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ON THE MAXIMALITY OF SUMS OF NONLINEAR MONOTONE OPERATORS

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1. **Introduction.** Let X be a real Banach space and let X^* be the dual of X, with $\langle x, x^*\rangle$ written in place of $x^*(x)$. A *monotone operator* from X to X^* is a (generally multivalued) mapping T such that

$$\langle x-y,\ x^*-y^*\rangle \geqq 0 \quad \text{whenever } x^*\in T(x),\ y^*\in T(y). \tag{1.1}$$

Such an operator T is said to be *maximal* if its *graph*, i.e. the set

$$G(T) = \{(x, x^*) \mid x\in X,\ x^*\in T(x)\} \subset X\times X^* \tag{1.2}$$

is not properly contained in the graph of any other monotone operator $T'\colon X \to X^*$. The set

$$D(T) = \{x\in X \mid T(x) \neq \varnothing\} \tag{1.3}$$

is called the *effective domain* of T, and T is said to be *locally bounded* at a point $x\in D(T)$ if there exists a neighborhood U of x such that the set

$$T(U) = \bigcup \{T(u) \mid u\in U\} \tag{1.4}$$

is a bounded subset of X.

It is apparent that, given any two monotone operators T_1 and T_2 from X to X^*, the operator T_1+T_2 is again monotone, where

$$\begin{aligned}(T_1+T_2)(x) &= T_1(x)+T_2(x)\\ &= \{x_1^*+x_2^* \mid x_1^*\in T_1(x),\ x_2^*\in T_2(x)\}.\end{aligned} \tag{1.5}$$

If T_1 and T_2 are maximal, it does not necessarily follow, however, that T_1+T_2 is maximal—some sort of condition is needed, since for example the graph of T_1+T_2 can even be empty (as happens when $D(T_1)\cap D(T_2)=\varnothing$).

The problem of determining conditions under which T_1+T_2 is maximal turns out to be of fundamental importance in the theory of monotone operators. Results in this direction have been proved by Lescarret [9] and Browder [5], [6], [7]. The strongest result which is known at present is:

THEOREM (BROWDER [6], [7]). *Let X be reflexive, and let T_1 and T_2 be monotone operators from X to X^*. Suppose that T_1 is maximal, $D(T_2)=X$, T_2 is single-valued*

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and hemicontinuous (i.e. continuous from line segments in X to the weak topology in X^*), and T_2 carries bounded sets into bounded sets. Then T_1+T_2 is a maximal monotone operator.*

The conditions here on T_2 imply in particular that T_2, like T_1, is maximal (Browder [3, Theorem 1.2]).

The purpose of this paper is to establish the following considerably more general result (where int and cl denote interior and (strong) closure, respectively).

THEOREM 1. *Let X be reflexive, and let T_1 and T_2 be maximal monotone operators from X to X^*. Suppose that either one of the following conditions is satisfied:*

(a) $D(T_1) \cap \operatorname{int} D(T_2) \neq \varnothing$, *or*

(b) *there exists an $x \in \operatorname{cl} D(T_1) \cap \operatorname{cl} D(T_2)$ such that T_2 is locally bounded at x.*

Then T_1+T_2 is a maximal monotone operator.

Conditions (a) and (b) of Theorem 1 are actually equivalent, as we have shown elsewhere [16, Theorem 1].

The derivation of Theorem 1 will rest heavily on the theory already developed by Browder in [6] and elsewhere. In the case where T_1 and T_2 are the subdifferentials of lower semicontinuous proper convex functions on X [12], [13], Theorem 1 could be deduced, however, directly from Fenchel's duality theorem and related results [11].

When X is finite-dimensional, Theorem 1 may be refined slightly in terms of relative interiors. (The *relative interior* of a subset C of X, denoted by ri C, is the interior of C relative to the *affine hull* of C, which is the intersection of all the hyperplanes containing C.)

THEOREM 2. *Let X be finite-dimensional, and let T_1 and T_2 be maximal operators from X to X^* such that*

$$\operatorname{ri} D(T_1) \cap \operatorname{ri} D(T_2) \neq \varnothing .$$

One of the main motivations behind Theorems 1 and 2 is that such results make it possible, as Browder has remarked [6, p. 92], to derive theorems about "variational inequalities" from fundamental theorems about the ranges and effective domains of (multivalued) maximal monotone operators. Some applications of this sort will be considered in §4. For the sake of applications to a class of variational inequalities studied in terms of hemicontinuity by Hartman-Stampacchia [8] and Browder [4], it will be useful to have, along with Theorems 1 and 2, the following special maximality theorem.

We shall say that an element $x^* \in X^*$ is *normal* to a convex subset K of X at a point x if

$$x \in K \quad \text{and} \quad \langle u-x, x^*\rangle \leqq 0, \qquad \forall\, u \in K. \tag{1.6}$$

For each $x \in X$, the set of all x^* normal to K at x is classically called the *normal* *cone* to K at x. The normal cone to K at x is a weak* closed convex cone in X^*, empty by definition when $x \notin K$, but containing at least the zero vector of X^* (and hence nonempty) when $x \in K$. The multivalued mapping from X to X^* which assigns to each $x \in X$ the normal cone to K at x will be referred to as the *normality operator* for K. (This mapping is actually the subdifferential of the indicator of K, so that it is a maximal monotone operator with effective domain K, if K is a nonempty closed convex set [12], [13].)

THEOREM 3. *Let K be a nonempty closed convex subset of X (where X is not necessarily reflexive), and let T_1: $X \to X^*$ be the normality operator for K. Let T_2: $X \to X^*$ be any single-valued monotone operator (not necessarily maximal) such that $D(T_2) \supset K$ and T_2 is hemicontinuous on K, i.e. continuous along each line segment in K with respect to the weak* topology of X^*. Then T_1+T_2 is a maximal monotone operator.*

2. **Preliminary results.** In this section we shall only be concerned with the case where X is reflexive. Asplund [1] has shown by means of a theorem of Lindenstrauss that, in this case, there exists an equivalent norm on X which is everywhere Gâteaux differentiable except at the origin and whose polar norm on X^* is everywhere Gâteaux differentiable except at the origin. (Under such a norm, the unit balls of X and X^* are strictly convex.) For notational simplicity, we may assume that the given norm on X already has these special properties. We denote by J the Gâteaux gradient of the function $h(x)=(1/2)\|x\|^2$. Thus J is the duality mapping which assigns to each $x \in X$ the unique $J(x) \in X^*$ such that

$$\langle x, J(x)\rangle = \|x\|^2 = \|J(x)\|^2. \tag{2.1}$$

(See [2].) As is known, J maps X one-to-one onto X^* and is continuous from the strong topology to the weak* topology. Also, J is a strictly monotone operator, i.e.

$$\langle x-y, J(x)-J(y)\rangle > 0 \quad \text{when } x \neq y. \tag{2.2}$$

For any monotone operator T from X to X^*, we define the mapping T^{-1} by

$$T^{-1}(x^*) = \{x \mid x^* \in T(x)\}. \tag{2.3}$$

It is obvious that T^{-1} is a monotone operator from X^* to X, and (assuming X to be reflexive) T^{-1} is maximal if and only if T is maximal. We denote the *range* of T by $R(T)$. Thus

$$R(T) = D(T^{-1}) = \bigcup \{T(x) \mid x \in X\}. \tag{2.4}$$

The main tool which we shall use in proving Theorem 1 is a generalization, essentially due to Browder [6], of the fundamental Hilbert space theorem of Minty [10].

PROPOSITION 1. *Let X be reflexive, and let J be the duality mapping defined above. Let T: $X \to X^*$ be any maximal monotone operator. Then, for any $\lambda > 0$, $R(T+\lambda J)$*

is all of X^* and $(T+\lambda J)^{-1}$ is a single-valued maximal monotone operator from X^* to X which is demicontinuous, i.e. continuous from the strong topology to the weak topology.

Proof. If $0 \in D(T)$, this follows from the argument given by Browder at the beginning of the proof of Theorem 4 of [6], except that Lemma 10 of [6] should be invoked in place of Lemma 8. We want to use this result, however, even in the case where $0 \notin D(T)$, so we must give a somewhat modified proof. Let $a \in D(T)$, and let T_1 and T_2 be the monotone operators defined by

$$T_1(x) = T(x+a), \qquad T_2(x) = \lambda J(x+a).$$

Then $0 \in D(T_1)$, and T_1 and T_2 satisfy the hypothesis of Theorem 1 of [6], as is easily verified. This implies that $R(T_1+T_2)$ is all of X^*, and of course $R(T_1+T_2)$ is the same as $R(T+\lambda J)$. Also, $T+\lambda J$ is a maximal monotone operator by Theorem 2 of [6](2). Thus $(T+\lambda J)^{-1}$ is a maximal monotone operator from X^* to X whose effective domain is all of X^*. We shall show that $(T+\lambda J)^{-1}$ is single-valued, and this will imply by Rockafellar [14, Corollary 1 to Theorem 1] that $(T+\lambda J)^{-1}$ is demicontinuous. Let x and y be elements of $(T+\lambda J)^{-1}(u^*)$, and let

$$x^* = u^* - \lambda J(x) \in T(x), \qquad y^* = u^* - \lambda J(y) \in T(y).$$

Since T is monotone, we have

$$\begin{aligned} 0 &= \langle x-y, u^*-u^*\rangle = \langle x-y, (x^*+\lambda J(x))-(y^*+\lambda J(Y))\rangle \\ &= \langle x-y, x^*-y^*\rangle + \lambda\langle x-y, J(x)-J(y)\rangle \\ &\geqq \lambda\langle x-y, J(x)-J(y)\rangle, \end{aligned}$$

and this implies $x=y$ by (2.2).

Corollary. *Let X be reflexive, and let J be the duality mapping defined above. Let T: $X \to X^*$ be a monotone operator. In order that T be maximal, it is necessary and sufficient that $R(T+J)$ be all of X^*.*

Proof. By Zorn's Lemma, there exists a maximal monotone operator T': $X \to X^*$ such that $T'(x) \supset T(x)$ for every x. Applying Proposition 1 to T' with $\lambda=1$, one sees that, for each $u^* \in X^*$, there exist unique elements $x \in X$ and $x^* \in X^*$ such that $x^* \in T'(x)$ and $x^*+J(x)=u^*$. Therefore, $T=T'$ if and only if each $u^* \in X^*$ can actually be expressed in the form $x^*+J(x)$ for some $x \in X$ and $x^* \in T(x)$, i.e. if and only if $R(T+J)=X^*$.

The next result may be regarded as a generalization of Theorem 2.2 of Browder [3]. Its proof is derived from Browder's proof of Theorem 3 of [6].

Proposition 2. *Let X be reflexive, and let T: $X \to X^*$ be a maximal monotone operator. Suppose there exists an $\alpha > 0$ such that*

$$\langle x, x^*\rangle \geqq 0 \quad \textit{whenever } \|x\| > \alpha,\ x \in D(T),\ x^* \in T(x). \tag{2.5}$$

(2) The hypothesis in Theorem 1 of [6] that T_1 have a dense domain is superfluous—it is nowhere used in the proof nor mentioned in the rest of the paper.

Then there exists an $x \in X$ *such that*

$$0 \in T(x). \tag{2.6}$$

Proof. We can assume that the given norm on X has the special properties described above. Let U_α be the closed ball of radius α around the origin of X. The set

$$T(U_\alpha) = \bigcup \{T(x) \mid \|x\| \leqq \alpha\} \tag{2.7}$$

is closed in X^*; see Rockafellar [14, Lemma 2]. Hence, to show that $0 \in T(U_\alpha)$, it suffices to show that $T(U_\alpha)$ meets every ball of positive radius about the origin of X^*. Let J be the duality mapping defined above. Given any $\varepsilon > 0$, there exists by Proposition 1 an $x \in X$ such that

$$0 \in (T+(\varepsilon/\alpha)J)(x).$$

Setting $x^* = -(\varepsilon/\alpha)J(x)$, we have $x^* \in T(x)$ and

$$0 = \langle x, x^* \rangle + (\varepsilon/\alpha)\langle x, J(x)\rangle,$$

or equivalently by (2.1)

$$\langle x, x^* \rangle = -(\varepsilon/\alpha)\|x\|^2. \tag{2.8}$$

If we had $\|x\| > \alpha$, the left side of (2.8) would be nonnegative by (2.5), whereas the right side would be negative, a contradiction. Therefore $\|x\| \leqq \alpha$ and

$$\|x^*\| = (\varepsilon/\alpha)\|J(x)\| = (\varepsilon/\alpha)\|x\| \leqq \varepsilon.$$

This shows that $T(U_\alpha)$ meets the closed ball of radius ε about the origin of X^*, and the proof is complete.

Besides the important device of perturbing a monotone operator T to an operator of the form $T + \lambda J$, we shall need to use a device of "truncation" in proving Theorem 1. For each $\alpha > 0$, we shall denote by B_α the subdifferential of the indicator function of the closed ball U_α of radius α about the origin in X (with respect to a norm on X having the special properties described above). Thus B_α is the normality operator for U_α, so that $B_\alpha(x) = \varnothing$ when $\|x\| > \alpha$, $B_\alpha(x)$ consists of solely the zero element of X^* when $\|x\| < \alpha$, and

$$B_\alpha(x) = \{\lambda J(x) \mid \lambda \geqq 0\} \quad \text{when } \|x\| = \alpha.$$

It is known that B_α is a maximal monotone operator (Rockafellar [12], [13]). In particular, therefore, if $T\colon X \to X^*$ is any monotone operator, $T + B_\alpha$ is a monotone operator. Note that

$$(T+B_\alpha)(x) = T(x) \quad \text{when } x \in \operatorname{int} U_\alpha, \tag{2.9}$$

$$D(T+B_\alpha) = D(T) \cap U_\alpha. \tag{2.10}$$

Proposition 3. *Let X be reflexive, and let $T\colon X \to X^*$ be a monotone operator* *such that* $0 \in D(T)$. *Suppose there exists an* $\alpha_0 > 0$ *such that the monotone operator* $T+B_\alpha$ *is maximal for every* $\alpha \geqq \alpha_0$, *where* B_α *is the mapping defined above. Then* T *is maximal.*

Proof. Subtracting a constant mapping from T if necessary, we can reduce to the case where $0 \in T(0)$. Let J be the duality mapping defined above, and let u^* be an arbitrary element of X^*. To prove that T is maximal, it suffices, according to the corollary to Proposition 1, to prove the existence of an $x \in X$ such that

$$u^* \in (T+J)(x). \tag{2.11}$$

Take any $\alpha \geqq \alpha_0$ such that $\alpha > \|u^*\|$. Since $T+B_\alpha$ is maximal by hypothesis, there exists by Proposition 1 a certain $x \in X$ such that

$$u^* \in (T+B_\alpha+J)(x) = (T+J)(x)+B_\alpha(x).$$

By the definition of B_α, this means that

$$u^* \in (T+J)(x)+\lambda J(x) = (T+(1+\lambda)J)(x), \tag{2.12}$$

where $\|x\| \leqq \alpha$, and λ is some nonnegative number, taken to be 0 if $\|x\| < \alpha$. We shall show that indeed $\|x\| < \alpha$, so that (2.12) reduces to (2.11) as desired. According to (2.12), there exists an $x^* \in T(x)$ such that

$$u^* = x^*+(1+\lambda)J(x).$$

In terms of this x^*, we have

$$\langle x, u^*\rangle = \langle x, x^*\rangle+(1+\lambda)\langle x, J(x)\rangle,$$

where $\langle x, x^*\rangle \geqq 0$ by the monotonicity of T and the fact that $0 \in T(0)$. It follows by (2.1) that

$$(1+\lambda)\|x\|^2 = (1+\lambda)\langle x, J(x)\rangle \leqq \langle x, u^*\rangle \leqq \|x\| \cdot \|u^*\|,$$

and hence that

$$\|x\| \leqq (1+\lambda)^{-1}\|u^*\| < (1+\lambda)^{-1}\alpha \leqq \alpha.$$

3. Proofs of the main results.

Proof of Theorem 1. Since assumptions (a) and (b) in Theorem 1 are equivalent by [14, Theorem 1], as already pointed out, we need only consider the case of assumption (a). We shall suppose, to begin with, that $D(T_2)$ is also a bounded subset of X. Subsequently we shall prove, using Proposition 3, that this boundedness assumption is unnecessary. It will be assumed, of course, that the norm on X which we work with has the special properties described at the beginning of §2.

Translating T_1 and T_2 by a common amount if necessary, and subtracting a constant mapping from T_1, we can assume that

$$0 \in T_1(0), \qquad 0 \in \operatorname{int} D(T_2). \tag{3.1}$$

Let J be the duality mapping defined above. We shall show that, for the monotone operator T_1+T_2, $R(T_1+T_2+J)$ is all of X^*, and this will establish that T_1+T_2 is maximal, according to the corollary to Proposition 1.

Given any $\bar{x}^* \in X^*$, we must show that $\bar{x}^* \in R(T_1+T_2+J)$. Subtracting a constant mapping from T_2 if necessary, we can reduce the argument to the case where $\bar{x}^*=0$. Thus we need only show the existence of an $x \in X$ such that

$$0 \in (T_1+T_2+J)(x). \tag{3.2}$$

Now x satisfies (3.2) if and only if there exists an $x^* \in X^*$ such that

$$-x^* \in (T_1+(1/2)J)(x) \quad \text{and} \quad x^* \in (T_2+(1/2)J)(x). \tag{3.3}$$

Define the mappings S_1 and S_2 from X^* to X by

$$S_1(x^*) = -(T_1+(1/2)J)^{-1}(-x^*), \tag{3.4}$$

$$S_2(x^*) = (T_2+(1/2)J)^{-1}(x^*). \tag{3.5}$$

The existence of an x and x^* satisfying (3.3) is then equivalent to the existence of an x^* satisfying

$$0 \in S_1(x^*)+S_2(x^*). \tag{3.6}$$

Therefore, to prove the existence of an x satisfying (3.2), it suffices to prove that

$$0 \in R(S_1+S_2). \tag{3.7}$$

To do this, we observe from Proposition 1 that S_1 and S_2 are single-valued maximal monotone operators, continuous from the strong topology of X^* to the weak topology of X, such that

$$D(S_1) = X^* = D(S_2). \tag{3.8}$$

Hence S_1+S_2 is a single-valued monotone operator, continuous from the strong topology to the weak topology, such that $D(S_1+S_2)=X^*$, and this implies by Browder [3, Theorem 1.2] that S_1+S_2 is maximal. Since $J(0)=0$, we have

$$0 \in (T_1+(1/2)J)(0) \tag{3.9}$$

by (3.1), and consequently $0 \in S_1(0)$. Therefore

$$\langle S_1(x^*), x^*\rangle \geqq 0, \qquad \forall\, x^* \in X^*, \tag{3.10}$$

by the monotonicity of S_1. Furthermore, $R(S_2)=D(T_2+(1/2)J)=D(T_2)$, so that $R(S_2)$ is a bounded set by our initial assumption and, by (3.1),

$$0 \in \operatorname{int} R(S_2). \tag{3.11}$$

We shall show that these properties of $R(S_2)$ imply the existence of an $\alpha>0$ such that

$$\langle S_2(x^*), x^*\rangle \geqq 0 \quad \text{whenever } \|x^*\| > \alpha. \tag{3.12}$$

This will establish (3.7) as desired, via Proposition 2, since by (3.10) and (3.12) we will have

$$\langle (S_1+S_2)(x^*), x^*\rangle \geqq 0 \quad \text{whenever } \|x^*\| > \alpha.$$

The proof that (3.12) holds for some $\alpha>0$ proceeds as follows. For any x^* and y^* in X^*, we have

$$\langle S_2(x^*)-S_2(y^*), x^*-y^*\rangle \geqq 0$$

by the monotonicity of S_2, in other words

$$\langle S_2(x^*), x^*\rangle \geqq \langle S_2(y^*), x^*\rangle + \langle S_2(x^*)-S_2(y^*), y^*\rangle. \tag{3.13}$$

Since $R(S_2)$ is bounded in X, $R(S_2)$ is contained in a certain ball of radius $\alpha_1>0$ about the origin, and hence

$$|\langle S_2(x^*)-S_2(y^*), y^*\rangle| \leqq 2\alpha_1\|y^*\| \tag{3.14}$$

in (3.13). On the other hand, (3.11) implies by [14, Theorem 1] that S_2^{-1} is locally bounded at 0. Thus there exist $\varepsilon>0$ and $\alpha_2>0$ such that

$$\{S_2(y^*) \mid \|y^*\| \leqq \alpha_2\} \supset \{y \mid \|y\| \leqq \varepsilon\}. \tag{3.15}$$

From (3.13) and (3.14), we have

$$\langle S_2(x^*), x^*\rangle \geqq \langle S_2(y^*), x^*\rangle - 2\alpha_1\alpha_2$$

for every y^* with $\|y^*\| \leqq \alpha_2$, so that by (3.15)

$$\begin{aligned}\langle S_2(x^*), x^*\rangle &\geqq \sup_{\|y\|\leqq\varepsilon} \{\langle y, x^*\rangle - 2\alpha_1\alpha_2\}\\ &= \varepsilon\|x^*\| - 2\alpha_1\alpha_2.\end{aligned}$$

The latter expression is nonnegative when $\|x^*\| \geqq 2\alpha_1\alpha_2/\varepsilon$, and therefore (3.12) holds, as claimed, for $\alpha \geqq 2\alpha_1\alpha_2/\varepsilon$.

The argument that we have given so far proves that Theorem 1 is valid under the additional hypothesis that $D(T_2)$ is bounded. We shall show now that this narrower version of Theorem 1 implies the general version. Let T_1 and T_2 be maximal monotone operators such that

$$D(T_1) \cap \operatorname{int} D(T_2) \neq \varnothing, \tag{3.16}$$

where $D(T_2)$ is not necessarily bounded. Translating the domains of T_1 and T_2 if necessary, we can assume that the origin belongs to the intersection (3.16). For each $\alpha>0$, the maximal monotone mapping B_α described in §2 then satisfies $D(T_2) \cap \operatorname{int} D(B_\alpha) \neq \varnothing$, and $D(B_\alpha)$ is bounded. The monotone operator T_2+B_α is therefore maximal by the narrower version of Theorem 1. Since

$$D(T_2+B_\alpha) = \{x \in D(T_2) \mid \|x\| \leqq \alpha\},$$

and the origin belongs to the intersection (3.16), we have $D(T_1) \cap \operatorname{int} D(T_2+B_\alpha)$

$\neq \varnothing$, where $D(T_2+B_\alpha)$ is again bounded. Thus the mapping $T_1+(T_2+B_\alpha)=(T_1+T_2)+B_\alpha$ is a maximal monotone operator for every $\alpha>0$ by the narrower version of Theorem 1, and we may conclude from Proposition 3 that T_1+T_2 is maximal. This proves Theorem 1 in the general case.

Proof of Theorem 2. Translating T_1 and T_2 by a common amount if necessary, we can suppose that

$$0 \in \operatorname{ri} D(T_1) \cap \operatorname{ri} D(T_2), \tag{3.17}$$

so that the affine hulls of $D(T_1)$ and $D(T_2)$ are certain subspaces L_1 and L_2 of X, respectively. Let $L_0=L_1 \cap L_2$, and for $i=0, 1, 2$ let

$$L_i^\perp = \{x^* \in X^* \mid \langle x, x^*\rangle = 0, \forall\, x \in L_i\}, \tag{3.18}$$

$$\begin{aligned} P_i(x) &= L_i^\perp \quad \text{if } x \in L_i, \\ &= \varnothing \quad \text{if } x \notin L_i. \end{aligned} \tag{3.19}$$

Each P_i is a maximal monotone operator (the subdifferential of the indicator of L_i), and

$$P_0 = P_1+P_2 = P_0+P_1 = P_0+P_2. \tag{3.20}$$

Given any $x \in D(T_1)$, $x^* \in T_1(x)$, $z^* \in L_1^\perp$, we have $\langle x-y, (x^*+z^*)-y^*\rangle = \langle x-y, x^*-y^*\rangle \geqq 0$ whenever $y \in D(T_1)$, $y^* \in T(y)$, because T_1 is a monotone operator with $D(T_1)\subset L_1$, and this implies by the maximality of T_1 that

$$x^*+z^* \in T_1(x).$$

Thus

$$T_1 = T_1+P_1. \tag{3.21}$$

Similarly

$$T_2 = T_2+P_2, \tag{3.22}$$

and it follows that

$$T_1+T_2 = T_1+T_2+P_1+P_2 = (T_1+P_0)+T_2. \tag{3.23}$$

In view of (3.20) and (3.21), we can regard T_1 and P_0 in a natural way as maximal monotone operators from the space L_1 to the quotient space $X^*/L_1^\perp$, which may be identified with the dual L_1^* of L_1. Theorem 1 is applicable to these mappings from L_1 to L_1^*, since by (3.17) the origin belongs to the intersection of $D(P_0)=L_0$ and the interior, relative to L_1, of $D(T_1)$. Thus T_1+P_0 must be a maximal monotone operator from X to X^*. Now we apply a similar argument to the space L_2. Since (3.22) holds and $(T_1+P_0)+P_2=T_1+P_0$, we can regard T_1+P_0 and T_2 as maximal monotone operators from L_2 to $X^*/L_2^\perp$, which may be identified with L_2^*. The interior of $D(T_2)$ relative to L_2 meets $D(T_1+P_0)=D(T_1) \cap L_2$ by (3.17), so $(T_1+P_0)+T_2$ is maximal by Theorem 1. This means by (3.23) that T_1+T_2 is a maximal monotone operator from X to X^*.

Proof of Theorem 3. As already mentioned in §1, T_1 is a maximal monotone operator, since it is the subdifferential of a certain closed proper convex function, namely the indicator of K [12], [13]. Therefore T_1+T_2 is, at all events, a monotone operator. Let $y \in X$ and $y^* \in X^*$ be such that

$$\langle x-y, x^*-y^*\rangle \geqq 0 \quad \text{whenever } x^* \in T_1(x)+T_2(x), \tag{3.24}$$

or in other words

$$\langle x-y, x_1^*\rangle+\langle x-y, T_2(x)-y^*\rangle \geqq 0 \quad \text{whenever } x \in K \text{ and } x_1^* \text{ is normal to } K \text{ at } x. \tag{3.25}$$

We shall demonstrate by a direct argument that

$$y^* \in T_1(y)+T_2(y), \quad \text{i.e. } y^*-T_2(y) \in T_1(y), \tag{3.26}$$

and this will prove that T_1+T_2 is maximal.

If x is any point of K and x_1^* is normal to K at x, then λx_1^* is likewise normal to K at x for every $\lambda \geqq 0$, so that by (3.25)

$$\lambda\langle x-y, x_1^*\rangle+\langle x-y, T_2(x)-y^*\rangle \geqq 0, \qquad \forall\, \lambda \geqq 0.$$

This implies that $\langle x-y, x_1^*\rangle \geqq 0$. Thus

$$\langle x-y, x_1^*-0\rangle \geqq 0 \quad \text{whenever } x_1^* \in T_1(x),$$

and since T_1 is a maximal monotone operator we may conclude that $0 \in T_1(y)$. Hence $y \in K$.

To complete the proof that (3.26) holds, i.e. that $y^*-T_2(y)$ is normal to K at y, we need only show that

$$\langle u-y, y^*-T_2(y)\rangle \leqq 0, \qquad \forall\, u \in K. \tag{3.27}$$

Fix any $u \in K$, and let

$$x_\lambda = \lambda u+(1-\lambda)y, \qquad 0 < \lambda < 1.$$

Since $y \in K$ and K is convex, we have $x_\lambda \in K$. Therefore (3.25) holds for $x=x_\lambda$ and $x_1^*=0$, and we have

$$\begin{aligned} 0 &\leqq \langle x_\lambda-y, T_2(x)-y^*\rangle = \lambda\langle u-y, T_2(x)-y^*\rangle \\ &= \lambda\langle u-y, T_2(x_\lambda)-T_2(y)\rangle-\lambda\langle u-y, y^*-T_2(y)\rangle. \end{aligned}$$

This implies that

$$\langle u-y, y^*-T_2(y)\rangle \leqq \langle u-y, T_2(x_\lambda)-T_2(y)\rangle, \qquad 0 < \lambda < 1.$$

Since T_2 is hemicontinuous, $T_2(x_\lambda)$ converges in the weak* topology to $T_2(y)$ as λ decreases to 0, and (3.27) must hold.

4. **Applications.** Theorems 1, 2 and 3 may be used to get new theorems asserting that $R(T_1+T_2)$ is all of X, or that $0 \in R(T_1+T_2)$, and so forth.

For example, Theorem 3 of Browder [6] leads to the following result, where a monotone operator $T: X \to X^*$ is said to be *coercive* if

$$\lim_{\alpha \to +\infty} (1/\alpha) \inf \{\langle x, x^* \rangle \mid x^* \in T(x), \|x\| \geqq \alpha\} = +\infty \tag{4.1}$$

(inf $\varnothing$ being $+\infty$ by convention).

THEOREM 4. *Let X be reflexive, and let T_1 and T_2 be monotone operators satisfying the hypothesis of Theorem* 1 (*or Theorem* 2 *or Theorem* 3), *such that T_1+T_2 is coercive. Then $R(T_1+T_2)=X^*$.*

(*In particular, T_1+T_2 is coercive if $0 \in D(T_1)$ and T_2 is coercive, or vice versa. Also, T_1+T_2 is coercive trivially if $D(T_1) \cap D(T_2)$ is bounded.*)

Proof. The monotone operator $T=T_1+T_2$ is maximal by Theorem 1 (or by Theorem 2 or Theorem 3, as the case may be), and since T is also coercive we have $R(T)=X^*$ according to Browder [6, Theorem 3]. (Browder's result assumes that the unit ball of X^* is strictly convex and that $0 \in D(T)$. However, the strict convexity assumption can be avoided by giving the proof in terms of a duality mapping J of the type employed in §2. This is permissible, since passage to an equivalent norm on X does not alter the coerciveness of the operator T. The assumption that $0 \in D(T)$ can then be avoided by invoking Proposition 1 of the present paper in the proof in place of Browder's Theorem 1.)

In the case where T_1 and T_2 satisfy the hypothesis of Theorem 3, Theorem 4 yields a result about variational inequalities proved independently by Browder [4] and Hartman-Stampacchia [8].

A condition for the existence of an x satisfying

$$0 \in T_1(x)+T_2(x) \tag{4.2}$$

can be obtained at once by combining Theorems 1, 2 and 3 with Proposition 2. In particular, taking T_1 or T_2 to be the normality operator associated with a convex set K, one obtains the following existence theorem for solutions to variational inequalities.

THEOREM 5. *Let X be reflexive, let K be a closed convex subset of X, and let $A: X \to X^*$ be a* (*possibly multivalued*) *monotone operator. Suppose there exist an $a \in K$ and an $\alpha > 0$ such that*

$$\langle x-a, x^* \rangle \geqq 0 \quad \textit{whenever } x \in D(A) \cap K,\ \|x\| > \alpha,\ x^* \in A(x). \tag{4.3}$$

Suppose also that one of the following five conditions is satisfied:

(a) *$D(A) \supset K$ and A is single-valued and hemicontinuous on K, or*

(b) *A is maximal and $K \cap \operatorname{int} D(A) \neq \varnothing$, or*

(c) *A is maximal and $D(A) \cap \operatorname{int} K \neq \varnothing$, or*

(d) *A is locally bounded at some $x \in K \cap \operatorname{cl} D(A)$ and maximal, or*

(e) *X is finite-dimensional, A is maximal and*

$$\operatorname{ri} D(A) \cap \operatorname{ri} K \neq \varnothing.$$

Then the variational inequality for A and K has a solution, i.e. there exists at least one $x \in D(A) \cap K$ such that, for some $x^* \in A(x)$, $-x^*$ is normal to K at x.

Proof. Replacing K and A by the translates $K' = K - a$ and $A'(x) = A(x+a)$ if necessary, we can reduce the theorem to the case where $a = 0$. In terms of the normality operator N for K, which has $D(N) = K$, the solutions to the variational inequality for A and K are simply the points x such that

$$0 \in N(x) + A(x) = (N+A)(x).$$

Such a solution exists by Proposition 2, if the monotone operator $T = N + A$ is maximal, since condition (2.5) for T is equivalent to condition (4.3) on K and A (when $0 = a \in K$). Each of the conditions (a), (b), (c), (d) and (e) is sufficient, in view of Theorems 1, 2 and 3, for T to be maximal.

When the set K is given by a system of convex inequality constraints,

$$K = \{x \mid f_i(x) \leqq 0, \; i = 1, \dots, m\}, \tag{4.4}$$

the normality condition in Theorem 5 can sometimes be expressed in terms of the subdifferentials ∂f_i of the convex functions f_i and certain *Lagrange multipliers.* In particular, there is the following result.

COROLLARY. *Let X be reflexive, and let $T\colon X \to X^*$ be a maximal monotone operator. Let $f_1, \dots, f_m$ be continuous real-valued convex functions on X. Suppose there exist an $a \in X$ and an $\alpha > 0$ such that*

$$a \in D(T) \quad \textit{and} \quad f_i(a) < 0 \quad \textit{for } i = 1, \dots, m, \tag{4.5}$$

$$\begin{aligned} \langle x - a, x^* \rangle \geqq 0 \quad \textit{whenever } x \in D(T), \quad x^* \in T(x), \\ \|x\| > \alpha \quad \textit{and} \quad f_i(x) \leqq 0 \textit{ for } i = 1, \dots, m. \end{aligned} \tag{4.6}$$

Then there exist real numbers $\lambda_1, \dots, \lambda_m$ (Lagrange multipliers) and an $x \in X$ such that

$$\lambda_i \geqq 0, \qquad f_i(x) \leqq 0, \qquad \lambda_i f_i(x) = 0, \qquad i = 1, \dots, m, \tag{4.7}$$

$$0 \in T(x) + \lambda_1 \partial f_1(x) + \cdots + \lambda_m \partial f_m(x). \tag{4.8}$$

Proof. For K as in (4.4), we have $a \in D(T) \cap \operatorname{int} K$ by (4.5), so that the hypothesis of Theorem 5 is satisfied under condition (c). The corollary then follows from the fact that (since the inequality system $f_i < 0$, $i = 1, \dots, m$, can be satisfied) the normal cone to K at a point $x \in K$ is the union of $\lambda_1 \partial f_1(x) + \cdots + \lambda_m \partial f_m(x)$ over all coefficients λ_i such that $\lambda_i \geqq 0$ for indices i such that $f_i(x) = 0$ and $\lambda_i = 0$ for indices i such that $f_i(x) < 0$. For the proof of the latter fact, see Rockafellar [11, p. 86]. (The argument in [11] concerns the case where $f_1, \dots, f_m$ are Gâteaux differentiable, but it is easily extended to the general case.)

REMARK. When certain of the functions f_i are actually *affine* (i.e. linear-plus-a-constant), the corresponding conditions $f_i(a) < 0$ in (4.5) may be weakened to

$f_i(a) \leqq 0$, provided that the condition $a \in D(T)$ is strengthened at the same time to $a \in \operatorname{int} D(T)$. This follows by the argument given in [11, p. 87]. With this modification, the corollary may be applied to cases where the definition of K involves constraints of the form $\langle x, b\rangle = \beta$, where $b \in X^*$, since such a constraint can always be re-expressed as a pair of affine inequality constraints:

$$f_1(x) = \langle x, b\rangle - \beta \leqq 0, \qquad f_2(x) = \beta - \langle x, b\rangle \leqq 0.$$

Observe that, according to (4.8) the "solution" x whose existence is asserted in the corollary is in particular a solution to:

$$0 \in S(x), \quad \text{where } S = T + \lambda_1 \partial f_1 + \cdots + \lambda_m \partial f_m. \tag{4.9}$$

Moreover, S is a maximal monotone operator by Theorem 1, since the subdifferentials ∂f_i are maximal monotone operators with $D(\partial f_i) = X$. (A nonnegative multiple of a maximal monotone operator with effective domain X is trivially another maximal monotone operator.) In particular, suppose in the corollary that T is single-valued on $D(T)$, and that each f_i is actually Gâteaux differentiable, so that the subdifferentials ∂f_i reduce to single-valued gradient mappings ∇f_i. Then S is a single-valued maximal monotone operator with $D(S) = D(T)$, and (4.9) becomes an equation:

$$0 = S(x) = T(x) + \lambda_1 \partial f_1(x) + \cdots + \lambda_m \partial f_m(x). \tag{4.10}$$

The Lagrange multipliers λ_i thus make it possible sometimes to reduce variational inequalities to operator equations of a simpler sort, which can be useful of course in the analysis of the solutions x, at least in cases where T is a differential or integral operator whose properties are well understood.

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