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ON n-PARAMETER DISCRETE AND CONTINUOUS SEMIGROUPS OF OPERATORS

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Abstract. We prove that n commuting operators on a Hilbert space can be uniquely simultaneously extended to doubly commuting coisometric operators if and only if they satisfy certain positivity conditions, which for the case $n=1$ state simply that the original operator is a contraction. Our proof establishes the connection between these positivity conditions and the backward translation semigroup on $l^2(\boldsymbol{Z}^{+n}, \mathscr{K})$. A semigroup of operators is unitarily equivalent to backward translation (or a part thereof) on $l^2(\boldsymbol{Z}^{+n}, \mathscr{K})$ if and only if the positivity conditions are satisfied and the individual operators are coisometries (or contractions) whose powers tend strongly to zero. Analogous results are proven in the continuous case $\boldsymbol{R}^{+n}$.

1. **Introduction.** In this paper we consider the question of which finite families of commuting operators on a Hilbert space can be simultaneously extended to a family of commuting coisometric operators. We generalize B. Sz-Nagy's Theorem [11] that every contraction has a unique minimal coisometric extension, by generalizing R. G. Douglas' proof [5]. In fact for commuting operators satisfying the positivity conditions, (ii) of Theorem 1, we actually construct their unique minimal doubly commuting coisometric extension, using the backward translation semigroup on $l^2(\boldsymbol{Z}^{+n}, \mathscr{K})$. B. Sz.-Nagy and C. Foiaş [13] have used these same positivity conditions to prove that the operators can be dilated in a particular way to a unique family of commuting unitary operators. Their proof is an existence proof depending on the theory of groups of operators of positive type, whereas our proof is constructive in nature. In the course of the proof we characterize the backward translation semigroup and its parts on $l^2(\boldsymbol{Z}^{+n}, \mathscr{K})$. In fact a semigroup of operators is unitarily equivalent to the backward translation semigroup if and only if it satisfies the positivity conditions (or equivalently: is doubly commuting) and the individual operators are coisometries whose powers tend strongly to zero. A semigroup of operators is unitarily equivalent to a part of backward translation if and only if it satisfies the positivity conditions and the powers of its operators

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tend strongly to zero. Thus backward translation is a "universal model" for contractive semigroups, the powers of whose operators tend strongly to zero, only when $n=1$. Our proof also characterizes doubly commuting coisometric semigroups. Completely analogous results are proven for strongly continuous real semigroups and backward translation on $L^2(\boldsymbol{R}^{+n}, \mathscr{K}, \text{Leb.})$. Our proof in the continuous case is a generalization of the proof of Theorem 2.1 in Lax and Phillips [10, p. 67], and is completely analogous to the proof in the discrete case.

We say that a family of n commuting operators $T_1, T_2, \ldots, T_n$ on a Hilbert space $\mathscr{H}$ can be *simultaneously extended* to commuting operators $E_1, E_2, \ldots, E_n$ on a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ if

$$T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} = E_1^{k_1} E_2^{k_2} \cdots E_n^{k_n}|_{\mathscr{H}}$$

for all nonnegative integers $k_1, k_2, \ldots, k_n$. This suggests a reformulation of the problem in terms of semigroups of operators. $\{T(g) : g \in \mathscr{S}\}$ is a *semigroup of operators* on $\mathscr{H}$ if $\mathscr{S}$ is a commutative semigroup with identity 0 and $T(g)$ are bounded linear operators on $\mathscr{H}$ which satisfy $T(0)=I$ and $T(g_1+g_2)=T(g_1)T(g_2)$ for $g_1, g_2 \in \mathscr{S}$. (In this paper $\mathscr{S}$ will be either $\boldsymbol{Z}^{+n}$ or $\boldsymbol{R}^{+n}$.) We say that $\{T(g) : g \in \mathscr{S}\}$ on $\mathscr{H}$ can be *extended* to $\{E(g) : g \in \mathscr{S}\}$ on $\mathscr{K} \supseteq \mathscr{H}$ if

$$T(g) = E(g)|_{\mathscr{H}} \quad \text{for all } g \in \mathscr{S},$$

in this case $\{T(g) : g \in \mathscr{S}\}$ is a *part* of $\{E(g) : g \in \mathscr{S}\}$. $\mathscr{H}$ is said to be a *full* invariant subspace for $\{E(g) : g \in \mathscr{S}\}$ if the smallest reducing subspace containing $\mathscr{H}$ is $\mathscr{K}$, in this case $\{E(g) : g \in \mathscr{S}\}$ is said to be *minimal.*

Instead of $\boldsymbol{Z}^{+n}$ and $\boldsymbol{R}^{+n}$ it will be convenient to consider the isomorphic semigroups $\boldsymbol{Z}^{+A}$ and $\boldsymbol{R}^{+A}$ of nonnegative integer and real valued functions on the set $A=\{1, 2, \ldots, n\}$. For $v \subseteq A$, define $\chi_v \in \boldsymbol{Z}^{+A}$ by

$$\begin{aligned} \chi_v(\omega) &= 1 \quad \text{if } \omega \in v, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

and let $|v| \equiv$ the number of elements of v. For a semigroup of operators $\{T(g) : g \in \boldsymbol{Z}^{+A}\}$ let $T_\omega = T(\chi_{\{\omega\}})$ for $\omega \in A$. We say that $\{T(g) : g \in \boldsymbol{Z}^{+A}\}$ is a *$*$-commuting* (or doubly commuting) semigroup if T_ω commutes with T_λ^* for all $\omega, \lambda \in A$, $\omega \neq \lambda$. Similar definitions can be given for $\boldsymbol{R}^{+A}$.

For the semigroups $\boldsymbol{Z}^{+A}$ and $\boldsymbol{R}^{+A}$ we define the Hilbert spaces $l^2(\boldsymbol{Z}^{+A}, \mathscr{K})$ and $L^2(\boldsymbol{R}^{+A}, \mathscr{K}, \text{Leb.})$ of equivalence classes of strongly measurable functions from $\boldsymbol{Z}^{+A}$ and $\boldsymbol{R}^{+A}$ into the Hilbert space $\mathscr{K}$. In either case the *backward translation semigroup* $\{B(g) : g \in \mathscr{S}\}$ is defined by $(B(g)x)(h)=x(h+g)$ for $x \in l^2$ or L^2 respectively. Notice that in either case $\{B(g): g \in \mathscr{S}\}$ is a $*$-commuting coisometric semigroup, the powers of whose elements tend strongly to zero.

2. The discrete case.

THEOREM 1. *Let $\{T(g) : g \in \boldsymbol{Z}^{+A}\}$ be a semigroup of operators on $\mathscr{H}$. Then the following are equivalent*:

(i) $\{T(g) : g \in Z^{+A}\}$ *is unitarily equivalent to a part of a unique minimal $*$-commuting coisometric semigroup* $\{V(g)^* : g \in Z^{+A}\}$.

(ii) *For all* $F \subseteq A$,

$$P_F \equiv \sum_{v \subseteq F} (-1)^{|v|} T(\chi_v)^* T(\chi_v) \quad \text{is } \geqq 0.$$

(iii) *For all* $g \in Z^{+A}$ $[(g \cdot \chi_v)(\omega) = g(\omega) \cdot \chi_v(\omega)]$,

$$P_g \equiv \sum_{v \subseteq \operatorname{supp}(g)} (-1)^{|v|} T(g \cdot \chi_v)^* T(g \cdot \chi_v) \quad \text{is } \geqq 0.$$

[For another equivalent condition concerning unitary dilations see [13, Theorem 9.1] and [4], where the case of arbitrary A is considered.]

Proof. If $\{V(g)^* : g \in Z^{+A}\}$ is a coisometric semigroup satisfying (i), then for all $F \subseteq A$, and $x \in \mathscr{H}$,

$$\begin{aligned}
(P_F x, x) &= \left(\left(\sum_{v \subseteq F} (-1)^{|v|} T(\chi_v)^* T(\chi_v)\right) x, x\right) \\
&= \left(\left(\sum_{v \subseteq F} (-1)^{|v|} P_{\mathscr{H}} V(\chi_v)|_{\mathscr{H}} V(\chi_v)^*|_{\mathscr{H}}\right) x, x\right) \\
&= \left(\left(\sum_{v \subseteq F} (-1)^{|v|} V(\chi_v) V(\chi_v)^*\right) x, x\right) \\
&= \left(\prod_{\omega \in F} (I - V_\omega V_\omega^*) x, x\right) \geqq 0
\end{aligned}$$

since $(I - V_\omega V_\omega^*)$ are commuting projections. Hence (i) implies (ii).

That (ii) implies (iii) is a technical detail which we omit. Although the proof that (iii) implies (i) is straightforward, it contains several complicated details. We therefore give the basic outline of the proof: (1) the establishment of the norm equation (2.2) which allows us to define the isometry Σ in (2.3); (2) the definition of $B_A(g)$, $C_F(g)$ to satisfying the first extension (2.4) of $T(g)$; (3) the proof of Lemma 1; (4) the verification that the $C_F(g)$ satisfy the hypotheses of Lemma 1; and (5) the use of Lemma 1 to extend $C_F(g)$ to obtain the second extension (2.7), which is the conclusion of the theorem.

We begin by noting that

$$\begin{aligned}
&\sum_{g \in \mathscr{S}} \|P_A^{1/2} T(g) x\|^2 \\
&\quad = \|x\|^2 - \sum_{i=1}^{n} \|R_i x\|^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \|R_{ij} x\|^2 + \cdots + (-1)^n \|R_{12\ldots n} x\|^2
\end{aligned} \tag{2.1}$$

where, for $l, \ldots, k$,

$$R_{\{l,\ldots,k\}} = R_{l,\ldots,k} = \left\{ \operatorname*{slim}_{p_l,\ldots,p_k \to \infty} (T_l^{p_l} \cdots T_k^{p_k})^* (T_l^{p_l} \cdots T_k^{p_k}) \right\}^{1/2}.$$

This limit exists because a monotone decreasing net of positive operators converges to a positive operator in the strong operator topology. (2.1) is proven by induction on $|A|$. For $|A|=1$, see [5].

For $F \subsetneqq A$, we define

$$Q_F = \sum_{v \subseteq F} (-1)^{|v|} R^2_{v \cup \{A \setminus F\}}$$

with $Q_\phi \equiv R_A^2$. Then $Q_F \geqq 0$. To see this, note that it is sufficient to show that

$$\sum_{v \subseteq F} (-1)^{|v|} R_v^2 \quad \text{is } \geqq 0.$$

But this follows since

$$\sum_{v \subseteq F} (-1)^{|v|} R_v^2 = \operatorname*{slim}_{\operatorname{supp}(g) = F; g \to \infty} P_g \geqq 0.$$

(Here $g \to \infty$ means $g(\omega) \to \infty$ for all $\omega \in F$.) We now rewrite (2.1) as

$$\|x\|^2 = \sum_{g \in Z^{+A}} \|P_A^{1/2} T(g) x\|^2 + \sum_{F \subsetneqq A} \|Q_F^{1/2} x\|^2. \tag{2.2}$$

[To see that (2.2) is the same equation as (2.1) compare the number of occurrences of $\|R_{ij\ldots k} x\|^2$ in each of the equations.]

Letting $\mathscr{H}_A = (P_A^{1/2}\mathscr{H})^-$, (2.2) allows us to define an isometry

$$\Sigma : \mathscr{H} \to l^2(Z^{+A}, \mathscr{H}_A) \oplus \sum_{F \subsetneqq A} \oplus (Q_F^{1/2}\mathscr{H})^-$$

by

$$\Sigma x = (P_A^{1/2} T(\cdot) x) \oplus \sum_{F \subsetneqq A} \oplus Q_F^{1/2} x. \tag{2.3}$$

If we define $\{B_A(g) : g \in Z^{+A}\}$ to be backward translation on $l^2(Z^{+A}, \mathscr{H}_A)$, and $\{C_F(g) : g \in Z^{+A}\}$ on $(Q_F^{1/2}\mathscr{H})^-$ by extending continuously from the formula

$$C_F(g) Q_F^{1/2} x = Q_F^{1/2} T(g) x \quad \text{for } x \in \mathscr{H},$$

then

$$\begin{aligned} \Sigma(T(g)x) &= (P_A^{1/2} T(\cdot + g) x) \oplus \sum_{F \subsetneqq A} \oplus Q_F^{1/2} T(g) x \\ &= B_A(g)(P_A^{1/2} T(\cdot) x) \oplus \sum_{F \subsetneqq A} \oplus C_F(g) Q_F^{1/2} x \\ &= (B_A(g) \oplus \sum_{F \subsetneqq A} \oplus C_F(g)) \Sigma x. \end{aligned}$$

Thus

$$T(g) = \Sigma^{-1} \left(B_A(g) \oplus \sum_{F \subsetneqq A} \oplus C_F(g) \right) \Big|_{\Sigma \mathscr{H}} \Sigma \tag{2.4}$$

where Σ is now considered to be a unitary operator from $\mathscr{H}$ onto $\Sigma\mathscr{H}$.

We now interrupt the proof of Theorem 1 to state and prove a lemma which is a special case of Theorem 1. We will need the following theorem of Ito [9]: Every isometric semigroup has a unique unitary extension. For $F\subseteq A$, it will be convenient to identify $\mathbf{Z}^{+F}$ as a subsemigroup of $\mathbf{Z}^{+A}$.

LEMMA 1. *Let* $\{C(g) : g\in \mathbf{Z}^{+A}\}$ *be a semigroup of operators on* $\mathscr{H}$ *such that, for a fixed* $F\subseteq A$,

(i) $S=\sum_{v\subseteq F}(-1)^{|v|}C(\chi_v)C(\chi_v)$ *is* $\geqq 0$,

(ii) *for* $g_1\in \mathbf{Z}^{+F}$, $C(g_1)$ *are contractions whose powers tend to zero strongly,*

(iii) *for* $g_2\in \mathbf{Z}^{+A\backslash F}$, $C(g_2)$ *are isometries.*

Then $\{C(g) : g\in \mathbf{Z}^{+A}\}$ *is unitarily equivalent to a part of a unique minimal* $*$*-commuting coisometric semigroup* $\{V(g)^* : g\in \mathbf{Z}^{+A}\}$ *such that*

(a) $\{V(g_1)^* : g_1\in \mathbf{Z}^{+F}\}$ *is backward translation on* $l^2(\mathbf{Z}^{+F},\mathscr{K})$ (*notation* $l^2(\mathbf{Z}^{+\phi},\mathscr{K})=\mathscr{K}$) *and*

(b) $\{V(g_2)^* : g_2\in \mathbf{Z}^{+A\backslash F}\}$ *is a unitary semigroup.*

Proof. The case $F=\phi$ is Ito's Theorem for $\mathscr{S}=\mathbf{Z}^{+A}$. Hence we assume $F\neq\phi$. Since $S\geqq 0$, we can form $S^{1/2}$ and obtain

$$\sum_{g_1\in \mathbf{Z}^{+F}} \|S^{1/2}C(g_1)x\|^2 = \|x\|^2 \tag{2.5}$$

for all $x\in\mathscr{H}$. To see this, we apply (2.1) with F in place of A to $\{C(g_1) : g_1\in \mathbf{Z}^{+F}\}$ and use (ii) to reduce the right-hand side to $\|x\|^2$.

Let $\mathscr{K}_1=(S^{1/2}\mathscr{H})^-$. For $g_2\in \mathbf{Z}^{+A\backslash F}$ we define an operator $W(g_2)$ on $(S^{1/2}\mathscr{H})$ by

$$W(g_2)S^{1/2}x = S^{1/2}C(g_2)x$$

and then extend continuously to all of $\mathscr{K}_1$. We can do this because

$$\{W(g_2) : g_2\in \mathbf{Z}^{+A\backslash F}\}$$

is isometric:

$$\begin{aligned}\|W(g_2)S^{1/2}x\|^2 &= \|S^{1/2}C(g_2)x\|^2\\ &= \sum_{v\subseteq F}(-1)^{|v|}\|C(\chi_v)C(g_2)x\|^2\\ &= \sum_{v\subseteq F}(-1)^{|v|}\|C(\chi_v)x\|^2 = \|S^{1/2}x\|^2\end{aligned}$$

since $C(g_2)$ is an isometry, commuting with $C(\chi_v)$. By Ito's Theorem there is a unique minimal unitary semigroup $\{U(g_2) : g_2\in \mathbf{Z}^{+A\backslash F}\}$ on $\mathscr{K}\supseteq\mathscr{K}_1$ which extends $\{W(g_2) : g_2\in \mathbf{Z}^{+A\backslash F}\}$. Define $\{V(g)^* : g\in \mathbf{Z}^{+A}\}$ on $l^2(\mathbf{Z}^{+F},\mathscr{K})$ by

$$(V(g_1)^*y)(h_1) = y(g_1+h_1) \quad \text{for } g_1\in \mathbf{Z}^{+F},$$

and

$$(V(g_2)^*y)(h_1) = U(g_2)y(h_1) \quad \text{for } g_2\in \mathbf{Z}^{+A\backslash F}.$$

By construction $V(g_1)$ commutes with $V(g_2)$ and $V(g_2)^*$ for $g_1\in \mathbf{Z}^{+F}$ and

$g_2 \in \boldsymbol{Z}^{+A\backslash F}$. Thus $\{V(g)^* : g \in \boldsymbol{Z}^{+A}\}$ is a $*$-commuting coisometric semigroup since $\{V(g_1)^* : g_1 \in \boldsymbol{Z}^{+F}\}$ and $\{V(g_2)^* : g_2 \in \boldsymbol{Z}^{+A\backslash F}\}$ are also $*$-commuting coisometric semigroups. If we define

$$\Sigma \colon \mathscr{H} \to l^2(\boldsymbol{Z}^{+F}, \mathscr{K}) \quad \text{by} \quad (\Sigma x)(h_1) = S^{1/2}C(h_1)x$$

then (2.5) states that Σ is an isometry, and Σ satisfies

$$C(g) = \Sigma^{-1}(V(g)^*|_{\Sigma\mathscr{H}})\Sigma$$

for all $g \in \boldsymbol{Z}^{+A}$. Here we are considering Σ as a unitary operator from $\mathscr{H}$ onto $\Sigma\mathscr{H}$. ∎

Returning to the proof of Theorem 1, we must show that for a fixed $F \subsetneqq A$, $C_F(g)$ are well defined and satisfy the hypotheses of Lemma 1. For $g_2 \in \boldsymbol{Z}^{+A\backslash F}$, $C_F(g_2)$ is an isometry, because

$$\|R_{v\cup\{A\backslash F\}}T(g_2)x\|^2 = \|R_{v\cup\{A\backslash F\}}x\|^2.$$

For $g_1 \in \boldsymbol{Z}^{+F}$, $C_F(g_1)$ is a contraction whose powers tend strongly to zero. To see this let $m \in F$. Then

$$\begin{aligned}\|Q_F^{1/2}x\|^2 - \|C_F(\chi_m)^p Q_F^{1/2}x\|^2 &= \|Q_F^{1/2}x\|^2 - \|Q_F^{1/2}T_m^p x\|^2 \\ &= \left(\left(\sum_{v\subseteq F} (-1)^{|v|}[R^2_{v\cup\{A\backslash F\}} - T_m^{*p}R^2_{v\cup\{A\backslash F\}}T_m^p]\right)x, x\right) \\ &\geqq 0\end{aligned}$$

because

$$\sum_{v\subseteq F} (-1)^{|v|}[R_v^2 - T_m^{*p}R_v^2T_m^p] = \operatorname*{slim}_{\operatorname{supp}(g)=F;\, g(m)=p;\, g\to\infty} P_g \geqq 0.$$

In fact

$$\lim_{p\to\infty} [\|Q_F^{1/2}x\|^2 - \|C_F(\chi_m)^pQ_F^{1/2}x\|^2] = \|Q_F^{1/2}x\|^2.$$

Thus

$$\lim_{p\to\infty} \|C_F(\chi_m)^pQ_F^{1/2}x\|^2 = 0.$$

It remains only to verify that

$$S_F = \sum_{v\subseteq F} (-1)^{|v|}C_F(\chi_v)^*C_F(\chi_v) \quad \text{is } \geqq 0.$$

But

$$\begin{aligned}(S_FQ_F^{1/2}x, Q_F^{1/2}x) &= \sum_{v\subseteq F} (-1)^{|v|}\|C_F(\chi_v)Q_F^{1/2}x\|^2 \\ &= \left(\left(\sum_{v\subseteq F} (-1)^{|v|}T(\chi_v)^*Q_FT(\chi_v)\right)x, x\right) \\ &= \left(\left(\sum_{v\subseteq F} (-1)^{|v|}T(\chi_v)^*R^2_{\{A\backslash F\}}T(\chi_v)\right)x, x\right) \\ &\geqq 0\end{aligned}$$

since $P_F \geqq 0$. The last equality follows because for all $\phi \subsetneqq u \subseteq F$

$$\sum_{v \subseteq F} (-1)^{|v|} T(\chi_v)^* R^2_{u \cup \{A \setminus F\}} T(\chi_v) = 0.$$

Hence we can apply Lemma 1, for each $F \subsetneqq A$, to obtain a $*$-commuting coisometric semigroup $\{V_F(g)^* : g \in \mathbf{Z}^{+A}\}$ on $\mathscr{H}_F \supseteq (Q_F^{1/2}\mathscr{H})^-$ such that

$$C_F(g) = \Sigma_F^{-1}(V_F(g)^*|_{\mathscr{M}_F})\Sigma_F \tag{2.6$_F$}$$

where $\mathscr{M}_F = \Sigma_F((Q^{1/2}\mathscr{H})^-)$.

If we now combine equations (2.4) and $(2.6)_F$, we obtain

$$T(g) = \Pi^{-1}\left(\left(B_A(g) \oplus \sum_{F \subsetneqq A} \oplus V_F(g)^*\right)\Big|_{\Pi\mathscr{H}}\right)\Pi, \tag{2.7}$$

where Π is a unitary operator when considered as an operator from $\mathscr{H}$ onto $\Pi\mathscr{H}$ obtained by a composition of Σ and Σ_F for $F \subsetneqq A$. Thus

$$V(g)^* = B_A(g) \oplus \sum_{F \subsetneqq A} \oplus V_F(g)^*$$

is the desired $*$-commuting coisometric semigroup.

Uniqueness follows from lemmas analogous to Lemmas 1 and 2 in [5]. In addition, it can be shown that our construction yields the unique minimal extension.

LEMMA 2. *Let V^* be a coisometry with invariant subspace $\mathscr{M}$. If $V^*|_{\mathscr{M}}$ is coisometric then $\mathscr{M}$ reduces V^*.*

Proof. See [3].

COROLLARY 1. *If $\{V(g)^* : g \in \mathbf{Z}^{+A}\}$ is a $*$-commuting coisometric semigroup on $\mathscr{H}$, then there exist unique reducing subspaces $\{\mathscr{H}_F\}$ such that $\mathscr{H} = \sum_{F \subseteq A} \oplus \mathscr{H}_F$ and for $F \subseteq A$:*

$\{V(g_1)^*|_{\mathscr{H}_F} : g_1 \in \mathbf{Z}^{+F}\}$ is unitarily equivalent to backward translation on $l^2(\mathbf{Z}^{+F}, \mathscr{H}_F)$, and

$\{V(g_2)^*|_{\mathscr{H}_F} : g_2 \in \mathbf{Z}^{+A \setminus F}\}$ is a unitary semigroup.

Proof. If we apply Theorem 1 to $\{V(g)^* : g \in \mathbf{Z}^{+A}\}$, then Lemma 2 says that $\Pi\mathscr{H}$ is reducing in addition to being full. Hence $\Pi\mathscr{H}$ is everything, or our original Π is a unitary implementing the required unitary equivalence. It is possible to specify the $\mathscr{H}_F$.

COROLLARY 2. *A semigroup $\{T(g) : g \in \mathbf{Z}^{+A}\}$ on $\mathscr{H}$ is unitarily equivalent to a part of backward translation on $l^2(\mathbf{Z}^{+A}, \mathscr{K})$ if and only if $P_A \geqq 0$ and, for all $\omega \in A$ and $x \in \mathscr{H}$,*

$$\|T_\omega^p x\| \to 0, \quad \text{as } p \to \infty.$$

Proof. This is merely a special case of Lemma 1. In this case all the positivity conditions follow from $P_A \geqq 0$. (See [7], [12].)

COROLLARY 3. *Let $\{T(g) : g \in \boldsymbol{Z}^{+A}\}$ be a semigroup of operators on $\mathscr{H}$. Then the following are equivalent*:

(a) *$\{T(g) : g \in \boldsymbol{Z}^{+A}\}$ is unitarily equivalent to backward translation on $l^2(\boldsymbol{Z}^{+A}, \mathscr{H})$.*

(b) *$\{T(g) : g \in \boldsymbol{Z}^{+A}\}$ is a $*$-commuting coisometric semigroup such that for all $\omega \in A$ and $x \in \mathscr{H}$*

$$\|T_\omega^p x\| \to 0 \quad \text{as } p \to \infty.$$

(c) *$\{T(g) : g \in \boldsymbol{Z}^{+A}\}$ is a coisometric semigroup such that $P_A \geqq 0$ and*

$$\|T_\omega^p x\| \to 0 \quad \text{as } p \to \infty$$

for all $\omega \in A$ and $x \in \mathscr{H}$. In fact $\mathscr{H} = (P_A^{1/2}\mathscr{H})^-$.

Proof. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are trivial. To see that (c) $\Rightarrow$ (a), apply Lemma 1 to $\{T(g) : g \in \boldsymbol{Z}^{+A}\}$ and then use Lemma 2 to see that Σ is actually a unitary operator implementing the unitary equivalence.

COROLLARY 4. *Every contraction has a unique coisometric extension. Every isometry is the direct sum of a unilateral shift and a unitary operator. Every contraction whose powers tend to zero strongly can be uniquely extended to the adjoint of a unilateral shift.*

Proof. This is merely the special case $|A| = 1$. Recall that U_+ is a unilateral shift if $\{U_+^n : n = 0, 1, 2, \ldots\}$ is forward translation on $l^2(\boldsymbol{Z}^+, \mathscr{H})$. The first statement is Sz.-Nagy's Theorem [11], [5], while the second statement is Von Neumann's Theorem [14]. Concerning the last statement see [1].

3. **The continuous case.** In this section we prove the continuous analogs of the results in §2, i.e., we consider the semigroup $\boldsymbol{R}^{+A}$ or $\boldsymbol{R}^{+n}$. The proof of Theorem 2 differs from that of Theorem 1 in that the concept of infinitesimal generator is introduced and that the isometry Σ (analogous to (2.3)) is first defined only on a dense set of $\mathscr{H}$, and then extended continuously to all of $\mathscr{H}$.

If $\mathscr{S}$ is a topological semigroup then a semigroup $\{T(g) : g \in \mathscr{S}\}$ on $\mathscr{H}$ is said to be *strongly continuous* if for every $g_0 \in \mathscr{S}$ and $x \in \mathscr{H}$

$$\|T(g)x - T(g_0)x\| \to 0$$

as $g \to g_0$. We remark that in Ito's Theorem if the isometric semigroup is strongly continuous then so is its unitary extension. The backward translation semigroup $\{B(g) : g \in \boldsymbol{R}^{+A}\}$ on $L^2(\boldsymbol{R}^{+A}, \mathscr{H}, \text{Leb.})$ is a strongly continuous $*$-commuting coisometric semigroup [8].

DEFINITION. Let $\{T(g) : g \in \boldsymbol{R}^{+A}\}$ be a strongly continuous semigroup of operators on $\mathscr{H}$. For $i \in A$ and $h > 0$ define $h_i \in \boldsymbol{R}^{+A}$ to be that function which is h at i and 0 elsewhere, and define $T_i(h) \equiv T(h_i)$. Define

$$B_i \equiv \operatorname*{slim}_{h \to 0^+} \frac{1}{h}(T_i(h) - I), \qquad \mathscr{D}_i \equiv \text{Domain of } B_i.$$

B_i is called the *infinitesimal generator* of the semigroup $\{T_i(h) : h \geqq 0\}$.

LEMMA 3. *With the hypotheses and notation of the definition, and with* $g \cong (t_1, \ldots, t_n)$:

(a) $\partial(T(g)x)/\partial t_i = B_i T(g)x = T(g)B_i x$ *for* $x \in \mathscr{D}_i$.

(b) $\mathscr{D} = \bigcap_{i=1}^n \mathscr{D}_i$ *is a dense linear manifold in* $\mathscr{H}$, *invariant for* $\{T(g) : g \in \boldsymbol{R}^{+A}\}$.

(c) B_i *commute on* $\mathscr{D}$.

(d) *For* $x, y \in \mathscr{D}$

$$\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\cdots\frac{\partial}{\partial t_n}(T(g)x, T(g)y) = \sum_{v \subseteq A} (B_v T(g)x, B_{A\setminus v} T(g)y) \tag{3.1}$$

where $B_v = B_l \cdots B_m$ *for* $v = \{l, \ldots, m\}$ *and* $B_\phi = I$.

Proof. (a) is in Lax and Phillips [10]. (b) and (c) are in Dunford and Segal [6]. (d) is proven by induction on $|A|$. For $|A| = 1$, see [3].

LEMMA 4. *Let* $\{T(g) : g \in \boldsymbol{R}^{+A}\}$ *be a strongly continuous semigroup of operators on* $\mathscr{H}$, *and suppose*

$$P_g \equiv \sum_{\operatorname{supp} v \subseteq (g)} (-1)^{|v|} T(g\cdot\chi_v)^* T(g\cdot\chi_v) \quad \textit{is } \geqq 0,$$

for all $g \in \boldsymbol{R}^{+A}$. *Then for* $x, y \in \mathscr{D}$

$$\lim_{k_1 \to 0+} \cdots \lim_{k_n \to 0+} \frac{1}{k_1 \cdots k_n} (P_k^{1/2} T(g)x, P_k^{1/2} T(g)y) = (-1)^n \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} (T(g)x, T(g)y), \tag{3.2}$$

where $k(i) = k_i > 0$, *for* $i \in A$.

Proof. Again the proof is by induction on $|A|$. For $|A| = 1$, see [3].

We now state the continuous analog of Theorem 1.

THEOREM 2. *Let* $\{T(g) : g \in \boldsymbol{R}^{+A}\}$ *be a strongly continuous semigroup of operators on* $\mathscr{H}$. *Then the following are equivalent*:

(i) $\{T(g) : g \in \boldsymbol{R}^{+A}\}$ *is unitarily equivalent to a part of a unique strongly continuous* $*$*-commuting coisometric semigroup.*

(ii) *For all* $g \in \boldsymbol{R}^{+A}$, P_g *is* $\geqq 0$.

Proof. That (i) implies (ii) follows as before.

Suppose that P_g is $\geqq 0$ for all $g \in \boldsymbol{R}^{+A}$. For $x, y \in \mathscr{D}$ let

$$(x, y)_1 = (-1)^n \sum_{v \subseteq A} (B_v x, B_{A\setminus v} y).$$

Then $(\cdot, \cdot)_1$ is a symmetric, positive semidefinite (combine (3.1) and (3.2) for $x = y$ an let $g \to 0$) bilinear functional on $\mathscr{D}$. Thus the Schwarz inequality holds, which implies that

$$\mathscr{N} = \{x \in \mathscr{D} : (x, x)_1 = 0\}$$

is a linear manifold. Form

$$\mathscr{H}_A = \text{completion of } \mathscr{D}/\mathscr{N} \text{ in } \|\cdot\|_1 \text{ norm.}$$

For $\alpha_1, \ldots, \alpha_n > 0$, let $\alpha \in \boldsymbol{R}^{+A}$ be such that $\alpha(i) = \alpha_i$. Then for $x \in \mathscr{D}$

$$\begin{aligned}(3.3)\qquad \int_0^{\alpha_1} \cdots \int_0^{\alpha_n} \|T(g)x\|_1^2\, dg &= (-1)^n \int_0^{\alpha_1} \cdots \int_0^{\alpha_n} \frac{\partial \cdots \partial}{\partial t_1 \cdots \partial t_n} \|T(g)x\|^2\, dg \\ &= \sum_{v \subseteq A} (-1)^{|v|} \|T(\alpha \cdot \chi_v)x\|^2\end{aligned}$$

by the Fundamental Theorem of Calculus, and induction on $|A|$.

If

$$R_{l \ldots k} = \Big(\operatorname*{slim}_{t_l, \ldots, t_k \to \infty} (T_l(t_l) \cdots T_k(t_k))^*(T_l(t_l) \cdots T_k(t_k))\Big)^{1/2}$$

then letting $\alpha_1, \ldots, \alpha_n \to \infty$ in (3.3) we obtain

$$\begin{aligned}(3.4)\qquad \int_{\boldsymbol{R}^{+A}} \|T(g)x\|_1^2\, dg &= \int_{\boldsymbol{R}^{+A}} \lim_{k \to 0+} \|P_k^{1/2} T(g)x\|^2\, dg \\ &= \|x\|^2 - \sum_{i=1}^{n} \|R_i x\|^2 + \cdots + (-1)^n \|R_{12\ldots n}x\|^2\end{aligned}$$

for all $x \in \mathscr{D}$. Noticing the similarity between (3.4) and (2.1), we define Q_F as before and arrive at an equation similar to (2.2), for all $x \in \mathscr{D}$. We then define

$$\Sigma \colon \mathscr{D} \to L^2(\boldsymbol{R}^{+A}, \mathscr{H}_A, \text{Leb.}) \oplus \sum_{F \neq A} \oplus\, (Q_F^{1/2}\mathscr{H})^-$$

by

$$\Sigma x = (T(\cdot)x) \oplus \sum_{F \subsetneq A} \oplus\, Q_F^{1/2} x$$

where we consider $T(\cdot)x$ to be an element of $\mathscr{D}/\mathscr{N}$. Since Σ is a densely defined (Lemma 3(b)) isometry (3.4), we can extend it continuously to all of $\mathscr{H}$.

If we define $\{B_A(g) : g \in \boldsymbol{R}^{+A}\}$ to be backward translation on $L^2(\boldsymbol{R}^{+A}, \mathscr{H}_A, \text{Leb.})$, and $\{C_F(g) : g \in \boldsymbol{R}^{+A}\}$ on $(Q_F^{1/2}\mathscr{H})^-$ by extending continuously from the formula

$$C_F(g) Q_F^{1/2} x = Q_F^{1/2} T(g) x,$$

then

$$\Sigma T(g) = \Big(B_A(g) \oplus \sum_{F \subsetneq A} \oplus\, C_F(g)\Big)\Sigma \quad \text{on } \mathscr{D},$$

and hence on $\mathscr{H}$. Thus

$$(3.5)\qquad T(g) = \Sigma^{-1}\Big(B_A(g) \oplus \sum_{F \subsetneq A} \oplus\, C_F(g)\Big)\Big|_{\Sigma\mathscr{H}} \Sigma$$

where Σ is now considered to be unitary operator from $\mathscr{H}$ onto $\Sigma\mathscr{H}$.

The proof now proceeds exactly as before. We first prove the continuous analog of Lemma 1, and then use it to extend each $\{C_F(g)\}$ to a strongly continuous $*$-commuting coisometric semigroup $\{V_F(g)^*\}$. Combining these extensions with (3.5) we obtain the desired conclusion. Uniqueness follows as before.

Again we have four corollaries to our theorem. Since they are completely analogous to those in §2 we only state them.

COROLLARY 5. *If* $\{V(g)^* : g \in \mathbf{R}^{+A}\}$ *is a strongly continuous* $*$*-commuting coisometric semigroup on* $\mathscr{H}$, *then there exists reducing subspaces* $\{\mathscr{H}_F\}$ *such that* $\mathscr{H} = \sum_{F\subseteq A} \bigoplus \mathscr{H}_F$ *and for* $F \subseteq A$

$\{V(g_1)^*|_{\mathscr{H}_F} : g_1 \in \mathbf{R}^{+F}\}$ *is unitarily equivalent to backward translation on* $L^2(\mathbf{R}^{+F}, \mathscr{K}_F, \text{Leb.})$, *and*

$\{V(g_2)^*|_{\mathscr{H}_F} : g_2 \in \mathbf{R}^{+A\setminus F}\}$ *is a unitary semigroup. Again it is possible to specify the* $\mathscr{H}_F$.

COROLLARY 6. *A strongly continuous semigroup* $\{T(g) : g \in \mathbf{R}^{+A}\}$ *on* $\mathscr{H}$ *is unitarily equivalent to a part of backward translation on* $L^2(\mathbf{R}^{+A}, \mathscr{K}, \text{Leb.})$ *if and only if* $P_g \geqq 0$ *for all* $g \in \mathbf{R}^{+A}$ *and for all* $\omega \in A$, $x \in \mathscr{H}$

$$\|T_\omega(h)x\| \to 0 \quad as\ h \to \infty.$$

COROLLARY 7. *Let* $\{T(g) : g \in \mathbf{R}^{+A}\}$ *be a strongly continuous semigroup of operators on* $\mathscr{H}$. *Then the following are equivalent*:

(a) $\{T(g) : g \in \mathbf{R}^{+A}\}$ *is unitarily equivalent to backward translation on*

$$L^2(\mathbf{R}^{+A}, \mathscr{K}, \text{Leb.}).$$

(b) $\{T(g) : g \in \mathbf{R}^{+A}\}$ *is a* $*$*-commuting coisometric semigroup such that for all* $\omega \in A$, $x \in \mathscr{H}$

$$\|T_\omega(h)x\| \to 0 \quad as\ h \to \infty.$$

(c) $\{T(g) : g \in \mathbf{R}^{+A}\}$ *is a coisometric semigroup such that* $P_g \geqq 0$ *for all* $g \in \mathbf{R}^{+A}$ *and*

$$\|T_\omega(h)x\| \to 0 \quad as\ h \to \infty$$

for all $\omega \in A$, $x \in \mathscr{H}$. *In fact* $\mathscr{K} = (\mathscr{D}/\mathscr{N})^-$.

COROLLARY 8. *Every strongly continuous contractive semigroup* $\{T(t) : t \geqq 0\}$ *on* $\mathscr{H}$ *has a unique coisometric extension. Every strongly continuous isometric semigroup is the direct sum of forward translation on* $L^2(\mathbf{R}^+, \mathscr{K}, \text{Leb.})$ *and a strongly continuous unitary semigroup. Every strongly continuous contractive semigroup* $\{T(t) : t \geqq 0\}$ *on* $\mathscr{H}$ *such that* $\|T(t)x\| \to 0$ *as* $t \to \infty$ *for all* $x \in \mathscr{H}$ *can be uniquely extended to backward translation on* $L^2(\mathbf{R}^+, \mathscr{K}, \text{Leb.})$.

Concerning the first statement see [3], [11]. The second statement is Cooper's Theorem [2]. We remark that we have given a new proof of Cooper's Theorem using our construction [3]. The last statement is Theorem 2.1 in [10].

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