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IDENTITIES INVOLVING THE COEFFICIENTS OF A CLASS OF DIRICHLET SERIES. IV

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Abstract. We consider a class of Dirichlet series satisfying a functional equation with gamma factors. We define a generalized Dirichlet series.that is analogous to the generalized zeta-function of Riemann. An analytic continuation for these generalized series is derived, and a few simple properties are established. Secondly, we prove a theorem on the Abel summation of Dirichlet series that satisfy Hecke's functional equation.

1. **Introduction.** We consider here two somewhat related problems associated with Dirichlet series satisfying a functional equation with Γ-factors. Let $\varphi(s) = \sum_{n=1}^{\infty} a(n)\lambda_n^{-s}$ be a Dirichlet series satisfying Hecke's functional equation, where $s=\sigma+it$ with σ and t both real, and $\sigma>\sigma_a$, the abscissa of absolute convergence of φ. In [2] we defined for $\sigma>\sigma_a$ and $a>0$ the generalized Dirichlet series

$$\varphi(s, a) = \sum_{n=1}^{\infty} a(n)(\lambda_n+a)^{-s},$$

which is analogous to the Hurwitz or generalized zeta-function of Riemann. We derived an analytic continuation for $\varphi(s, a)$ and determined a few of its properties. In [3] we found an easier derivation of our analytic continuation. The method we used in [2] is used here to derive an analytic continuation and some simple properties for generalized Dirichlet series arising from Dirichlet series satisfying a much more general functional equation, but we do not, in general, obtain a simple formula for the analytic continuation of $\varphi(s, a)$. In particular, we shall obtain a new and simple method of continuing the Hurwitz zeta-function $\zeta(s, a)$.

In the above problem we perturbed the Dirichlet series by replacing λ_n by λ_n+a. Next, we consider a different type of problem where we perturb the series by multiplying the terms of the series by $\exp(-\lambda_n\delta)$, $\delta>0$. As we shall see, our result can be interpreted as a modified form of Abel summation. Unfortunately, in this problem our method is only applicable to series which are solutions to Hecke's functional equation.

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2. **Notation, definition, and preliminary results.** In the sequel we write $z=x+iy$ with x and y both real. A always denotes a positive constant, not necessarily the same with each occurrence. The summation sign $\sum$ appearing with no indices will always mean $\sum_{n=1}^{\infty}$. If b is real, we write $\int_{(b)}$ for $\int_{b-i\infty}^{b+i\infty}$.

We suppose that

$$\varphi(s) = \sum a(n)\lambda_n^{-s}, \qquad \psi(s) = \sum b(n)\mu_n^{-s}$$

are solutions of the functional equation

$$\Delta(s)\varphi(s) = \Delta(r-s)\psi(r-s) \tag{2.1}$$

where r is real and

$$\Delta(s) = \prod_{k=1}^{N} \Gamma(\alpha_k s+\beta_k),$$

where $\alpha_k>0$ and β_k is complex, $k=1,\ldots,N$. For a more complete definition of these series see [3] or [5]. If $\Delta(s)=\Gamma(s)$, we have Hecke's functional equation

$$\Gamma(s)\varphi(s) = \Gamma(r-s)\psi(r-s). \tag{2.2}$$

For $x>0$ and $0<c<\sigma$ [6, p. 311],

$$\frac{1}{2\pi i}\int_{(c)} \Gamma(s-z)\Gamma(z)x^{-z}\,dz = \Gamma(s)(1+x)^{-s}. \tag{2.3}$$

If $F(\alpha,\beta;\gamma;z)$ denotes the hypergeometric function, for $\sigma>0$, [6, p. 310],

$$\int_0^b \frac{x^{s-1}\,dx}{(1+x)^{\nu}} = \frac{b^s}{s} F(\nu, s; s+1; -b). \tag{2.4}$$

Also [7, p. 1043],

$$F(\nu, s; s+1; -1/x) = \frac{s}{s-\nu}x^{\nu}F(\nu, \nu-s; \nu+1-s; -x)+\frac{\Gamma(s+1)\Gamma(\nu-s)}{\Gamma(\nu)}x^{s}. \tag{2.5}$$

3. **Generalized Dirichlet series.**

THEOREM 3.1. *Let $\varphi(s)$ satisfy* (2.1) *and let $\varphi(s,a)$, $a>0$, be its generalized Dirichlet series. Assume that the singularities of φ are poles. (The number of poles is finite, as they are contained in a compact set.) Let σ_a and σ_a^* denote the abscissas of absolute convergence of φ and ψ, respectively. Choose $c>\sup(0,\sigma_a,\sigma_a^*)$ such that the line $x=r-c$ does not contain a pole of $\Gamma(z)\varphi(z)$. Let $R(s,a)$ denote the sum of the residues of $\Gamma(s-z)\Gamma(z)\varphi(z)a^z$ at the poles of $\Gamma(z)\varphi(z)$ in the strip $r-c<x<c$. Then, for $\sigma>r-c$,*

$$\Gamma(s)a^s\varphi(s,a) = a^r \sum b(n)f(s,a\mu_n)+R(s,a), \tag{3.1}$$

where

$$f(s,w) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(z+s-r)\Gamma(r-z)\Delta(z)}{\Delta(r-z)}\,w^{-z}\,dz.$$

REMARKS. By taking c large, the analytic continuation of $\varphi(s, a)$ may be taken to the left as far as we wish. The assumption that the singularities of φ are at most poles easily insures the analytic continuation of $R(s, a)$.

Proof. For $\sigma > c > \sup(0, \sigma_a, \sigma_a^*)$, by (2.3) we have

$$\frac{1}{2\pi i}\int_{(c)} \Gamma(s-z)\Gamma(z)\varphi(z)a^z\,dz = \Gamma(s)a^s\varphi(s, a), \tag{3.2}$$

where the inversion in order of summation and integration is justified by absolute convergence. We now move the line of integration to $x = r-c$ by integrating around the boundary of the rectangle with vertices $c \pm iY$ and $r-c\pm iY$ and then letting Y tend to ∞. With the use of Stirling's formula and a Phragmén-Lindelöf theorem it is easy to show that the integrals over the horizontal sides tend to 0 as Y tends to ∞. We arrive at for $\sigma > c$,

$$\frac{1}{2\pi i}\int_{(c)} \Gamma(s-z)\Gamma(z)\varphi(z)a^z\,dz = I(s, a)+R(s, a), \tag{3.3}$$

where

$$\begin{aligned} I(s, a) &= \frac{1}{2\pi i}\int_{(r-c)} \Gamma(s-z)\Gamma(z)\varphi(z)a^z\,dz \\ &= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s+z-r)\Gamma(r-z)\Delta(z)\psi(z)}{\Delta(r-z)}\,a^{r-z}\,dz \\ &= a^r \sum b(n)f(s, a\mu_n). \end{aligned}$$

Here we have replaced z by $r-z$, used (2.1), and then inverted the order of integration and summation by absolute convergence.

Now, $f(s, w)$ is an analytic function of s in the rectangle R defined by $r-c+\varepsilon \leqq \sigma \leqq \gamma$ and $-T \leqq t \leqq T$, where $\varepsilon > 0$, $\gamma > c$, and $T > 0$. Choose Y so that $Y-T \geqq 1$ and $|-b_k/\alpha_k| < Y$, where $b_k = \operatorname{Im}(\beta_k)$, $k = 1, \ldots, N$. Clearly, for s in R,

$$\left|\int_{c-iY}^{c+iY} \frac{\Gamma(z+s-r)\Gamma(r-z)\Delta(z)}{\Delta(r-z)}(a\mu_n)^{-z}\,dz\right| \leqq A\mu_n^{-c},$$

where A is independent of s in R. By Stirling's formula for s in R,

$$\begin{aligned} &\left|\int_{c+iY}^{c+i\infty} \frac{\Gamma(z+s-r)\Gamma(r-z)\Delta(z)}{\Delta(r-z)}(a\mu_n)^{-z}\,dz\right| \\ &\leqq A\mu_n^{-c}\int_Y^{\infty} e^{-\pi y}(y+t)^{c+\sigma-r-1/2}y^{r-c-1/2}\prod_{k=1}^{N}(\alpha_k y+b_k)^{\alpha_k(2c-r)}\,dy \\ &\leqq A\mu_n^{-c}, \end{aligned}$$

where A is independent of s in R. A similar estimate holds for the integral over $(c-iY, c-i\infty)$. Thus,

$$|I(s, a)| \leqq A\sum |b(n)|\mu_n^{-c},$$

where A is independent of s in R. Hence, by Weierstrass' M-test $I(s, a)$ converges uniformly on R and represents an analytic function there since $f(s, a\mu_n)$ is analytic, $n=1, 2, \ldots$. Thus, by analytic continuation, (3.2) and (3.3), we have shown (3.1) for s in $\{s : \sigma > c\} \cup R$. Since $\varepsilon > 0$ can be taken arbitrarily small and T can be chosen arbitrarily large, (3.1) is valid for $\sigma > r-c$.

The following corollaries are easily deduced from Theorem 3.1

COROLLARY 3.2. *If $\varphi(s)$ is entire, then $\varphi(s, a)$ is entire.*

COROLLARY 3.3. *Let $\varphi(s)$ have a pole of order n at $s=k$. If k is a positive integer, then $\varphi(s, a)$ has poles of order n at $s=1, \ldots, k$ and poles of order $n-1$ at $s=0, -1, -2, \ldots$. If k is not a positive integer, then $\varphi(s, a)$ has poles of order n at $s=k, k-1, k-2, \ldots$.*

Proof. The conclusion follows immediately from an examination of $R(s, a)/\Gamma(s)$, for $\Gamma(s-k)$ and its derivatives have poles at $s=k, k-1, k-2, \ldots$, and $1/\Gamma(s)$ has simple zeros at $s=0, -1, -2, \ldots$.

EXAMPLE 3.1. Let $\varphi(s)=\pi^{-s/2}\zeta(s)$, where $\zeta(s)$ is the Riemann zeta-function. Let $c=m+\frac{1}{2}$, where m is a positive integer. Replacing a by $\pi^{1/2}a$, we have the following representation for the Hurwitz zeta-function $\zeta(s, a)$ for $\sigma > \frac{1}{2}-m$:

$$\zeta(s, a)-a^{-s} = \frac{a^{-s}}{\Gamma(s)}\left\{\Gamma(s-1)a+\sum_{j=0}^{m-1}\frac{(-1)^j\Gamma(s+j)\zeta(-j)}{j!\,a^j}+F(s)\right\}, \tag{3.4}$$

where $F(s)$ is analytic for $\sigma > \frac{1}{2}-m$. We see from (3.4) that $\zeta(s, a)$ is analytic except for a simple pole at $s=1$ with residue 1. (3.4) yields immediately a simple formula for $\zeta(n, a)$, where n is a nonpositive integer. Thus, $\zeta(0, a)=-a+\frac{1}{2}$, $\zeta(-1, a) =-\frac{1}{2}a^2+\frac{1}{2}a-1/12$, etc. For other methods of continuing $\zeta(s, a)$ and calculating $\zeta(n, a)$ see [9, p. 37] and [10, pp. 266–267].

EXAMPLE 3.2. Let χ be a nonprincipal, primitive character modulo k. Then, the Dirichlet L-function $L(s, \chi)$ satisfies the functional equation

$$R(s, \chi) = (\pi/k)^{(s+b)/2}\Gamma(\tfrac{1}{2}\{s+b\})L(s, \chi) = \varepsilon(\chi)R(1-s, \bar{\chi}),$$

where $b=0$ if $\chi(-1)=1$, $b=1$ if $\chi(-1)=-1$, and $|\varepsilon(\chi)|=1$. Let c be as in Example 3.1 and replace a by $(\pi/k)^{1/2}a$. Then, putting

$$L(s, \chi, a) = \sum \chi(n)(n+a)^{-s} \qquad (\sigma > 0),$$

we have for $\sigma > \frac{1}{2}-m$

$$L(s, \chi, a) = \frac{a^{-s}}{\Gamma(s)}\left\{\sum_{j=0}^{m-1}\frac{(-1)^j\Gamma(s+j)L(-j, \chi)}{j!\,a^j}+G(s)\right\},$$

where $G(s)$ is analytic. $L(s, \chi, a)$ is an entire function, and we can easily calculate $L(n, \chi, a)$, where n is a nonpositive integer. In particular, $L(0, \chi, a)=L(0, \chi)$.

EXAMPLE 3.3. Let K be an algebraic number field of degree r_1+2r_2, where r_1 is the number of real conjugates of K and $2r_2$ the number of imaginary conjugates.

Then there exists a positive constant B, depending upon K, such that the Dedekind zeta-function $\zeta_K(s)$ satisfies the functional equation

$$\xi(s) = B^{-s}\Gamma^{r_1}(\tfrac{1}{2}s)\Gamma^{r_2}(s)\zeta_K(s) = \xi(1-s)$$

and is analytic save for a simple pole at $s=1$. The generalized Dedekind zeta-function

$$\zeta_K(s, a) = \sum F(n)(n+a)^{-s} \qquad (\sigma > 1),$$

where $F(n)$ is the number of nonzero integral ideals of norm n in K, has an analytic continuation that is analytic except for a simple pole at $s=1$. If n is a nonpositive integer, $\zeta_K(n, a)$ can be easily calculated in the same manner as above.

4. **Hecke's functional equation and Abel summation.** Let φ and ψ satisfy (2.2), and for $x>0$ let

$$\Phi(x) = \sum a(n) \exp[-\lambda_n x], \qquad \Psi(x) = \sum b(n) \exp[-\mu_n x].$$

Bochner [4] has shown that

$$\Phi(x) = x^{-r}\Psi(1/x)+P(x), \tag{4.1}$$

where

$$P(x) = \frac{1}{2\pi i}\int_C \chi(s)x^{-s}\,ds,$$

$\chi(s)=\Gamma(s)\varphi(s)$ and C denotes a curve, or curves, encircling the singularities of χ. Our next theorem is an easy consequence of (4.1).

THEOREM 4.1. *Suppose that χ has at most simple poles. Let the poles be at $s_1, \ldots, s_m$ with residues $\rho_1, \ldots, \rho_m$, respectively. For every $\delta>0$ suppose that $\tilde{\varphi}(s, \delta)= \sum a(n)\lambda_n^{-s} \exp[-\lambda_n\delta]$ converges for $\sigma>\sigma_0$, where $\sigma_0 \leqq \sigma_a$. Then, for $\sigma>\sigma_0$,*

$$\varphi(s) = \lim_{\delta\to 0}\left\{\tilde{\varphi}(s, \delta) - \sum_{k=1}^{m} \frac{\rho_k\Gamma(s_k-s)}{\Gamma(s_k)}\,\delta^{s-s_k}\right\}. \tag{4.2}$$

REMARKS. The assumption that χ has simple poles is not strictly necessary. However, the calculations become harder, otherwise. Using a representation of $\zeta(s)$ as a contour integral, Atkinson [1] proved a result like that of (4.2) when $a(n)=1$ and $\lambda_n=n$.

Proof. For $\sigma>\sup(0, \sigma_a)$, we have by (4.1),

$$\begin{aligned}\Gamma(s)\tilde{\varphi}(s, \delta) &= \int_0^\infty x^{s-1}\Phi(x+\delta)\,dx\\ &= \int_0^1 x^{s-1}(x+\delta)^{-r}\Psi(1/(x+\delta))\,dx\\ &\quad+\int_0^1 x^{s-1}P(x+\delta)\,dx+\int_1^\infty x^{s-1}\Phi(x+\delta)\,dx.\end{aligned} \tag{4.3}$$

Since

$$P(y) = \sum_{k=1}^{m} \rho_k y^{-s_k}, \tag{4.4}$$

we find that

$$\int_0^1 x^{s-1}P(x+\delta)\,dx = \sum_{k=1}^{m} \rho_k \delta^{s-s_k} \int_0^{1/\delta} x^{s-1}(x+1)^{-s_k}\,dx$$
$$= s^{-1} \sum_{k=1}^{m} \rho_k \delta^{-s_k} F(s_k, s; s+1; -1/\delta)$$

from (2.4). It follows from (4.3) that

$$\begin{aligned} \Gamma(s)\tilde{\varphi}(s,\delta) - s^{-1} \sum_{k=1}^{m} \rho_k \delta^{-s_k} F(s_k, s; s+1; -1/\delta) \\ = \int_0^1 x^{s-1}(x+\delta)^{-r}\Psi(1/(x+\delta))\,dx + \int_1^\infty x^{s-1}\Phi(x+\delta)\,dx. \end{aligned} \tag{4.5}$$

Since $\tilde{\varphi}(s, \delta)$ converges uniformly for $|\arg(s-\sigma_0)| \leqq \frac{1}{2}\pi - \varepsilon$ for every $\varepsilon > 0$, by analytic continuation (4.5) is valid for $\sigma > \sigma_0$. Since (4.3) is valid also for $\delta = 0$, we have from (4.4) for $\sigma > \sup(0, \sigma_a)$,

$$\int_0^1 x^{s-r-1}\Psi(1/x)\,dx + \int_1^\infty x^{s-1}\Phi(x)\,dx = \Gamma(s)\varphi(s) - \sum_{k=1}^{m} \frac{\rho_k}{s-s_k}. \tag{4.6}$$

By analytic continuation, (4.6) is valid for all s. Hence, (4.5) and (4.6) yield for $\sigma > \sigma_0$,

$$\begin{aligned} \lim_{\delta \to 0} \left\{ \Gamma(s)\tilde{\varphi}(s,\delta) - s^{-1} \sum_{k=1}^{m} \rho_k \delta^{-s_k} F(s_k, s; s+1; -1/\delta) \right\} \\ = \Gamma(s)\varphi(s) - \sum_{k=1}^{m} \frac{\rho_k}{s-s_k}. \end{aligned} \tag{4.7}$$

Upon the use of (2.5) and the fact that $\lim_{\delta \to 0} F(\alpha, \beta; \gamma; \delta) = 1$, (4.7) becomes for $\sigma > \sigma_0$,

$$\lim_{\delta \to 0} \left\{ \Gamma(s)\tilde{\varphi}(s,\delta) - s^{-1} \sum_{k=1}^{m} \rho_k \frac{\Gamma(s+1)\Gamma(s_k - s)}{\Gamma(s_k)} \delta^{s-s_k} \right\} = \Gamma(s)\varphi(s),$$

which is clearly equivalent to (4.2).

EXAMPLES. Suppose that $f(s) = \sum a(n)n^{-s}$ is a Dirichlet series of signature (λ, r, γ) [8], where $\lambda > 0$, $r > 0$ and $\gamma = \pm 1$. Then, $\lambda_n = \mu_n = 2\pi n/\lambda$ and $b(n) = \gamma a(n)$. Also, f has at most one simple pole, that at $s = r$ with residue ρ, say. Replacing $2\pi\delta/\lambda$ by δ, we find that (4.2) yields for all s,

$$f(s) = \lim_{\delta \to 0} \left\{ \sum a(n)n^{-s}e^{-n\delta} - \rho\Gamma(r-s)\delta^{s-r} \right\}. \tag{4.8}$$

We give two illustrations of (4.8). Let $r_k(n)$ denote the number of representations of n as a sum of k squares and consider $\zeta_k(s)=\sum r_k(n)n^{-s}$ which has signature $(2, \frac{1}{2}k, 1)$ and a simple pole at $s=\frac{1}{2}k$ with residue $\pi^{k/2}$. Thus, for all s,

$$\zeta_k(s) = \lim_{\delta\to 0} \left\{\sum r_k(n)n^{-s}e^{-n\delta}-\pi^{k/2}\Gamma(\tfrac{1}{2}k-s)\delta^{s-k/2}\right\}.$$

Secondly, let $\tau(n)$ denote Ramanujan's arithmetical function. Then, $f(s)=\sum \tau(n)n^{-s}$ has signature (1, 12, 1) and is entire. Hence, for all s,

$$f(s) = \lim_{\delta\to 0} \sum \tau(n)n^{-s}e^{-n\delta}.$$

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