J. B. Diaz and F. T. Metcalf, *On the structure of the set of subsequential limit points of successive approximations*, Bull. Amer. Math. Soc. **73** (1967), 516–519. MR **35** #2268.

3. N. Dunford and J. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

4. W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67** (1949), 217–240. MR **12**, 112.

5. M. Edelstein, *A remark on a theorem of M. A. Krasnoselski*, Amer. Math. Monthly **73** (1966), 509–510. MR **33** #3072.

6. M. A. Krasnosel'skiĭ, *Two remarks on the method of successive approximations*, Uspehi Mat. Nauk **10** (1955), no. 1 (63), 123–127. (Russian) MR **16**, 833.

7. W. Robert Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510. MR **14**, 988.

8. S. Mazur, *Über die kleinste konvexe Menge, die eine gegebene kompakte Menge enthält*, Studia Math. **2** (1930), 7–9.

9. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597. MR **35** #2183.

10. C. L. Outlaw and C. W. Groetsch, *Averaging iteration in a Banach space*, Notices Amer. Math. Soc. **15** (1968), 180. Abstract #653–342.

11. B. J. Pettis, *A proof that every uniformly convex space is reflexive*, Duke Math. J. **5** (1939), 249–253.

12. H. Schaefer, *Über die Methode sukzessiver Approximationen*, Jber. Deutsch. Math.-Verein. **59** (1957), 131–140. MR **18**, 811.

13. J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Math. **2** (1930), 171–180.

North Carolina State University,
Raleigh, North Carolina 27607