IDENTITIES INVOLVING THE COEFFICIENTS OF
A CLASS OF DIRICHLET SERIES. IV

BY
BRUCE C. BERNDT(1)

Abstract. We consider a class of Dirichlet series satisfying a functional equation with gamma factors. We define a generalized Dirichlet series that is analogous to the generalized zeta-function of Riemann. An analytic continuation for these generalized series is derived, and a few simple properties are established. Secondly, we prove a theorem on the Abel summation of Dirichlet series that satisfy Hecke’s functional equation.

1. Introduction. We consider here two somewhat related problems associated with Dirichlet series satisfying a functional equation with \( \Gamma \)-factors. Let \( \varphi(s) = \sum_{n=1}^{\infty} a(n) \lambda_n^{-s} \) be a Dirichlet series satisfying Hecke’s functional equation, where \( s = \sigma + it \) with \( \sigma \) and \( t \) both real, and \( \sigma > \sigma_a \), the abscissa of absolute convergence of \( \varphi \). In [2] we defined for \( \sigma > \sigma_a \) and \( \alpha > 0 \) the generalized Dirichlet series

\[
\varphi(s, \alpha) = \sum_{n=1}^{\infty} a(n)(\lambda_n + \alpha)^{-s},
\]

which is analogous to the Hurwitz or generalized zeta-function of Riemann. We derived an analytic continuation for \( \varphi(s, \alpha) \) and determined a few of its properties. In [3] we found an easier derivation of our analytic continuation. The method we used in [2] is used here to derive an analytic continuation and some simple properties for generalized Dirichlet series arising from Dirichlet series satisfying a much more general functional equation, but we do not, in general, obtain a simple formula for the analytic continuation of \( \varphi(s, \alpha) \). In particular, we shall obtain a new and simple method of continuing the Hurwitz zeta-function \( \zeta(s, \alpha) \).

In the above problem we perturbed the Dirichlet series by replacing \( \lambda_n \) by \( \lambda_n + \alpha \). Next, we consider a different type of problem where we perturb the series by multiplying the terms of the series by \( \exp(-\lambda_n \delta) \), \( \delta > 0 \). As we shall see, our result can be interpreted as a modified form of Abel summation. Unfortunately, in this problem our method is only applicable to series which are solutions to Hecke’s functional equation.
2. Notation, definition, and preliminary results. In the sequel we write $z = x + iy$ with $x$ and $y$ both real. $A$ always denotes a positive constant, not necessarily the same with each occurrence. The summation sign $\sum$ appearing with no indices will always mean $\sum_{n=1}^{\infty}$. If $b$ is real, we write $\int_{b}^{1+i\infty}$.

We suppose that

$$
\varphi(s) = \sum a(n)\lambda_n^{-s}, \quad \psi(s) = \sum b(n)\mu_n^{-s}
$$

are solutions of the functional equation

$$
(2.1) \quad \Delta(s)\varphi(s) = \Delta(r-s)\psi(r-s)
$$

where $r$ is real and

$$
\Delta(s) = \prod_{k=1}^{N} \Gamma(\alpha_k s + \beta_k),
$$

where $\alpha_k > 0$ and $\beta_k$ is complex, $k = 1, \ldots, N$. For a more complete definition of these series see [3] or [5]. If $A(s) = \Gamma(s)$, we have Hecke's functional equation

$$
(2.2) \quad \Gamma(s)\varphi(s) = \Gamma(r-s)\psi(r-s).
$$

For $x > 0$ and $0 < c < \sigma$ [6, p. 311],

$$
(2.3) \quad \frac{1}{2\pi i} \int_{(c)} \Gamma(s-z)\Gamma(z)x^{-z} \, dz = \Gamma(s)(1+x)^{-z}.
$$

If $F(\alpha, \beta; \gamma; z)$ denotes the hypergeometric function, for $\sigma > 0$, [6, p. 310],

$$
(2.4) \quad \int_{0}^{\infty} \frac{x^{s-1}}{(1+x)^\gamma} \, dx = \frac{b^s}{s} F(v, s; s+1; -b).
$$

Also [7, p. 1043],

$$
(2.5) \quad F(v, s; s+1; -1/x) = \frac{s}{s-v} x^v F(v, v-s); v+1-s; -x) + \frac{\Gamma(s+1)\Gamma(v-s)}{\Gamma(v)} x^s.
$$


**Theorem 3.1.** Let $\varphi(s)$ satisfy (2.1) and let $\varphi(s, a), a > 0$, be its generalized Dirichlet series. Assume that the singularities of $\varphi$ are poles. (The number of poles is finite, as they are contained in a compact set.) Let $\sigma_\alpha$ and $\sigma_\psi^*$ denote the abscissas of absolute convergence of $\varphi$ and $\psi$, respectively. Choose $c > \sup (0, \sigma_\alpha, \sigma_\psi^*)$ such that the line $x = r - c$ does not contain a pole of $\Gamma(z)\varphi(z)$. Let $R(s, a)$ denote the sum of the residues of $\Gamma(s-z)\Gamma(z)\varphi(z)a^z$ at the poles of $\Gamma(z)\varphi(z)$ in the strip $r - c < x < c$. Then, for $\sigma > r - c$,

$$
(3.1) \quad \Gamma(s)\alpha^s\varphi(s, a) = \alpha^s \sum b(n)f(s, a\mu_n) + R(s, a),
$$

where

$$
f(s, w) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(z+s-r)\Gamma(r-z)\Delta(z)}{\Delta(r-z)} w^{-z} \, dz.
$$
Remarks. By taking $c$ large, the analytic continuation of $\varphi(s, a)$ may be taken to the left as far as we wish. The assumption that the singularities of $\varphi$ are at most poles easily insures the analytic continuation of $R(s, a)$.

Proof. For $\sigma > c > \text{sup} (0, \sigma_0, \sigma_0^*)$, by (2.3) we have

$$\frac{1}{2\pi i} \int_{(c)} \Gamma(s - z) \Gamma(z) \varphi(z) a^z \, dz = \Gamma(s) a^s \varphi(s, a),$$

where the inversion in order of summation and integration is justified by absolute convergence. We now move the line of integration to $x = r - c$ by integrating around the boundary of the rectangle with vertices $c + i Y$ and $r - c + i Y$ and then letting $Y$ tend to $\infty$. With the use of Stirling’s formula and a Phragmén-Lindelöf theorem it is easy to show that the integrals over the horizontal sides tend to 0 as $Y$ tends to $\infty$. We arrive at for $\sigma > c$,

$$\frac{1}{2\pi i} \int_{(c)} \Gamma(s - z) \Gamma(z) \varphi(z) a^z \, dz = I(s, a) + R(s, a),$$

where

$$I(s, a) = \frac{1}{2\pi i} \int_{(r - c)} \Gamma(s - z) \Gamma(z) \varphi(z) a^z \, dz$$

$$= \frac{1}{2\pi i} \int_{(c)} \Gamma(s + z - r) \Gamma(r - z) \varphi(z) a^z \frac{\Delta(r)}{\Delta(r - z)} \, dz$$

$$= a^r \sum b(n) f(s, a) \mu_n.$$

Here we have replaced $z$ by $r - z$, used (2.1), and then inverted the order of integration and summation by absolute convergence.

Now, $f(s, w)$ is an analytic function of $s$ in the rectangle $R$ defined by $r - c + \epsilon \leq \sigma \leq \gamma$ and $-T \leq t \leq T$, where $\epsilon > 0, \gamma > c$, and $T > 0$. Choose $Y$ so that $Y > T$ and $| - b_k / \alpha_k | < Y$, where $b_k = \text{Im} (\beta_k), k = 1, \ldots, N$. Clearly, for $s$ in $R$,

$$\left| \int_{c - iY}^{c + iY} \Gamma(z + s - r) \Gamma(r - z) \varphi(z) a^z \frac{\Delta(z)}{\Delta(r - z)} \, dz \right| \leq A \mu_n^{-\epsilon},$$

where $A$ is independent of $s$ in $R$. By Stirling’s formula for $s$ in $R$,

$$\left| \int_{c + iY}^{c + i\infty} \Gamma(z + s - r) \Gamma(r - z) \varphi(z) a^z \frac{\Delta(z)}{\Delta(r - z)} \, dz \right| \leq A \mu_n^{-\epsilon} \int_{Y}^{\infty} e^{-\pi u (y + t)^{c + s - r - 1/2} \Gamma(r - c - 1/2)} \prod_{k=1}^{N} \left( \alpha_k y + b_k \right)^{\varphi_k(2c - r)} \, dy$$

$$\leq A \mu_n^{-\epsilon},$$

where $A$ is independent of $s$ in $R$. A similar estimate holds for the integral over $(c - i Y, c - i \infty)$. Thus,

$$|I(s, a)| \leq A \sum |b(n)| \mu_n^{-\epsilon},$$
where $A$ is independent of $s$ in $R$. Hence, by Weierstrass’ $M$-test $I(s, a)$ converges uniformly on $R$ and represents an analytic function there since $f(s, a, u_n)$ is analytic, $n = 1, 2, \ldots$. Thus, by analytic continuation, (3.2) and (3.3), we have shown (3.1) for $s \in \{s : \sigma > c\} \cup R$. Since $\varepsilon > 0$ can be taken arbitrarily small and $T$ can be chosen arbitrarily large, (3.1) is valid for $\sigma > r - c$.

The following corollaries are easily deduced from Theorem 3.1

**COROLLARY 3.2.** If $\varphi(s)$ is entire, then $\varphi(s, a)$ is entire.

**COROLLARY 3.3.** Let $\varphi(s)$ have a pole of order $n$ at $s = k$. If $k$ is a positive integer, then $\varphi(s, a)$ has poles of order $n$ at $s = 1, \ldots, k$ and poles of order $n - 1$ at $s = 0, -1, -2, \ldots$. If $k$ is not a positive integer, then $\varphi(s, a)$ has poles of order $n$ at $s = k, k - 1, k - 2, \ldots$.

**Proof.** The conclusion follows immediately from an examination of $R(s, a)/\Gamma(s)$, for $\Gamma(s-k)$ and its derivatives have poles at $s = k, k - 1, k - 2, \ldots$, and $1/\Gamma(s)$ has simple zeros at $s = 0, -1, -2, \ldots$.

**Example 3.1.** Let $\varphi(s) = \pi^{-s/2} \zeta(s)$, where $\zeta(s)$ is the Riemann zeta-function. Let $c = m + \frac{1}{2}$, where $m$ is a positive integer. Replacing $a$ by $\pi^{1/2} a$, we have the following representation for the Hurwitz zeta-function $\zeta(s, a)$ for $\sigma > \frac{1}{2} - m$:

$$
\zeta(s, a) - a^{-s} = \frac{a^{-s}}{\Gamma(s)} \left( \Gamma(s-1)a + \sum_{j=0}^{n-1} \frac{(-1)^j \Gamma(s+j) \zeta(-j)}{j! a^j} + F(s) \right),
$$

where $F(s)$ is analytic for $\sigma > \frac{1}{2} - m$. We see from (3.4) that $\zeta(s, a)$ is analytic except for a simple pole at $s = 1$ with residue 1. (3.4) yields immediately a simple formula for $\zeta(n, a)$, where $n$ is a nonpositive integer. Thus, $\zeta(0, a) = -a + \frac{1}{2}, \quad \zeta(-1, a) = -\frac{1}{2} a^2 + \frac{1}{2} a - 1/12$, etc. For other methods of continuing $\zeta(s, a)$ and calculating $\zeta(n, a)$ see [9, p. 37] and [10, pp. 266–267].

**Example 3.2.** Let $\chi$ be a nonprincipal, primitive character modulo $k$. Then, the Dirichlet L-function $L(s, \chi)$ satisfies the functional equation

$$
R(s, \chi) = (\pi/k)^{(s+b)/2} \Gamma(\frac{1}{2} s + b) L(s, \chi) = e(\chi) R(1-s, \bar{\chi}),
$$

where $b = 0$ if $\chi(-1) = 1$, $b = 1$ if $\chi(-1) = -1$, and $|e(\chi)| = 1$. Let $c$ be as in Example 3.1 and replace $a$ by $(\pi/k)^{1/2} a$. Then, putting

$$
L(s, \chi, a) = \sum \chi(n)(n+a)^{-s} \quad (\sigma > 0),
$$

we have for $\sigma > \frac{1}{2} - m$

$$
L(s, \chi, a) = \frac{a^{-s}}{\Gamma(s)} \left( \sum_{j=0}^{n-1} \frac{(-1)^j \Gamma(s+j) L(-j, \chi)}{j! a^j} + G(s) \right),
$$

where $G(s)$ is analytic. $L(s, \chi, a)$ is an entire function, and we can easily calculate $L(n, \chi, a)$, where $n$ is a nonpositive integer. In particular, $L(0, \chi, a) = L(0, \chi)$.

**Example 3.3.** Let $K$ be an algebraic number field of degree $r_1 + 2r_2$, where $r_1$ is the number of real conjugates of $K$ and $2r_2$ the number of imaginary conjugates.
Then there exists a positive constant $B$, depending upon $K$, such that the Dedekind zeta-function $\zeta_K(s)$ satisfies the functional equation

$$
\zeta(s) = B^{-s} \Gamma(1/2 \sigma) \Gamma(\sigma) \zeta_K(s) = \xi(1-s)
$$

and is analytic save for a simple pole at $s=1$. The generalized Dedekind zeta-function

$$
\zeta_K(s, a) = \sum F(n)(n+a)^{-s} \quad (\sigma > 1),
$$

where $F(n)$ is the number of nonzero integral ideals of norm $n$ in $K$, has an analytic continuation that is analytic except for a simple pole at $s=1$. If $n$ is a nonpositive integer, $\zeta_K(n, a)$ can be easily calculated in the same manner as above.

4. Hecke's functional equation and Abel summation. Let $\varphi$ and $\psi$ satisfy (2.2), and for $x>0$ let

$$
\Phi(x) = \sum a(n) \exp \left[ -\lambda_n x \right], \quad \Psi(x) = \sum b(n) \exp \left[ -\mu_n x \right].
$$

Bochner [4] has shown that

$$
\Phi(x) = x^{-\eta} \Gamma(1/x) + P(x),
$$

where

$$
P(x) = \frac{1}{2\pi i} \int_C \chi(s)x^{-s} \, ds,
$$

$\chi(s) = \Gamma(s)\varphi(s)$ and $C$ denotes a curve, or curves, encircling the singularities of $\chi$.

Our next theorem is an easy consequence of (4.1).

**Theorem 4.1.** Suppose that $\chi$ has at most simple poles. Let the poles be at $s_1, \ldots, s_m$ with residues $\rho_1, \ldots, \rho_m$, respectively. For every $\delta > 0$ suppose that $\varphi(s, \delta) = \sum a(n) \lambda_n^{-s} \exp \left[ -\lambda_n \delta \right]$ converges for $\sigma > \sigma_0$, where $\sigma_0 \leq \sigma$. Then, for $\sigma > \sigma_0$,

$$
\varphi(s) = \lim_{\delta \to 0} \left\{ \varphi(s, \delta) - \sum_{k=1}^m \frac{\rho_k \Gamma(s_k - \delta)}{\Gamma(s_k)} \delta^{-s_k} \right\}.
$$

**Remarks.** The assumption that $\chi$ has simple poles is not strictly necessary. However, the calculations become harder, otherwise. Using a representation of $\zeta(s)$ as a contour integral, Atkinson [1] proved a result like that of (4.2) when $a(n) = 1$ and $\lambda_n = n$.

**Proof.** For $\sigma > \sup (0, \sigma_0)$, we have by (4.1),

$$
\Gamma(s)\varphi(s, \delta) = \int_0^{\infty} x^{\delta-1} \Phi(x+\delta) \, dx
$$

(4.3)

$$
= \int_0^1 x^{\delta-1} (x+\delta)^{-\eta} \Gamma(1/(x+\delta)) \, dx
$$

$$
+ \int_1^\infty x^{\delta-1} P(x+\delta) \, dx + \int_1^\infty x^{\delta-1} \Phi(x+\delta) \, dx.
$$
Since
\begin{equation}
P(y) = \sum_{k=1}^{m} \rho_k y^{-s_k},
\end{equation}
we find that
\begin{align*}
\int_{0}^{1} x^s \log \frac{1}{(1+x/\delta)} dx &= \sum_{k=1}^{m} \rho_k \log \frac{1}{(1+x/\delta)}
\end{align*}
from (2.4). It follows from (3.3) that
\begin{align*}
\Gamma(s)\phi(s, \delta) - s^{-1} \sum_{k=1}^{m} \rho_k \log \frac{1}{(1+x/\delta)} &= \int_{0}^{1} x^s \log \frac{1}{(1+x/\delta)} dx + \int_{1}^{\infty} x^s \Phi(x) dx.
\end{align*}
Since \( \phi(s, \delta) \) converges uniformly for \(|\arg (s-\sigma_0)| \leq \frac{1}{2} \pi - \epsilon \) for every \( \epsilon > 0 \), by analytic continuation (4.5) is valid for \( \sigma > \sigma_0 \). Since (4.3) is valid also for \( \delta = 0 \), we have from (4.4) for \( \sigma > \sup (0, \sigma_0) \),
\begin{equation}
\int_{0}^{1} x^s \log \frac{1}{(1+x/\delta)} dx + \int_{1}^{\infty} x^s \Phi(x) dx = \Gamma(s)\phi(s) - \sum_{k=1}^{m} \frac{\rho_k}{s-s_k}.
\end{equation}
By analytic continuation, (4.6) is valid for all \( s \). Hence, (4.5) and (4.6) yield for \( \sigma > \sigma_0 \),
\begin{equation}
\lim_{\delta \to 0} \left\{ \Gamma(s)\phi(s, \delta) - s^{-1} \sum_{k=1}^{m} \rho_k \log \frac{1}{(1+x/\delta)} \right\}
= \Gamma(s)\phi(s) - \sum_{k=1}^{m} \frac{\rho_k}{s-s_k}.
\end{equation}
Upon the use of (2.5) and the fact that \( \lim_{\delta \to 0} F(\alpha, \beta; \gamma; \delta) = 1 \), (4.7) becomes for \( \sigma > \sigma_0 \),
\begin{equation}
\lim_{\delta \to 0} \left\{ \Gamma(s)\phi(s, \delta) - s^{-1} \sum_{k=1}^{m} \rho_k \frac{\Gamma(s+1)\Gamma(s_k-s)}{\Gamma(s_k)} \delta^{s-s_k} \right\} = \Gamma(s)\phi(s),
\end{equation}
which is clearly equivalent to (4.2).

EXAMPLES. Suppose that \( f(s) = \sum a(n)n^{-s} \) is a Dirichlet series of signature \( (\lambda, r, \gamma) \) [8], where \( \lambda > 0 \), \( r > 0 \) and \( \gamma = \pm 1 \). Then, \( \lambda_k = \mu_n = 2\pi n/\lambda \) and \( b(n) = \gamma a(n) \). Also, \( f \) has at most one simple pole, that at \( s=r \) with residue \( \rho \), say. Replacing \( 2\pi \delta/\lambda \) by \( \delta \), we find that (4.2) yields for all \( s \),
\begin{equation}
f(s) = \lim_{\delta \to 0} \left\{ \sum a(n)n^{-s} e^{-n\delta} - \rho \Gamma(r-s) \delta^{s-r} \right\}.
\end{equation}
We give two illustrations of (4.8). Let \( r_a(n) \) denote the number of representations
of $n$ as a sum of $k$ squares and consider $\zeta_k(s) = \sum r_k(n)n^{-s}$ which has signature $(2, \frac{1}{2}k, 1)$ and a simple pole at $s = \frac{1}{2}k$ with residue $\pi^{k/2}$. Thus, for all $s$,

$$\zeta_k(s) = \lim_{\delta \to 0} \left\{ \sum r_k(n)n^{-s}e^{-\pi \delta} - \pi^{k/2}\Gamma(\frac{1}{2}k - s)\delta^{s-k/2} \right\}.$$ 

Secondly, let $\tau(n)$ denote Ramanujan’s arithmetical function. Then, $f(s) = \sum \tau(n)n^{-s}$ has signature $(1, 12, 1)$ and is entire. Hence, for all $s$,

$$f(s) = \lim_{\delta \to 0} \sum \tau(n)n^{-s}e^{-\pi \delta}.$$ 

REFERENCES


UNIVERSITY OF ILLINOIS, Urbana, Illinois 61801