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ZERO-ONE LAWS FOR GAUSSIAN PROCESSES(1)

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Abstract. Some zero-one laws are proved for Gaussian processes defined on linear spaces of functions. They are generalizations of a result for Wiener measure due to R. H. Cameron and R. E. Graves. The proofs exploit an interesting relationship between a Gaussian process and its reproducing kernel Hilbert space. Applications are discussed.

1. **Introduction.** In their 1951 paper [2], R. H. Cameron and R. E. Graves have proved a remarkable property of the Wiener process, that every measurable r-module in Wiener space can only have Wiener measure either zero or one. The proof, which relies heavily on analysis connected with the Fourier-Hermite development of functionals on Wiener space, would lead one to wonder whether this result is a peculiarity of the Wiener process. It is the aim of this paper to show that the zero-one law mentioned above is true for all Gaussian stochastic processes with a continuous covariance function. The precise assumptions on the probability space of such a process are given in the next section.

The statement of our first main result (Theorem 1) and the completion of its proof are given in §4. Preparatory lemmas which carry the major burden of the proof, and bring out its basic ideas are given in §3.

In Theorem 2 we give a different version of a zero-one law also valid for general Gaussian processes and pertaining to groups instead of r-modules.

Some interesting consequences of Theorem 1 are discussed in §6. Specifically, Theorem 3 is a generalization of a zero-one law for Wiener processes recently derived by L. A. Shepp [12] and subsequently extended by D. E. Varberg [13]. Theorem 4 provides information—new as far as we know—concerning the uniform convergence with probability one, of the orthogonal expansion of a Gaussian process.

The author wishes to thank Professor Robert Cameron for his helpful discussions on these questions. It was his seminar talk on his own paper that was the starting point of the present work.

2. **Notation and preliminaries.** Let P_0 be a Gaussian probability measure given on the measurable space $(X, \boldsymbol{B}(X))$, where X is a family of real valued functions x

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defined on set T and $\boldsymbol{B}(X)$ is the σ-field of subsets of X generated by sets of the form

(2.1) $$E = \{x \in X : [x(t_1), \ldots, x(t_n)] \in B^n\}$$

($t_1, \ldots, t_n \in T$ and B^n is an n-dimensional Borel set). We shall denote the convariance function of P_0 by R and assume the mean function to be zero, i.e.,

(2.2) $$\int_X x(t)P_0(dx) = 0 \qquad (t \in T)$$

and for each t and s in T

(2.3) $$\int_X x(t)x(s)P_0(dx) = R(t, s).$$

Let $H(R)$ denote the reproducing kernel ($r \cdot k$) Hilbert space determined by R. It is a Hilbert space of real valued functions k on T with the following properties:

(2.4) $$R(\cdot, t) \text{ belongs to } H(R) \text{ for each } t \text{ in } T;$$

if $\langle\ ,\ \rangle$ is the inner product in $H(R)$, then for every k in $H(R)$

(2.5) $$\langle k, R(\cdot, t)\rangle = k(t).$$

For a discussion of $r \cdot k$ Hilbert spaces see [1]. Their application to the study of Gaussian stochastic processes is to be found in many recent papers, notably [11]. The Gaussian process considered in this paper will be represented by the triplet $(X, \boldsymbol{B}_0(X), P_0)$, where $\boldsymbol{B}_0(X)$ is the completion of $\boldsymbol{B}(X)$ under P_0 and where the following basic assumptions will be made.

(2.6) T is a complete separable metric space.

(2.7) X is a linear space of functions under the usual operation of addition of functions and multiplication by real scalars.

(2.8) R is a continuous function on $T \times T$.

(2.9) $$H(R) \subset X.$$

Assumption (2.8) implies that the elements of $H(R)$ are continuous functions on T and also that $H(R)$ is separable [11].

For $m \in X$, the transformation $\sigma_m: X \to X$ defined by

(2.10) $$\sigma_m x = x + m$$

clearly sends $\boldsymbol{B}(X)$-sets into $\boldsymbol{B}(X)$-sets. The measure P_m given by

(2.11) $$P_m(F) = P_0(\sigma_m^{-1}F) \qquad (F \in \boldsymbol{B}(X))$$

is Gaussian with mean function m and the same covariance R as P_0. We shall use the well-known fact, [11], that under (2.6) and (2.8)

(2.12) $$P_m \equiv P_0$$

relative to $\boldsymbol{B}(X)$, if and only if

(2.13) $$m \in H(R).$$

The notation $P_m \equiv P_0$ means that P_m and P_0 are mutually absolutely continuous. If m is in $H(R)$ it follows from (2.12) that the completion $\boldsymbol{B}_m(X)$ of $\boldsymbol{B}(X)$ under P_m coincides with $\boldsymbol{B}_0(X)$. Hence for all m in $H(R)$ the transformation σ_m of (2.10) is $\boldsymbol{B}_0(X)$ measurability preserving and the family of complete Gaussian measures $\{P_m, m \in H(R)\}$ is given on the same measurable space $(X, \boldsymbol{B}_0(X))$.

We shall write $L^2(P_0)$ for $L^2(X, \boldsymbol{B}_0(X), P_0)$, the Hilbert space of $\boldsymbol{B}_0(X)$-measurable, real valued functions square integrable with respect to P_0. Two subspaces of $L^2(P_0)$ will be of special interest for us: (i) $\mathscr{L}_0(X)$ the subspace of a.e. constant random variables in $L^2(P_0)$, and (ii) $\mathscr{L}_1(X)$, the closed linear subspace of $L^2(P_0)$ spanned by all finite linear combinations of the form $\sum_{i=1}^n c_i x(t_i)$ where the c_i's are real constants, $t_i \in T$ and $x \in X$.

The proof of Theorem 1 is achieved by means of a series of auxilliary results which have been arranged to bring out the ideas underlying the main result and also to indicate the possibility of generalizations to non-Gaussian processes.

3. **Lemmas.** Let us denote the Radon-Nikodým derivative of P_m with respect to P_0 by ρ_m. Note that $\rho_0 = 1$ a.s. (P_0). In what follows we shall write $(\ ,\ )$ and $\|\ \|$ and $\langle\ ,\ \rangle$, $|||\cdot|||$ for the inner product and norm of the two spaces $L^2(P_0)$ and $H(R)$ respectively.

LEMMA 1. *For every* $m \in H(R)$

$$\rho_m \in L^2(P_0); \tag{3.1}$$

and if $m_1, m_2 \in H(R)$

$$(\rho_{m_1}, \rho_{m_2}) = \exp(\langle m_1, m_2\rangle). \tag{3.2}$$

Proof. (3.1) and (3.2) are almost immediate consequences of the following two facts [11]. There is an inner product preserving isomorphism between $\mathscr{L}_1(X)$ and $H(R)$ which we shall denote by $\leftrightarrow$. If $u(x) \in \mathscr{L}_1(X)$ and $u \leftrightarrow m$ then ρ_m is given by the expression

$$\rho_m(x) = \exp\{u(x) - \tfrac{1}{2}|||m|||^2\} \quad \text{a.s. } P_0. \tag{3.3}$$

The validity of (3.1) and (3.2) now becomes clear from the following steps. If $u_i \in \mathscr{L}_1(X)$, $u_i \leftrightarrow m_i$ $(i=1, 2)$ we have

$$\begin{aligned}\int_X \rho_{m_1}(x)\rho_{m_2}(x)P_0(dx) &= \int_X \exp(u_1(x) - \tfrac{1}{2}|||m_1|||^2)\exp(u_2(x) - \tfrac{1}{2}|||m_2|||^2)P_0(dx)\\ &= \int_X \exp(u_1(x)+u_2(x) - \tfrac{1}{2}|||m_1+m_2|||^2 + \langle m_1, m_2\rangle)P_0(dx)\\ &= \exp(\langle m_1, m_2\rangle)\int_X \rho_{m_1+m_2}(x)P_0(dx)\\ &= \exp(\langle m_1, m_2\rangle)P_{m_1+m_2}(X)\\ &= \exp(\langle m_1, m_2\rangle).\end{aligned} \tag{3.4}$$

LEMMA 2. *ρ_m is a continuous function from $H(R)$ to $L^2(P_0)$.*

The proof is obvious from (3.2).

The next lemma occurs in the literature on Gaussian processes in contexts not very different from our own (see, e.g., [7], [8]). The form of the lemma suitable for our purpose is as given by R. LePage in his thesis [10].

LEMMA 3. *The family $\{\rho_m, m \in H(R)\}$ spans $L^2(P_0)$.*

Proof. Let $g \in L^2(P_0)$ be such that

$$(g, \rho_m) = 0 \tag{3.5}$$

for all $m \in H(R)$. Fix a finite subset $S=\{t_1, \ldots, t_n\}$ of T and choose m in $H(R)$ of the following form

$$m(\cdot) = \sum_{j=1}^{n} c_j R(\cdot, t_j). \tag{3.6}$$

Then the random variable $\sum_{j=1}^{n} c_j x(t_j)$ is the element of $\mathscr{L}_1(X)$ which corresponds to m given by (3.6). It then follows from (3.3), (3.5) and (3.6) that

$$\int_X g(x) \exp\left(\sum_{j=1}^{n} c_j x(t_j)\right) P_0(dx) = 0. \tag{3.7}$$

Let $\boldsymbol{B}^S(X)$ be the sub σ-field of $\boldsymbol{B}(X)$ generated by the random variables $\{x(t), t \in S\}$. From (3.7) and the fact that the conditional expectation relative to $\boldsymbol{B}^S(X)$

$$E\left\{g(x) \exp\left(\sum_{j=1}^{n} c_j x(t_j)\right) \middle| \boldsymbol{B}^S(X)\right\} = E\{g(x)|\boldsymbol{B}^S(X)\} \exp\left(\sum_{j=1}^{n} c_j x(t_j)\right)$$

we obtain

$$\int_X h[x(t_1), \ldots, x(t_n)] \exp\left(\sum_{j=1}^{n} c_j x(t_j)\right) P_0(dx) = 0. \tag{3.8}$$

Here $h[x(t_1), \ldots, x(t_n)] = E\{g(x)|\boldsymbol{B}^S(X)\}$, so that $h[a_1, \ldots, a_n]$ is a Borel function of the n real variables $(a_1, \ldots, a_n)$. Further h satisfies equation (3.8) for all real numbers $c_j, j=1, \ldots, n$. Using the property that P_0 is a Gaussian measure it can be deduced from (3.8) that

$$h = 0 \quad \text{a.s. } P_0. \tag{3.9}$$

The details of the argument leading from (3.8) to (3.9) can be found in [9, p. 132]. Thus

$$E\{g(x)|\boldsymbol{B}^S(X)\} = 0 \quad \text{a.s. } P_0 \tag{3.10}$$

for every finite subset S of T. From the definition of $\boldsymbol{B}(X)$ [3, p. 604] and (3.10) we have

$$E\{g(x)|\boldsymbol{B}(X)\} = 0 \quad \text{a.s. } P_0. \tag{3.11}$$

Since g is measurable with respect to $\boldsymbol{B}_0(X)$ it follows from (3.11) that $g(x)=0$ a.s. P_0 and the lemma is proved.

LEMMA 4. *Let $g \in L^2(P_0)$. Then*

$$g \in \mathscr{L}_0(X) \tag{3.12}$$

if and only if

$$(g, \rho_m) = (g, \rho_0) \tag{3.13}$$

for each $m \in H(R)$.

Proof. It is obvious that (3.12) implies (3.13). Conversely, writing $\rho_m=(\rho_m-\rho_0)+\rho_0$, noting that $\rho_0=1$ spans $\mathscr{L}_0(X)$ and that

$$(\rho_m-\rho_0, \rho_0) = \int_X \rho_m(x)P_0(dx)-\int_X \rho_0(x)P_0(dx) = P_m(X)-P_0(X) = 0$$

we have

$$\rho_m-\rho_0 = \mathscr{P}_{L^2(P_0)\ominus\mathscr{L}_0(X)}\rho_m \tag{3.14}$$

where $\mathscr{P}_E$ denotes the projection operator onto the subspace E of $L^2(P_0)$. Hence from (3.13) and (3.14)

$$(g, \mathscr{P}_{L^2(P_0)\ominus\mathscr{L}_0(X)}\rho_m) = 0, \tag{3.15}$$

i.e.,

$$(\mathscr{P}_{L^2(P_0)\ominus\mathscr{L}_0(X)}g, \rho_m) = 0 \tag{3.16}$$

for all m in $H(R)$. From Lemma 3 and (3.16) it follows that

$$\mathscr{P}_{L^2(P_0)\ominus\mathscr{L}_0(X)}g = 0, \tag{3.17}$$

which implies that $g \in \mathscr{L}_0(X)$.

The following two lemmas which together form the kernel of the proof of Theorem 1, generalizes the approach of Cameron and Graves (see, e.g., Lemma 3 and Theorem 2 of [2]).

DEFINITION. A subset M of X is said to be a module over the rationals (or an r-module) if for every x_1 and x_2 in M and rational numbers r_1 and r_2,

$$r_1x_1+r_2x_2 \in M. \tag{3.18}$$

LEMMA 5. *Let M be an r-module such that*

$$M \in \boldsymbol{B}_0(X) \tag{3.19}$$

and

$$P_0(M) > 0. \tag{3.20}$$

Then

(3.21) $$M \text{ contains } H(R).$$

Proof. First we make the following observation. For any $\boldsymbol{B}_0(X)$ measurable set F and real number α write

(3.22) $$F_\alpha = \{x \in X : x = y + \alpha m,\ y \in F\}$$

where m is a fixed element in $H(R)$. Then since $-\alpha m \in H(R)$ and $F_\alpha = \sigma^{-1}_{(-\alpha m)}F$, we have

$$P_0(F_\alpha) = P_{(-\alpha m)}(F) = \int_F \rho_{(-\alpha m)}(x) P_0(dx).$$

Hence, from the Schwartz inequality

(3.23) $$|P_0(F_\alpha) - P_0(F)| \leqq \|\rho_{(-\alpha m)} - \rho_0\|.$$

We conclude from (3.23) and Lemma 2 that

(3.24) $$\lim_{\alpha \to 0} P_0(F_\alpha) = P_0(F).$$

To prove the lemma, suppose (3.21) does not hold. If r is any rational number let $M = F$ and M_r denote the set defined in (3.22) with $\alpha = r$ and m in $H(R)$ such that $m \notin M$. The fact that M is an r-module and $m \notin M$ implies that the sets M_{r_1} and M_{r_2} are disjoint whenever $r_1 \neq r_2$. Hence the sets $M_{1/n}$ $(n = 1, 2, \ldots)$ are mutually disjoint. We then have

$$\sum_{n=1}^{\infty} P_0(M_{1/n}) = P_0\left(\bigcup_{n=1}^{\infty} M_{1/n}\right) \leqq 1,$$

so that

(3.25) $$P_0(M_{1/n}) \to 0 \quad \text{as } n \to \infty.$$

But this is impossible since from (3.24), $P_0(M_{1/n}) \to P_0(M)$ which is positive by (3.20). Hence $H(R) \subset M$.

Lemma 6. *Let $\{e_j\}_1^\infty$ be a complete orthonormal system (C.O.N.S.) in $H(R)$ and g a $\boldsymbol{B}_0(X)$-measurable real function such that for each x in X and every rational r*

(3.26) $$g(x + re_j) = g(x) \qquad (j = 1, 2, \ldots).$$

Then

(3.27) $$g(x) = \text{constant } a.s.\ P_0.$$

Proof. We shall first assume that $g \in L^2(P_0)$. Let m be an arbitrary element of $H(R)$. Then, using the separability of $H(R)$ it is easy to find a sequence $\{m^{(p)}\}$, $m^{(p)} \in H(R)$ such that

(3.28) $$m^{(p)} = \sum_{j=1}^{n_p} c_j^{(p)} e_j,$$

where $c_j^{(p)}$ are rationals, the sum in (3.28) is finite and

$$\||m^{(p)}-m\|| \to 0 \quad \text{as } p \to \infty. \tag{3.29}$$

By repeated application of (3.26) we obtain

$$g(x+m^{(p)}) = g(x) \tag{3.30}$$

for every x in X and $p=1, 2, \ldots$. Hence from (3.30) and the standard formula for change of variable under a measurable transformation (see Halmos [6, p. 163]) we find that for every positive integer p,

$$\begin{aligned} \int_X g(x)P_0(dx) &= \int_X g(x+m^{(p)})P_0(dx) \\ &= \int_X g(x)\rho_{m^{(p)}}(x)P_0(dx). \end{aligned} \tag{3.31}$$

As $p \to \infty$, the right-hand side of (3.31) converges to $\int_X g(x)\rho_m(x)P_0(dx)$ because of Lemma 2 and (3.29). The resulting relation from (3.31) can then be written as

$$(g, \rho_0) = (g, \rho_m). \tag{3.32}$$

Since (3.32) holds for every m in $H(R)$ it at once follows from Lemma 4 that $g \in \mathscr{L}_0(X)$. In other words, $g(x)=\text{constant}$ a.s. P_0.

Next suppose that g is $\boldsymbol{B}_0(X)$-measurable. If N is any positive integer define $g_N(x)=g(x)$ if $|g(x)| \leqq N$, and $=0$ if $|g(x)|>N$. Then since $g_N \in L^2(P_0)$ and satisfies (3.26) it follows from the first part of the proof that $g_N(x)=\text{constant}$ a.s. P_0. Since N is arbitrary we have $g(x)=\text{constant}$ a.s. P_0.

4. **A zero-one law for r-modules.** We are now in a position to state and prove our first main theorem.

THEOREM 1. *If M is a $\boldsymbol{B}_0(X)$-measurable r-module then*

$$P_0(M) = 0 \text{ or } 1. \tag{4.1}$$

Proof. If $P_0(M)>0$, then $M \supset H(R)$ by Lemma 5. Let $\{e_j\}_1^\infty$ be a C.O.N.S. in $H(R)$. Then $e_j \in M$. Since M is an r-module it is easy to see that $x+re_j \in M$ if and only if $x \in M$. Letting I_M be the characteristic function of M it then follows that

$$I_M(x+re_j) = I_M(x) \qquad (j = 1, 2, \ldots) \tag{4.2}$$

for all x in X and every rational r. Applying Lemma 6 to the $\boldsymbol{B}_0(X)$ measurable function $I_M(x)$ we have

$$I_M(x) = \text{constant a.s. } P_0. \tag{4.3}$$

Finally since $I_M(x)=0$ or 1 and $P_0(M)$ is positive, (4.3) implies that $P_0(M)=1$. This completes the proof of the theorem.

Before we proceed further some remarks on the scope of assumptions (2.6)–(2.9) seem desirable. In most applications (Example 4 given below is an exception)

T is either the real line or an interval of the real line, so that (2.6) is fulfilled. Assumption (2.8) concerning the continuity of the covariance R (which is equivalent to the continuity in quadratic mean of the process $x(t)$) is a reasonable restriction. Only the assumptions (2.7) and (2.9) invite specific comment. We hope that the following examples will show that for nearly every Gaussian process it is possible to find a realization in a space X for which (2.7) and (2.9) are satisfied.

EXAMPLE 1. $X=\boldsymbol{R}^T$, the set of all real valued functions on T. (2.7) and (2.9) obviously hold.

EXAMPLE 2. Let $T=(-\infty, \infty)$ or a finite interval $[a, b]$, and let $X=C(T)$ the space of real continuous functions on T. This case covers all Gaussian processes with (almost all) continuous sample functions including, of course, the special case of the Wiener process. The validity of (2.7) and (2.9) is again obvious.

EXAMPLE 3. As above let $T=(-\infty, \infty)$ or $[a, b]$ and let $f(t, \omega)$ be a measurable, Gaussian process defined on a probability space $(\Omega, \mathscr{A}, \mu)$. Letting $\tilde{P}_0$ be the Gaussian measure induced on $(\boldsymbol{R}^T, \boldsymbol{B}(\boldsymbol{R}^T))$ it is easy to verify that X, the set of all real Lebesgue measurable functions on T, is a subset of $\boldsymbol{R}^T$ of outer $\tilde{P}_0$ measure one. The probability space $(X, \boldsymbol{B}(X), P_0)$ where P_0 is defined appropriately on sets of $\boldsymbol{B}(X)$, then defines a Gaussian process equivalent to $f(t, \omega)$. X is obviously a linear space. Also $H(R)$ is contained in X since all the functions in $H(R)$ are continuous.

EXAMPLE 4. This example shows that the results of this paper are applicable to certain Gaussian generalized stochastic processes, e.g., those studied in [7]. Let $T=\Phi$, where Φ is a countably Hilbertian nuclear space and let $X=\Phi'$, the dual space of Φ. Let P_0 be a Gaussian measure on $\boldsymbol{B}(\Phi')$ with continuous covariance R. It can then be shown that

$$\Phi \subset H(R) \subset \Phi'. \tag{4.4}$$

Thus all the assumptions (2.6), (2.7) and (2.9) are seen to hold. For the verification of (4.4), definitions of terms not explained here and other details concerning nuclear spaces and Gaussian measures on them we refer the reader to the book by I. M. Gel'fand and N. Ja. Vilenkin [5, Chapter 1, p. 138].

5. **A zero-one law for subgroups.** We shall now consider extending the zero-one law of the last section to groups. If instead of being an r-module the set M of Theorem 1 is merely a subgroup of the additive group X, the proof of Lemma 5 as given above does not work. In the lemmas that follow the essential differences between the proofs of Theorem 1 and Theorem 2 are noted. There is, however, one respect in which the result for subgroups does not generalize Theorem 1. For the latter, we have to consider subgroups G which are measurable with respect to $\boldsymbol{B}(X)$ and not with respect to $\boldsymbol{B}_0(X)$, the completion of $\boldsymbol{B}(X)$ under P_0. The reason for this is that Lemma 10 fails to hold if $\boldsymbol{B}(X)$ is replaced by its completion (e.g., if X is the space of continuous functions on [0, 1] and P_0 is standard Wiener measure).

In the following we shall assume that the reader is familiar with the terminology and facts from elementary number theory and group theory. The use of these ideas as well as Lemmas 8 and 9 are suggested from the book on abelian groups by L. Fuchs [4]. In fact the proof of Lemma 9 essentially follows [4, p. 158], but is given here because it is simpler to prove it here than to give a direct reference.

Let p be any prime and let Q_p be the ring of all rational numbers $r=m/n$ where n and p are relatively prime. (It is always assumed that m and n have no common factors.) Since by (2.7) X is a linear vector space of functions, X is a group under addition. Now suppose that G is a subgroup of X and define

$$G_p = \bigcup_{a \in J_p} \left(\frac{1}{a} G\right), \tag{5.1}$$

where J_p is the set of all integers which are relatively prime to p. Note that it makes sense to define the sets $(1/a)G=\{y : y(t)=(1/a)x(t), x \in G\}$ and that $G_p \subset X$ again an account of (2.7).

LEMMA 8. *G_p is a module over Q_p. That is, if x and y are in G_p and $r_1, r_2 \in Q_p$ then*

$$r_1 x + r_2 y \in G_p. \tag{5.2}$$

Proof. Let x, y, r_1 and r_2 be as given above. Then for some a and $b \in J_p$, $x=x_1/a$, $y=y_1/b$ where $x_1, y_1 \in G$. If $r_i=m_i/n_i$ $(i=1, 2)$ we have

$$r_1x+r_2y = (\nu_1x_1+\nu_2y_1)/L, \tag{5.3}$$

where ν_1 and ν_2 are integers and L is the l.c.m. of n_1a and n_2b. Since G is a group and ν_1 and ν_2 are integers,

$$\nu_1x_1+\nu_2y_1 = x' \in G. \tag{5.4}$$

Since the integers a, b, n_1, n_2 are all relatively prime to p it follows that L is also relatively prime to p. Hence from (5.3) and (5.4)

$$r_1x+r_2y = x'/L \tag{5.5}$$

where $L \in J_p$, i.e., $r_1x+r_2y \in G_p$.

LEMMA 9. *Let π denote the set of all primes. Then*

$$\bigcap_{p \in \pi} G_p = G. \tag{5.6}$$

Proof. Suppose x belongs to the left-hand side of (5.6). Then $x=y/a$, where $y \in G$ and a is relatively prime to p. Let

$$a = p_1^{l_1} \cdots p_s^{l_s} \tag{5.7}$$

be the prime factorization of a. Now $x \in G_{p_j}$ $(j=1, \ldots, s)$, so that there exist integers a_j prime to p_j such that

$$a_j x \in G. \tag{5.8}$$

From (5.7) it follows that the integers $a, a_1, \ldots, a_s$ are relatively prime. Hence there are integers $m, m_1, \ldots, m_s$ such that $ma+m_1a_1+\cdots+m_sa_s=1$. From (5.8) and since $ax \in G$ we have $x=(ma+m_1a_1+\cdots+m_sa_s)x \in G$. Thus we have shown that

$$\bigcap_{p\in\pi} G_p \subset G. \tag{5.9}$$

Conversely let $x \in G$. If p is any prime and $a \in J_p$, then $x \in G_p$ since $x=(1/a)(ax)$, $ax \in G$. Hence

$$G \subset \bigcap_{p\in\pi} G_p. \tag{5.10}$$

Thus (5.6) is established.

LEMMA 10. *$G \in \boldsymbol{B}(X)$ implies $G_p \in \boldsymbol{B}(X)$.*

Proof. The transformation $Tx=ax$, $(a\neq 0)$ maps X into itself and furthermore, since the inverse image under T of a cylinder set in $\boldsymbol{B}(X)$ of the form

$$\{x \in X : a_j \leqq x(t_j) \leqq b_j,\ j = 1, \ldots, n\}$$

is again in $\boldsymbol{B}(X)$ it follows that $T^{-1}A \in \boldsymbol{B}(X)$ for all $A \in \boldsymbol{B}(X)$. Hence $G \in \boldsymbol{B}(X)$ implies that $(1/a)G=T^{-1}G \in \boldsymbol{B}(X)$. It follows therefore that

$$G_p = \bigcup_{a\in J_p} \left(\frac{1}{a}\, G\right)$$

is $\boldsymbol{B}(X)$ measurable.

LEMMA 11. *Let G be a $\boldsymbol{B}(X)$-measurable subgroup such that*

$$P_0(G) > 0. \tag{5.11}$$

Then

$$G_p \text{ contains } H(R). \tag{5.12}$$

Proof. First, it follows from Lemmas 8, 9 and 10 that G_p is a $\boldsymbol{B}(X)$-measurable module over Q_p with $P_0(G_p)>0$. Now suppose there exists an $m \in H(R)$ and $\notin G_p$. For each positive integer n write $k_n=p(p+1)\cdots(np)$ and define the sequence of sets

$$M_n = \{x : x = y+k_n^{-1}m,\ y \in G_p\}. \tag{5.13}$$

We shall show that if $n\neq\nu$ the sets M_n and M_ν are disjoint. If M_n and M_ν have an element in common, then for some x_1, x_2 in G_p,

$$x_1+k_n^{-1}m = x_2+k_\nu^{-1}m, \tag{5.14}$$

so that

$$m = k_nk_\nu(x_2-x_1)/(k_\nu-k_n). \tag{5.15}$$

We may assume, obviously, that $\nu > n$. Then $k_\nu - k_n = k_n \cdot q$, where

$$q = (np+1)(np+2)\cdots(\nu p)-1. \tag{5.16}$$

(5.15) becomes

$$m = k_\nu(x_2-x_1)/q. \tag{5.17}$$

Now $k_\nu(x_2-x_1) \in G_p$ since k_ν is an integer and G_p is a group. Also it is easy to verify from (5.16) that q and p are relatively prime. Hence $k_\nu(x_2-x_1)=x'/r$, for some $x' \in G$ and $r \in J_p$, and from (5.17) $m=x'/rq$, where it is easy to see that rq is relatively prime to p, i.e., that $rq \in J_p$. This proves $m \in G_p$, which is a contradiction. Hence M_n and M_ν are disjoint. Clearly, the sets M_n are $\boldsymbol{B}(X)$-measurable and as in Lemma 2 $P(M_n) \to P(G_p)$ as n (and hence k_n) $\to \infty$. But since $P(G_p)>0$ and $\sum_{n=1}^{\infty} P(M_n) \leqq 1$ we have a contradiction as in the proof of Lemma 5. Thus (5.12) is proved.

THEOREM 2. *If G is a $\boldsymbol{B}(X)$-measurable subgroup of X, then*

$$P_0(G) = 0 \text{ or } 1. \tag{5.18}$$

Proof. Suppose $P_0(G)>0$. Then by Lemma 11, for every prime p G_p contains $H(R)$. Let $\{e_j\}_1^\infty$ be a C.O.N.S. in $H(R)$. From the fact that G_p is a group and $H(R) \subset G_p$ it follows that $x \in G_p$ if and only if $x+re_j \in G_p$ for all rational numbers r and $j=1, 2, \ldots$. In other words for each integer j and rational r we have

$$I_{G_p}(x+re_j) = I_{G_p}(x) \tag{5.19}$$

for all $x \in X$. Lemma 6 now applies without any essential change to the $\boldsymbol{B}(X)$-measurable function $g(x)=I_{G_p}(x)$ and we have

$$I_{G_p}(x) = \text{constant} \quad \text{a.s. } P_0. \tag{5.20}$$

Since $P_0(G_p)$ is positive, (5.20) implies that

$$P_0(G_p) = 1. \tag{5.21}$$

Finally, from Lemma 9 and (5.21) we have $P_0(G)=1$, and the proof of the theorem is complete.

6. **Some applications of the zero-one law.** The following application of Theorem 1 was suggested by R. H. Cameron. In [12] L. A. Shepp has proved a zero-one law for a Wiener process to the effect that if $x(t)$ denotes the sample function of a Wiener process defined on $C[a, b]$ and $f(t)$ is a nonrandom Lebesgue integrable function on $[a, b]$ then either $\int_a^b |f(t)|x^2(t)\,dt<\infty$ a.s. or $=\infty$ a.s. Recently D. Varberg has shown that this result holds for any sample continuous Gaussian process [13]. We shall show that the zero-one law of Shepp and Varberg, in fact an extension of it, follows as an immediate consequence of Theorem 1.

THEOREM 3. *Let $x(t)$, $(a \leqq t \leqq b)$ be a sample continuous Gaussian stochastic process with zero mean function, and let $f(t)$ be an arbitrary measurable function* *and p a positive number. Then either*

(6.1) $$f(t)[x(t)]^p \in L^1[a, b], \text{ a.s.},$$

or

(6.2) $$f(t)[x(t)]^p \notin L^1[a, b], \text{ a.s.}$$

Proof. Let the Gaussian process be given by $(X, \boldsymbol{B}_0(X), P_0)$ where $X = C[a, b]$ is the space of real continuous functions on $[a, b]$, P_0 the probability measure induced by the continuous convariance R, and the random variables of the process are given by the coordinate variables $x(t)$, $x \in X$. Define the set

(6.3) $$M = \left\{x \in X : \int_a^b |f(t)| \cdot |x(t)|^p \, dt < \infty\right\}.$$

M is obviously $\boldsymbol{B}(X)$-measurable. Further, let x and y be in M and let r_1, r_2 be any two rational numbers. From the elementary inequalities

$$|r_1 x + r_2 y|^p \leqq |r_1|^p |x|^p + |r_2|^p |y|^p \quad \text{if } 0 < p \leqq 1,$$

and

$$\leqq 2^{p-1}(|r_1 x|^p + |r_2 y|^p) \quad \text{if } p > 1$$

it follows that $r_1 x + r_2 y \in M$. Hence M defined by (6.3) is an r-module and Theorem 1 immediately yields the result $P_0(M) = 0$ or 1. Theorem 3 is thus proved.

For the next application of Theorem 1 let $x(t)$ be a Gaussian process as in Theorem 3. Define the random variables

(6.4) $$\xi_n(x) = (\lambda_n)^{-1/2} \int_a^b x(t)\phi_n(t) \, dt$$

where $\{\lambda_n\}_1^\infty$ and $\{\phi_n\}_1^\infty$ are the eigenvalues and eigenfunctions of the continuous covariance R. Then $\{\xi_n(x)\}$ are independent Gaussian random variables with zero mean and unit variance. The series

(6.5) $$\sum_{n=1}^{\infty} \lambda_n^{1/2} \xi_n(x) \phi_n(t)$$

which for each t, converges almost surely is called the orthogonal or Karhunen-Loève expansion associated with the process $x(t)$. The conditions under which (6.5) converges uniformly with respect to t almost surely have been discussed recently in the literature (see e.g., [14]). Without additional assumptions on $x(t)$ we can deduce from Theorem 1 the following zero-one law.

THEOREM 4. *Let $(X, \boldsymbol{B}_0(X), P_0)$, $(X = C[a, b])$ be the probability space of the Gaussian process $x(t)$ whose mean is zero and covariance function is R. Then the P_0-probability that the series* (6.5) *converges uniformly with respect to t in* $[a, b]$ *is either zero or one.*

Proof. Write

$$M = \left\{x \in X : \sum_{n=1}^{\infty} \lambda_n^{1/2}\xi_n(x)\phi_n(t) \text{ converges uniformly with respect to } t \text{ in } [a, b]\right\}. \tag{6.6}$$

Since from (6.4)

$$\xi_n(r_1x_1+r_2x_2) = r_1\xi_n(x_1)+r_2\xi_n(x_2), \qquad (x_1, x_2 \in X, \text{ and } r_1, r_2 \text{ rationals}),$$

it is obvious that the measurable set M is an r-module. The desired conclusion then is an immediate consequence of Theorem 1.

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