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MARKUSCHEVICH BASES AND DUALITY THEORY

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Abstract. Several duality theorems concerning Schauder bases in locally convex spaces have analogues in the theory of Markuschevich bases. For example, a locally convex space with a Markuschevich basis is semireflexive iff the basis is shrinking and boundedly complete.

The strong existence Theorem III.1 for Markuschevich bases allows us to show that a separable Banach space is isomorphic to a conjugate space iff it admits a boundedly complete Markuschevich basis, and that a separable Banach space has the metric approximation property iff it admits a Markuschevich basis which is a generalized summation basis in the sense of Kadec.

I. **Introduction.** In recent years a number of papers have discussed applications of Schauder bases to the duality theory of locally convex spaces. (For example, see [2], [5], [9], [10], and [11].) However, the lack of a good existence theorem for Schauder bases severely limits the applicability of these results. In this paper we discuss a generalization of Schauder bases (called Markuschevich bases or M-bases) for which there are good existence theorems. In fact, Markuschevich [8] showed that every separable Banach space admits a M-basis and Theorem III.1 gives a better existence theorem for general linear topological spaces.

In §II we introduce the concepts of shrinking and boundedly complete Markuschevich bases. The main results of this section are that a locally convex space with a M-basis is semireflexive iff the M-basis is shrinking and boundedly complete (Theorem II.6) and that a Banach space which admits a boundedly complete M-basis is canonically isomorphic to the adjoint of the coefficient space of the M-basis (Theorem II.5). Of course, these theorems have analogues in Schauder basis theory (see [5], [9], and [1]).

Theorem III.1 shows that every strongly separable, strongly closed, total subspace of the adjoint of a separable linear topological space X is the coefficient space of some countable M-basis for X. (This result is perhaps implicit in the results of [4], but we include a proof for completeness.) This theorem has several interesting applications. For example, a separable locally convex space has a strongly separable adjoint iff the space admits a countable shrinking M-basis (Corollary III.3). A separable Banach space is isomorphic to a conjugate Banach space iff the space admits a boundedly complete M-basis (Theorem III.4).

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In §IV we discuss generalized summation bases, a type of Markuschevich bases introduced by Kadec [6]. We use Theorem III.1 to show that a separable Banach space admits a generalized summation basis iff the space has the metric approximation property.

We use the notation and terminology of [7]. X always represents a Hausdorff linear topological space and X^* represents the set of continuous linear functionals on X. We assume that X^* is endowed with the topology of uniform convergence on $w(X, X^*)$ bounded subsets of X, and call this topology on X^* the strong topology. When X^* is total over X, the natural embedding of X into X^{**} is denoted by "^".

If f is a function on a set Z and Y is a subset of Z, $f|_Y$ denotes the restriction of f to Y. The linear span of a subset Y of a linear space is denoted by sp (Y). The domain, range, and null space of a linear operator T are denoted, respectively, by $D(T)$, $R(T)$, and ker (T).

II. **Applications of M-bases to duality theory.** Let (X, T) be a linear topological space. Recall that a biorthogonal collection $\{x_i, f_i\}_{i\in I}$ in (X, X^*) is a Markuschevich basis (M-basis) for X iff $\{x_i\}_{i\in I}$ is fundamental in (X, T) and $\{f_i\}_{i\in I}$ is total over X. The strong closure of sp $(\{f_i\}_{i\in I})$ in X^* is called the coefficient space of the M-basis $\{x_i, f_i\}$.

Definition II.1. Let $\{x_i, f_i\}_{i\in I}$ be a M-basis for X. $\{x_i, f_i\}$ is said to be shrinking iff $\{f_i\}_{i\in I}$ is strongly fundamental in X^*. (Equivalently, $\{x_i, f_i\}$ is shrinking iff $\{f_i, \hat{x}_i\}$ is a M-basis for X^* when X^* is endowed with the strong topology.)

Definition II.2. Let $\{x_i, f_i\}_{i\in I}$ be a M-basis for X. $\{x_i, f_i\}$ is said to be boundedly complete iff whenever $\{Y_d\}$ is a bounded net in X such that for each i in I, $\lim_d f_i(Y_d)$ exists, there is x in X such that for each $i \in I, f_i(x) = \lim_d f_i(Y_d)$.

Let $\{x_i, f_i\}_{i=1}^{\infty}$ be a Schauder basis for a locally convex space X. If $\{x_i, f_i\}$ is shrinking as a Schauder basis, then obviously it is shrinking as a M-basis. It is also easy to see that if $\{x_i, f_i\}$ is boundedly complete as a M-basis, then it is boundedly complete as a Schauder basis. The converses of these statements are true for uniformly bounded Schauder bases.

Theorem II.3. *Let $\{x_i, f_i\}_{i=1}^{\infty}$ be a uniformly bounded Schauder basis for a locally convex space X.* (1) *If $\{x_i, f_i\}$ is boundedly complete as a Schauder basis, then it is boundedly complete as a M-basis.* (2) *If $\{x_i, f_i\}$ is shrinking as a M-basis, then it is shrinking as a Schauder basis.*

Proof. Let $\{S_n\}_{n=1}^{\infty}$ be the partial sum operators associated with $\{x_i, f_i\}$. (That is, $S_n(x) = \sum_{i=1}^{n} f_i(x)x_i$.) We are assuming that $\{S_n\}_{n=1}^{\infty}$ is uniformly bounded.

To prove (1), we let $\{Y_d : d \in D\}$ be as in Definition II.2. For each n, $\{S_n(Y_d) : d \in D\}$ is a Cauchy net in the finite dimensional Hausdorff space $(R(S_n), w(R(S_n), \{f_i\}_{i=1}^{n}))$, hence $\lim_d S_n(Y_d)$ exists. Since $\{S_n\}$ is uniformly bounded and $\{Y_d\}$ is bounded, $\{\lim_d S_n(Y_d)\}_{n=1}^{\infty}$ is bounded. Since $\{x_i, f_i\}$ is boundedly complete as a Schauder basis, $\{\lim_d S_n(Y_d)\}_{n=1}^{\infty}$ must converge to, say, x. Clearly $f_i(x)$

$=\lim_d f_i(Y_d)$, for all $i=1, 2, 3, \ldots$, so $\{x_i, f_i\}$ is boundedly complete as a M-basis.

To prove (2), we note that the uniform boundedness of $\{S_n\}$ implies that $\{S_n^*\}_{n=1}^{\infty}$ is equicontinuous on X^*. Hence $A=\{x : \lim_n S_n^*(x)=x\}$ is closed in X^*. Since A obviously contains $\operatorname{sp}(\{f_i\})$ and $\{f_i\}$ is fundamental in X^*, A must equal X^*. Then $\{f_i, \hat{x}_i\}_{i=1}^{\infty}$ is a Schauder basis for X^*, which is to say that $\{x_i, f_i\}$ is a shrinking Schauder basis for X.

The hypothesis in Theorem II.3 that $\{x_i, f_i\}_{i=1}^{\infty}$ is a uniformly bounded Schauder basis is necessary. Indeed, let Y be any separable, infinite dimensional Banach space and let $\{x_i, f_i\}_{i=1}^{\infty}$ be any M-basis for Y which is not a Schauder basis for Y. Let $X=\{x \in Y : \{\sum_{i=1}^{n} f_i(x)x_i\}_{n=1}^{\infty}$ converges weakly to $x\}$. Let X be endowed with the $w(X, Y^*)$ topology. It is easy to see that $\{x_i, f_i\}_{i=1}^{\infty}$ is boundedly complete as a Schauder basis for X. $\{x_i, f_i\}_{i=1}^{\infty}$ is not boundedly complete as a M-basis for X because X is a proper subspace of Y. Also, $\{x_i, f_i\}_{i=1}^{\infty}$ is obviously a shrinking M-basis for X. $\{x_i, f_i\}_{i=1}^{\infty}$ is not a shrinking Schauder basis for X because $\{f_i\}_{i=1}^{\infty}$ is not even a basic sequence in $Y^*(=X^*)$ and $s(Y^*, Y)=s(Y^*, X)$.

Results from Schauder basis theory suggest that "shrinking" and "boundedly complete" should be dual concepts. Under certain circumstances, this is the case.

THEOREM II.4. *Let $\{x_i, f_i\}_{i\in I}$ be a shrinking M-basis for a locally convex evaluable space X. Then $\{f_i, \hat{x}_i\}_{i\in I}$ is a boundedly complete M-basis for X^*.*

Proof. Let $\{Y_d\}$ be a bounded net in X^* such that for each i in I, $\lim_d \hat{x}_i(Y_d)$ exists. Note that $\{Y_d\}$ is equicontinuous on X because X is evaluable, hence $\{Y_d\}$ has a weak* cluster point, say, y. Clearly $\hat{x}_i(y)=\lim_d \hat{x}_i(Y_d)$, for all i in I, so $\{f_i, \hat{x}_i\}$ is boundedly complete.

THEOREM II.5. *Let $\{x_i, f_i\}_{i\in I}$ be a boundedly complete M-basis for a Banach space X. Let Y be the coefficient space of the basis $\{x_i, f_i\}$. Then the canonical embedding of X into Y^* is an isomorphism of X onto Y^*. Hence $\{f_i, \hat{x}_i|_Y\}_{i\in I}$ is a shrinking M-basis for Y.*

Proof. Note that the canonical embedding of X into Y^* is one-to-one, because Y is total over X. It is norm decreasing, hence continuous. We show that it is onto Y^* (and hence an isomorphism by the open mapping theorem).

Let G be in Y^*. Let D be the collection of finite subsets of I, and direct D by inclusion. By Helley's theorem (cf., e.g., [12, p. 103]), for each d in D there is Y_d in X such that for each i in d, $f_i(Y_d)=G(f_i)$, and $\|Y_d\| \leqq \|G\|+1$. Since $\{x_i, f_i\}$ is boundedly complete, there is x in X such that for each i in I, $\lim_d f_i(Y_d)=f_i(x)$. Clearly $f_i(x)=G(f_i)$, for all i in I. Since $\{f_i\}$ is fundamental in Y and both $\hat{x}$ and G are continuous on Y, $f(x)=G(f)$, for all f in Y. Thus the canonical embedding of X into Y^* is onto.

Since $\{x_i\}$ is fundamental in X, $\{\hat{x}_i|_Y\}$ is fundamental in Y^*, and thus $\{f_i, \hat{x}_i|_Y\}$ is shrinking. This completes the proof.

Historically, a major reason for considering shrinking and boundedly complete

Schauder bases was to characterize reflexivity (see [5] and [9]). Similarly, semireflexivity is characterized by the existence of a boundedly complete, shrinking M-basis.

THEOREM II.6. *Let $\{x_i, f_i\}_{i\in I}$ be a M-basis for a locally convex space X. X is semireflexive iff $\{x_i, f_i\}$ is both shrinking and boundedly complete.*

Proof. Suppose first that X is semireflexive. $\{f_i\}$ is total over X, hence is weak* fundamental in X^*, hence is weakly fundamental in X^*, hence is fundamental in X^*. Thus $\{x_i, f_i\}$ is shrinking. Now let $\{Y_d\}$ be a bounded net as in Definition II.2. Since X is semireflexive, $\{Y_d\}$ has a weak cluster point, say, x. Clearly $f_i(x) = \lim_d f_i(Y_d)$, for all i in I, so that $\{x_i, f_i\}$ is boundedly complete.

To go the other way, suppose that $\{x_i, f_i\}$ is shrinking and boundedly complete. To show that X is semireflexive, it is sufficient to show that bounded, weakly Cauchy nets in X are weakly convergent. Let $\{Y_d\}$ be a bounded, weakly Cauchy net in X. Clearly $\lim_d f_i(Y_d)$ exists for each i in I, hence by the boundedly complete assumption there is x in X such that for all i in I, $f_i(x) = \lim_d f_i(Y_d)$. We need to show that $\{Y_d\}$ weakly converges to x. Now $\{Y_d\}$ is bounded, so $\{\hat{Y}_d\}$ is equicontinuous on X^*. $\{\hat{Y}_d\}$ converges to $\hat{x}$ pointwise on the subset $\{f_i\}$ of X^*. By the shrinking assumption, $\{f_i\}$ is fundamental in X^*, hence $\{\hat{Y}_d\}$ converges to $\hat{x}$ pointwise on X^*. That is, $\{Y_d\}$ converges weakly to x. This completes the proof.

III. **Existence theorem for countable Markuschevich bases.**

Unfortunately, there is no general existence theorem for Schauder bases. In contrast to this situation, Theorem III.1 provides a very fine existence theorem for M-bases.

THEOREM III.1. *Let X be separable and let Y be a closed, separable, total subspace of X^*. Then X admits a M-basis $\{y_i, g_i\}_{i=1}^\infty$ whose coefficient space is Y.*

Proof. Let $\{x_i\}_{i=1}^\infty$ be a fundamental subset of X and let $\{f_i\}_{i=1}^\infty$ be a fundamental subset of Y. Note that $\{f_i\}_{i=1}^\infty$ is total over X. Hence we can assume, without loss of generality, that $f_1(x_1) \neq 0$.

Let $y_1 = x_1$, $g_1 = f_1/f_1(x_1)$, $k(1) = 1$.

Suppose that $\{y_i, g_i\}_{i=1}^{k(n)}$ have been defined so that

(1) $\{y_i, g_i\}_{i=1}^{k(n)}$ is biorthogonal;

(2) $\operatorname{sp}(\{y_i\}_{i=1}^{k(n)}) \supset \operatorname{sp}(\{x_i\}_{i=1}^{n})$;

(3) $Y \supset \operatorname{sp}(\{g_i\}_{i=1}^{k(n)}) \supset \operatorname{sp}(\{f_i\}_{i=1}^{n})$.

A. If x_{n+1} is in $\operatorname{sp}(\{y_i\}_{i=1}^{k(n)})$, we let $k = k(n)$ and proceed to B. If x_{n+1} is not in $\operatorname{sp}(\{y_i\}_{i=1}^{k(n)})$, we let $k = k(n)+1$, and let $y_k = x_{n+1} - \sum_{i=1}^{k(n)} g_i(x_{n+1}) y_i$. (1) implies that $g_i(y_k) = 0$, for each $i \leqq k(n)$. By using the Hahn-Banach theorem in the space $(X, w(X, Y))$, we can find g_k in Y such that $g_k(y_k) = 1$ and $g_k(y_i) = 0$ for each $i \leqq k(n)$. Note that $\{y_i, g_i\}_{i=1}^{k}$ is biorthogonal, $\operatorname{sp}(\{y_i\}_{i=1}^{k}) \supset \operatorname{sp}(\{x_i\}_{i=1}^{n+1})$, and $Y \supset \operatorname{sp}(\{g_i\}_{i=1}^{k})$. Now proceed to B.

B. If f_{n+1} is in $\operatorname{sp}(\{g_i\}_{i=1}^{k})$, let $k(n+1) = k$. If f_{n+1} is not in $\operatorname{sp}(\{g_i\}_{i=1}^{k})$, let $k(n+1)$

$=k+1$, and let $g_{k(n+1)}=f_{n+1}-\sum_{i=1}^{k} f_{n+1}(y_i)g_i$. By using the Hahn-Banach theorem in the space $(Y, w(Y, X))$, we can find $y_{k(n+1)}$ in X such that $g_{k(n+1)}(y_{k(n+1)})=1$ and $g_i(y_{k(n+1)})=0$ for all $i \leqq k$.

We have thus extended the sequence $\{y_i, g_i\}_{i=1}^{k(n)}$ to a sequence $\{y_i, g_i\}_{i=1}^{k(n+1)}$ such that $\{y_i, g_i\}_{i=1}^{k(n+1)}$ satisfies (1), (2), and (3) if $n+1$ is substituted for n. The sequence $\{y_i, g_i\}_{i=1}^{\infty}$ obviously satisfies the conclusion of the theorem.

REMARK III.2. Theorem III.1 shows that every separable linear topological space which admits a countable total family of continuous linear functionals must also admit a countable M-basis. Unfortunately, there are separable locally convex spaces which do not admit a countable total family of continuous linear functionals. (An example is the product of $\aleph_1$ copies of the real line with the product topology.) Of course, such spaces do not admit countable M-bases.

One instance of Theorem III.1 is of particular interest:

COROLLARY III.3. *Suppose that X is separable and X^* is total over X. Then X admits a countable shrinking M-basis iff X^* is strongly separable.*

THEOREM III.4. *Let X be a separable Banach space. X is isomorphic to a conjugate Banach space iff X admits a boundedly complete M-basis.*

Proof. If X is isomorphic to Y^*, then Y is separable, so that the conclusion follows from Corollary III.3 and Theorem II.4. Conversely, if X admits a boundedly complete M-basis, the conclusion follows from Theorem II.5.

Let X be a Banach space. Recall that a subspace Y of X^* is norming iff there is $k>0$ such that for all x in X, $\|x\| \leqq k \sup \{|f(x)| : f \in Y, \|f\| \leqq 1\}$. Equivalently, Y is norming iff the canonical mapping of X into Y^* is an isomorphism of X into Y^*. Note that Theorem II.5 shows that the coefficient space of a boundedly complete M-basis is norming. However, there are M-bases whose coefficient spaces are not norming. This follows from Theorem III.1 and the well-known fact that there are closed total subspaces of c_0^* which are not norming. On the other hand, if X is separable it is easy to see that X^* contains a separable norming subspace. It thus follows from Theorem III.1 that every separable Banach space admits a M-basis whose coefficient space is norming.

For many applications, it is desirable to have a stronger existence theorem for M-bases than Theorem III.1. In particular, which nonseparable Banach spaces admit M-bases? (Dyer [3] has noted that for T uncountable, $m(T)$ has no M-bases.) Let X be a separable Banach space. Is there a M-basis $\{x_i, f_i\}$ for X such that $\{x_i\}$ is bounded in X, $\{f_i\}$ is bounded in X^*, and the coefficient space of $\{x_i, f_i\}$ is norming?

IV. Generalized summation bases in spaces with the metric approximation property.

In this section we assume that X is a separable Banach space, $\{x_i, f_i\}_{i=1}^{\infty}$ is a M-basis for X, and $\{S_n\}_{n=1}^{\infty}$ is the sequence of operators defined by $S_n(x)=\sum_{i=1}^{n} f_i(x)x_i$. I is the identity operator on X.

Following Kadec [6] we say that $\{x_i, f_i\}$ is a generalized summation basis (g.s.b.) for X iff there is a sequence $\{T_n\}_{n=1}^{\infty}$ of linear operators with $R(T_n) \subset D(T_n) = \mathrm{sp}\,(\{x_i\}_{i=1}^{n})$ such that the sequence $\{T_n S_n\}_{n=1}^{\infty}$ of linear operators on X converges pointwise to I. Kadec pointed out that not every countable M-basis is a g.s.b. Indeed, this follows from the comments at the end of §III and the easily verified fact that the coefficient space of a g.s.b. is norming (see [6] and [4]).

It is not known whether every separable Banach space admits a g.s.b. Note that the existence of a g.s.b. for X implies that X has the metric approximation property—i.e., that there is a sequence of continuous linear operators of finite range (but not necessarily of norm 1) on X which converges pointwise to I. In fact, Theorem IV.1 shows that the metric approximation property is equivalent to the existence of a g.s.b.

THEOREM IV.1. *Let X be a separable Banach space which has the metric approximation property. Then X admits a generalized summation basis.*

Proof. Let $\{L_n\}_{n=1}^{\infty}$ be a sequence of linear operators of finite range on X which converges pointwise to I. Let $\{x_i, f_i\}_{i=1}^{\infty}$ be any M-basis for X such that the coefficient space of $\{x_i, f_i\}$ contains $\bigcup_{n=1}^{\infty} R(L_n^*)$. (By Theorem III.1, such a M-basis exists.)

Write $L_1(x) = \sum_{i=1}^{p} g_i(x) y_i$, where $\{y_i\}_{i=1}^{p} \subset R(L_1)$ and $\{g_i\}_{i=1}^{p} \subset R(L_1^*)$. If $\varepsilon > 0$, there is a positive integer $n(1)$ such that for each $i \leqq p$, there are $\bar{x}_i$ in $\mathrm{sp}\,(\{x_i\}_{i=1}^{n(1)})$ and $\bar{f}_i$ in $\mathrm{sp}\,(\{f_i\}_{i=1}^{n(1)})$ such that $\|\bar{x}_i - y_i\| < \varepsilon$ and $\|\bar{f}_i - g_i\| < \varepsilon$.

Let $T_{n(1)}(x) = \sum_{i=1}^{p} \bar{f}_i(x)\bar{x}_i$. Note that

$$\begin{aligned} \|T_{n(1)}(x) - L_1(x)\| &= \left\| \sum_{i=1}^{p} (\bar{f}_i(x) - g_i(x))\bar{x}_i + \sum_{i=1}^{p} g_i(x)(\bar{x}_i - y_i) \right\| \\ &\leqq \left[\sum_{i=1}^{p} \|\bar{f}_i - g_i\| \, \|\bar{x}_i\| + \|g_i\| \, \|\bar{x}_i - y_i\| \right] \|x\|. \end{aligned}$$

Thus if ε is sufficiently small, we can assure that $\|T_{n(1)} - L_1\| < 1$.

Note that $T_{n(1)} = T_{n(1)} S_{n(1)}$ and $R(T_{n(1)}) \subset \mathrm{sp}\,(\{x_i\}_{i=1}^{n(1)})$.

A simple extension of this argument shows that there is an increasing sequence $\{n(i)\}_{i=1}^{\infty}$ of positive integers and a sequence $\{T_{n(i)}\}_{i=1}^{\infty}$ of linear operators on X such that for each i,

$$\|T_{n(i)} - L_i\| < 1/i, \qquad R(T_{n(i)}) \subset \mathrm{sp}\,(\{x_j\}_{j=1}^{n(i)}), \quad \text{and} \quad T_{n(i)} = T_{n(i)} S_{n(i)}.$$

$\{T_{n(i)}\}_{i=1}^{\infty}$ converges pointwise to I because $\{\|T_{n(i)} - L_i\|\}_{i=1}^{\infty}$ converges to 0 and $\{L_i\}_{i=1}^{\infty}$ converges pointwise to I.

We complete the sequence $\{T_j\}_{j=1}^{\infty}$ by defining

$$\begin{aligned} T_j &= S_j, & &\text{if } j < n(1), \\ &= T_{n(i)}, & &\text{if } n(i) \leqq j < n(i+1). \end{aligned}$$

Note that for each j, $T_j = T_j S_j$. We have seen that $\{T_j\}_{j=1}^{\infty}$ converges pointwise to I, so $\{T_jS_j\}_{j=1}^{\infty}$ converges pointwise to I. Clearly $R(T_j) \subset \text{sp}(\{x_i\}_{i=1}^{j})$, so we have shown that $\{x_i, f_i\}_{i=1}^{\infty}$ is a g.s.b. for X.

Theorem II.6 and the proof of Theorem IV.1 yield a (slight) strengthening of Theorem 4 of [6].

COROLLARY IV.2. *Let X be a separable reflexive Banach space which has the metric approximation property. Then every M-basis for X is a g.s.b.*

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