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THE HOMOTOPY TYPE OF FREDHOLM MANIFOLDS(1)

BY
KALYAN K. MUKHERJEA(2)

Abstract. Banach manifolds whose tangent bundles admit a reduction to the Fredholm group have been intensively studied in the last few years. Here we show that such a manifold (under appropriate smoothness and separability restrictions) is homotopy equivalent to the union of a nested sequence of closed finite-dimensional submanifolds.

Introduction. Banach manifolds whose tangent bundles admit a reduction to the Fredholm group have been intensively studied in the last few years. Here we show that such a manifold (under appropriate smoothness and separability restrictions) is homotopy equivalent to the union of a nested sequence of closed finite-dimensional submanifolds.

This result allows one to develop a fairly elaborate algebraic topology of such manifolds and the proper Fredholm maps between such manifolds. In particular, duality theorems, degree theory, and a version of Lefschetz's coincidence theorem have been obtained [9], [10]. In a different direction, this result is the starting point of Eells and Elworthy's theorem that every smooth, paracompact, separable Hilbert manifold is diffeomorphic to an open subset of l^2, [4].

1. **Preliminaries.** In what follows E, F will always denote infinite-dimensional Banach spaces (over the real numbers) and $L(E, F)$ the space of continuous, linear maps from E to F with the norm topology. We shall denote $L(E, E)$ by $L(E)$. We use the following notation and nomenclature for the subspaces of $L(E, F)$ or $L(E)$, below:

$C(E, F)\equiv$ the completely continuous or *compact operators* in $L(E, F)$.

$C(E)\equiv C(E, E)$.

$Lis(E, F)\equiv$ the bijective isomorphisms in $L(E, F)$.

$GL(E)\equiv Lis(E, E)$, called the *general linear group* of E.

$GC(E)\equiv\{\alpha\in GL(E) \mid \alpha=I+\alpha';\ \alpha'\in C(E)\}$.

$GC(E)$ is a closed normal subgroup of $GL(E)$ of index 2 and we shall call it the *Fredholm group* of E.

$\Phi(E, F)\equiv\{\alpha\in L(E, F) \mid \dim(\text{Kernel } \alpha)<\infty,\ \dim(\text{Coker } \alpha)<\infty\}$.

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We call an element of $\Phi(E, F)$ a *Fredholm operator* from E to F. If $\alpha \in \Phi(E, F)$ we define its *index* $\iota(\alpha)$ by

$$\iota(\alpha) = \dim(\text{Kernel } \alpha) - \dim(\text{Cokernel } \alpha).$$

We put $\Phi_n(E, F) \equiv \{\alpha \in \Phi(E, F) \mid \iota(\alpha) = n\}$, $\Phi_n(E) \equiv \Phi_n(E, E)$. We recall that $\Phi_n(E, F)$ is open in $L(E, F)$ and if $\alpha \in \Phi_n(E, F)$, $\beta \in C(E, F)$, then $(\alpha+\beta) \in \Phi_n(E, F)$.

DEFINITION 1.1. Let E be a Banach space. A *flag* in E is a double sequence of closed linear spaces:

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots \subset E, \qquad E \supset E^{\infty-1} \supset E^{\infty-2} \supset \cdots \supset E^{\infty-n} \supset \cdots$$

such that:

(i) $\dim E_n = n$, $\operatorname{codim} E^{\infty-n} = n$.

(ii) We have isomorphisms $E_n \oplus E^{\infty-n} \cong E$.

(iii) $\bigcup_n E_n$ is dense in E.

REMARK. Clearly a separable Banach space with a Schauder basis admits a flag. If E is a Banach space with a flag, there are obvious inclusion maps

$$GL(E_n) \to GC(E).$$

THEOREM. *The natural inclusion map*

$$\operatorname{dir} \lim_n GL(E_n) \to GC(E)$$

is a homotopy equivalence.

Proof. See [3], [5].

COROLLARY. *$\pi_0(GC(E)) \cong Z_2$, setwise. Indeed the Leray-Schauder index induces a group-structure on $\pi_0(GC(E))$ and an isomorphism $\pi_0(GC(E)) \cong Z_2$. The component containing the unit of $GC(E)$ is denoted by $GC^+(E)$.*

Let E be a separable, C^p-smooth Banach space and M a C^p-smooth manifold on E.

DEFINITION 1.2. A *Fredholm structure* on M is an integrable reduction of its principle bundle $\pi: PM \to M$ to $GC(E)$. A *Fredholm manifold* is a Banach manifold together with a Fredholm structure.

In other words, a Fredholm structure (which we shall write as "Φ-structure") is a maximal atlas on M, such that the derivative of the coordinate changes always lies in $GC(E)$.

A *Fredholm atlas* is a subatlas of a Φ-structure. As usual, any Fredholm atlas can be uniquely extended to a Φ-structure.

REMARK. Results of Elworthy and Tromba imply that given any reduction of $\pi: PM \to M$ to $GC(E)$ there is a Φ-structure equivalent to this reduction. This together with Kuiper's theorem that $GL(E)$ is contractible for a large class of Banach spaces—all Hilbert spaces, for example—shows that the class of Banach manifolds admitting a Φ-structure is sufficiently rich to be interesting.

DEFINITION 1.3. Let $f: M \to N$ be a C^p $(p \geqq 1)$ map of connected Banach manifolds. We say f is a *Fredholm map* if for each $x \in M$,

$$Df(x): T_x M \to T_{f(x)} N$$

is a Fredholm operator.

The index of $Df(x)$ is independent of x and is defined to be the index of f. A Fredholm map of index n will be called a Φ_n-map. The following result is the starting point of the theory of Fredholm manifolds:

THEOREM (SMALE-SARD). *If X, Y are connected, separable C^r-manifolds and if $\varphi: X \to Y$ is a Φ_k-map which is C^s-smooth, $s \leqq r$ and $s > \max(0, k)$, the regular values of φ form a residual subset of Y. (See* [13].)

The next result due to Elworthy and Tromba, shows that manifolds which admit a Φ-structure are precisely the manifolds on which Fredholm maps may be defined:

THEOREM 1.4 (a). *Let M be a paracompact, Fredholm manifold of class C^p, modelled on a C^p-smooth Banach space E, $p \geqq 1$. Then there is a C^p, Φ_0-map $f: M \to E$ such that if (U, φ) is a chart of this Φ-structure,*

$$D(f \circ \varphi^{-1})(x): E \to E$$

is in $GC(E)$ for all $x \in \varphi(U)$.

(b) *If $f: M \to N$ is a Φ_0-map and N a Fredholm manifold modelled on E, there exists a unique Φ-structure (modelled on E) on M such that if (U, φ), (V, ψ) are charts of these Φ-structures on M and N respectively, for each $x \in \varphi(U)$*

$$D(\psi \circ f \circ \varphi^{-1})(x): E \to E$$

is in $GC(E)$ (whenever $\psi \circ f \circ \varphi^{-1}$ is defined).

Proof. See [5], [14].

If E were stable, i.e. $E \times \boldsymbol{R}^n \cong E$ for all $n \geqq 1$, then 1.4(b) would hold if $f: M \to N$ was a Φ_n-map.

2. **Filtrations of Fredholm manifolds.** Throughout the rest of this paper, we assume that E is a C^∞-smooth, separable Banach space with a flag and M is a connected, C^∞, paracompact manifold modelled on E.

DEFINITION 2.1. A *filtration* of M is a sequence of closed submanifolds $M_i \subset M$ $(i \geqq n_0)$ such that

(i) $M_n \subset M_{n+1}$;

(ii) $\dim M_n = n$;

(iii) if $M_\infty = \bigcup \{M_n \mid n \geqq n_0\}$ is the union of the M_n's with the direct limit topology, the natural map $i_\infty: M_\infty \to M$ is a homotopy equivalence.

EXAMPLES. 1. A flag in E is a filtration of E.

2. If $U \subset E$ is open, $U_n = U \cap E_n$ where $E_n \subset E$ are the elements of a flag, then $\{U_n\}_{n \geqq 1}$ constitute a filtration of E. This is known as the Palais-Svarc lemma.

Our objective is the following:

THEOREM 2.2. *Let M be a paracompact, connected, C^∞-manifold modelled on E. If M admits a Φ-structure then M admits of a filtration.*

The rest of this section is devoted to a proof of this result.

LEMMA 2.3([3]). *Let $f\colon M \to E$ be a C^∞, Φ_0-map and $\{E_n, E^{\infty-n}\}_{n\geq 1}$ be a flag in E. For each $p \in E$, let $f_p(x)=f(x)+p$.*

Let $E_f=\{p \in E \mid f_p$ is transversal to E_n for all $n\}$. Then E_f is residual in E.

Proof. Let $\pi_n\colon E \to E^{\infty-n}$ be the canonical projection. Then kernel $\pi_n=E_n$. Now $f_n \equiv \pi_n \circ f\colon M \to E^{\infty-n}$ is a C^∞, Φ_n-map and hence by the Smale-Sard theorem, the regular values, V_n, of f_n are residual in $E^{\infty-n}$. Hence $U_n=\pi_n^{-1}(V_n)$ is residual in E. If $p \in U_n$, $\pi_n \circ f_{(-p)}\colon M \to E^{\infty-n}$ is regular at the origin and hence f_{-p} is transversal to E_n. Hence, $E_f^n=\{p \in E \mid f_p$ transversal to $E_n\}$ is residual in E. Hence $E_f=\bigcap_{n\geq 1} E_f^n$ is residual in E. Q.E.D.

Proof of 2.2.

Step 1. *The construction of the manifolds, M_n.* Choose a Φ-structure on M. Then by 1.4, there is a C^∞, Φ_0-map $f\colon M \to E$.

In view of 2.3 we may assume w.l.o.g. that f is tranversal to all the E_n's of the flag in E. Then put $M_n=f^{-1}(E_n)$. M_n when nonempty is a closed n-dimensional submanifold of M.

Step 2. *$\bigcup_n M_n$ is dense in M.* First note that if F is any closed subspace of E of finite codimension, there is an n_0 such that if $n \geq n_0$, $E_n+F=E$. For let $F_n=E_n+F$ for $n=1, 2, \dots$ and let $d(n)=\dim(E/F_n)$. Then $d(1)\geq d(2)\geq \cdots \geq d(n) \geq \cdots \geq 0$. Hence there is an integer n_0 such that $d(n_0+i)=d(n_0)$ for all $i \geq 0$. Since each F_n is closed this means $F_{n_0}=F_{n_0+i}$ for all $i\geq 0$. Hence $E_{n_0+i}\subset F_{n_0}$ for $i \geq 0$. But $\bigcup E_n$ is dense in E and F_{n_0} closed. Hence $F_{n_0}=E$.

Now let $x \in M$ and $U \subset M$ any open set containing x. We shall show that $U \cap M_n \neq \varnothing$ if n is large enough. By the local representation theorem [1, p. 4], there is an open set $V \subset U$ and a diffeomorphism

$$\alpha\colon V \to V_1 \times V_2 \subset F_1 \times F_2 = E$$

such that:

(i) V_i is a ball in the subspace F_i of E for $i=1, 2$, F_1 having finite dimension and F_2 finite codimension,

(ii) $F_2=\operatorname{Im} Df(x)$, $\alpha(x)=f(x)$,

(iii) $f \circ \alpha^{-1}(x_1, x_2)=(\eta(x_1, x_2), x_2)$.

Choose n so large that $E_n \cap (V_1 \times V_2) \neq \varnothing$ and $E_n+F_2=E$. Then, let F be a complement of $E_n \cap F_2$ in E_n. We can now find a diffeomorphism

$$\beta\colon V \to W_1 \times W_2 \subset F \times F_2$$

([3]) This result of F. Quinn [11] leads to an elegant construction of filtrations (see Step 1, below). My original method was clumsy though adequate.

such that $\beta(x)=f(x)$ and $f\circ\beta^{-1}(x_1, x_2)=(\bar{\eta}(x_1, x_2), x_2)$. Choose $y_2 \in W_2$ such that $y_2 \in E_n$. Then $\beta^{-1}(y_1, y_2) \in M_n$ for any $y_1 \in W_1$. Hence $U \cap M_n \neq \varnothing$.

Step 3. $M_\infty = \bigcup_{n\geqq 1} M_n$ with the direct limit topology has the homotopy type of a CW-complex. This follows immediately from the corollary in p. 153 of [8]. So by Whitehead's theorem, if we can show that $i_\infty\colon M_\infty \to M$ induces an isomorphism of homotopy groups, we will have demonstrated that the M_n's constitute a filtration of M. This follows at once from:

THEOREM 2.3. *Let $M_n \subset M$ be a filtration of a connected, C^∞-Fredholm manifold constructed by the above procedure. Then there exist open tubular neighborhoods Z_n of M_n in M and open sets $U_n \subset Z_n$ such that*

$$U_n \subset U_{n+1} \quad \text{and} \quad \bigcup_n U_n = M.$$

We need,

LEMMA 2.4. *Let ρ be a metric on M, compatible with its topology. Let $U_n \supset M_n$ be open sets and let $\varepsilon\colon M \to \mathbf{R}_+$ be a continuous function. Suppose for each $x \in M_n$ and $\delta < \varepsilon(x)$, $B_\delta(x)$ (the ball of radius δ centered at x) is contained in U_n. Then $\bigcup_n U_n = M$.*

Proof. Let $x_0 \in M_n$, and $\varepsilon(x_0)=\varepsilon_0$. Then there exists an $\eta>0$ such that if $\rho(x_0, y) < 2\eta$, $\varepsilon(y) > \varepsilon_0/2$. Since the M_n's are dense in M, we can choose an n such that $\rho(x_0, M_n) < \min(\varepsilon_0/2, \eta/2)/8$. Choose a $p \in M_n$ such that $\rho(p, x_0) < \min(\varepsilon_0/2, \eta/2)/2$. Then $B_{\varepsilon(p)/2}(p) \subset U_n$ by hypothesis. But $x \in B_{\varepsilon(p)/2}(p)$. Hence $x \in U_n$ i.e. $\bigcup_n U_n = M$. Q.E.D.

We may now proceed with the proof of Theorem 2.3.

Step 1. The result is true if TM admits a spray S, such that with respect to the charts of the Fredholm structure, S has principal part of the form:

$$S(x, v) = (v, 0).$$

(Here, as in what follows, we use the notation and terminology of Lang [7].)

Proof. First choose once and for all a Finsler structure on TM. We shall use the corresponding Finsler metric ρ in applying Lemma 2.4 later on. If $x_0 \in M_n$, the local representation of transversality [1, p. 45] implies that there exists a neighborhood V_{x_0} of x_0 and a diffeomorphism

$$\psi_{x_0}\colon V_{x_0} \to C^1_{x_0} \times C^2_{x_0} \subset E_n \times E^{\infty-n}$$

such that:

(i) $C^1_{x_0}$, $C^2_{x_0}$ are open balls in E_n, $E^{\infty-n}$ respectively.

(ii) $\psi_{x_0}(x_0)=f(x_0)$, $f\circ\psi_{x_0}^{-1}(y_1, y_2)=(\eta(y_1, y_2), y_2)$ for all $(y_1, y_2) \in C^1_{x_0}\times C^2_{x_0}$.

If $m \geqq n$, we can decompose $\psi_{x_0}(V_{x_0})$ as a product of balls in $E_m \times E^{\infty-m}$ and thus $\psi_{x_0}| V_{x_0} \cap M_m$ followed by projection onto E_m yields submanifold charts for M_m, $m \geqq n$.

We have an exact sequence of vector bundles,

$$0 \to TM_n \to TM|M_n \to \nu(M_n) \to 0$$

where $\nu(M_n)$ is the normal bundle of M_n in M. It is easy to see that $\nu(M_n) \simeq M_n \times E^{\infty - n}$. We split the exact sequence by an injection

$$\sigma: \nu(M_n) = M_n \times E^{\infty - n} \to TM|M_n$$

defined as follows: Let $(U_\alpha, \varphi_\alpha)$ be a chart of the Φ-structure of M. Then with respect to the local trivialization of TM over U_α, σ has the local representation:

$$\sigma(\varphi_\alpha(y), v) = (\varphi_\alpha(y), D(\varphi_\alpha \circ \psi_y^{-1})(\varphi_\alpha(y)) \circ (0, v))$$

where $y \in U_\alpha \cap M_n$ and $v \in E^{\infty - n}$.

Let $\exp_S: TM \to M$ be the exponential map associated with the spray S.

There is a commutative diagram of vector bundles and vector bundle morphisms for each $z \in M_n$:

$$\begin{array}{ccccc} \nu(M_n)|V_z \cap M_n & \xrightarrow{\sigma} & TM|V_z \cap M_n & \longrightarrow & TM|V_z \\ \cong \Big| & & \Big| \cong & & \Big| \cong \\ C_z^1 \times E^{\infty - n} & \longrightarrow & C_z^1 \times E & \longrightarrow & C_z^1 \times C_z^2 \times E. \end{array}$$

Let the composition in the lower row be denoted $F: C_z^1 \times E^{\infty - n} \to C_z^1 \times C_z^2 \times E$. Let $y \in V_z \cap M_n$. Then for $v \in E^{\infty - n}$

$$(*) \qquad 2 \cdot \| D(\psi_z \circ \psi_y^{-1})(\psi_z(y)) \circ \pi_n \| \cdot \|v\| < \text{radius } (C_z^1 \times C_z^2)$$

$G \equiv \exp_S \circ F: C_z^1 \times E^{\infty - n} \to C_z^1 \times C_z^2$ is given by

$$\exp_S \circ F(y, v) = (y, D(\psi_z \circ \psi_y^{-1})(\psi_z(y))(0, v)).$$

Choose $r < 1/2$ radius $(C_z^1 \times C_z^2)$, such that if $\|(z, 0) - (y, v)\| < 4r$ and v satisfies (*):

$$\| DG(z, 0) \circ \pi_n - DG(y, v) \circ \pi_n \| < \| DG(z, 0) \circ \pi_n \| / 2.$$

Then, an easy calculation using Lang's estimates [7, p. 12], shows that

$$G: G^{-1}(B_{r/2}(z, 0)) \to B_{r/2}(z, 0)$$

is a diffeomorphism. Hence the local tubular neighborhood of M_n obtained from S will contain $B_{r/2}(z, 0)$. But the choice of r precludes the possibility that in constructing the global tubular neighborhood, we may have to make it "thinner" than $B_{r/2}(z, 0)$. Now it is clear that r may be chosen to be locally uniformly bounded away from zero. And hence, it is possible to find inside a tubular neighborhood Z_n of M_n an open set U_n satisfying the hypothesis of Lemma 2.4.

Now it is easy to see that U_{n+1} may be constructed such that $U_n \subset U_{n+1} \subset U_{n+2} \subset \cdots$. Thus Step 1 is proven.

Step 2. *M arbitrary*. Let Ξ be any spray on M. We shall regard the trajectories associated with this spray as "perturbed trajectories" of the trivial spray constructed before. Precisely speaking, let (U, φ) be a Fredholm chart and suppose with respect to the local trivializations over U, Ξ has principal part

$$\Xi(x, v) = (v, \xi(x, v)), \qquad x \in \varphi(U),\ v \in E.$$

Let $\mathscr{D}$ be the open subset of $\varphi(U) \times E$ such that $(x, v) \in \mathscr{D}$ implies $\|\xi(x, v)\| < \varepsilon$ and $\|D\xi(x, v)\| < \delta$. Let $\alpha_{(x,v)}(t)$ be the trajectories of the spray with initial condition (x, v) and $\Gamma_{(x,v)}(t)$ the trajectories of the trivial spray with the same initial condition. If $t \leqq 1$ and ε, δ so chosen that $\alpha_{(x,v)}(t)$ is defined for $t \leqq 1$,

$$\|\alpha_{(x,v)}(t) - \Gamma_{(x,v)}(t)\| < \|v\|, \qquad \|D\alpha_{(x,v)}(t) - D\Gamma_{(x,v)}(t)\| < \delta$$

where D represents derivative with respect to the parameters (x, v). This is immediate from Robbin [12]. Recalling that $\xi(x, v)$ is of degree 2 in v, we can find a continuous function $\lambda\colon M \to \boldsymbol{R}_+$ such that if $B_{\lambda(x)} \subset E$ is the ball of radius $\lambda(x)$ centered at the origin and

$$\mathscr{E}_{\lambda,n} = \{(x, v) \in M \times E^{\infty - n} \mid v \in B_{\lambda(x)} \cap E^{\infty - n}\}$$

then:

(1) $\exp_\Xi$ is defined on $\sigma(\mathscr{E}_{\lambda,n})$.

(2) $\|\exp_S \circ F(x, v) - \exp_\Xi \circ F(x, v)\| < \frac{1}{8}\|G(x, v)\|$ for $x \in M_n$.

(3) $\|D(\exp_S \circ F)(x, v) - D(\exp_\Xi \circ F)(x, v)\| < \frac{1}{8}$.

Now we can repeat the procedure in Step 1 and using Lang's estimates show that tubular neighborhoods Z_n can be constructed using Ξ, containing open sets U_n satisfying the hypotheses of Lemma 2.4. Q.E.D.

REMARKS. 1. In [9], Theorem 2.2 was proved independently of 2.3. Theorem 2.3 was obtained following a request of Eells who needed a slightly stronger version of 2.3 for his attack (with Elworthy) on the "open embedding problem." Subsequently the original proof of 2.2 was found erroneous and rather than mend it we have used 2.3 to prove 2.2. Eells and Elworthy have proved (in two entirely different ways) somewhat stronger versions of 2.3, [4].

2. One should note that loosely speaking 2.2 and 2.3 are analogous results—one belonging to homotopy theory, the other to differential topology. Given a filtration, the homotopy type of a Fredholm manifold is known. Results of Kuiper and Burghelea imply that (at least for Hilbert manifolds) homotopy equivalent manifolds are C^∞ diffeomorphic [2]. 2.3 says that the "simplest open sets" containing M_n—open subsets of $M_n \times E^{\infty - n}$—determine the C^∞-structure.

3. Using the fact that there is always a proper, bounded map $f\colon M \to E$ we can obtain filtrations $\{M_n\}$ for which each M_n is compact.

3. **Orientations.**

The method of constructing filtrations described in the previous section is sufficiently important to have a special name.

DEFINITION 3.1. A filtration $\{M_n\}$ of a Fredholm manifold is called a *Fredholm* *filtration* if it is obtained as the inverse image of a flag under a Φ_0-map from M, which is transversal to each element of the flag.

We shall use the properties of Fredholm filtrations in studying the idea of orientations of Fredholm manifolds.

DEFINITION 3.2. Let M be a Fredholm manifold. If its principal bundle

$$\pi: PM \to M \qquad (\text{a } GC(E)\text{-bundle})$$

admits a further reduction to $GC^+(E)$, we say that the Fredholm structure is *orientable*. An *oriented* Fredholm structure is a specific reduction to $GC^+(E)$.

Clearly M is orientable if and only if there is a section of the principal $\boldsymbol{Z}_2$-bundle:

$$\tilde{M} \equiv PM/GC^+(E) \to M.$$

$\tilde{M}$ is called the *orientable double cover of M.*

PROPOSITION. *M is orientable if and only if $\tilde{M}$ has two components.*

Proof. Trivial.

Let S_H, P_H be the unit sphere and projective space of separable, infinite-dimensional Hilbert space H. Then the natural quotient $q: S_H \to P_H$ is a classifying bundle for $\boldsymbol{Z}_2$-bundles.

Thus there is a map $\theta: M \to P_H$ (unique up to homotopy) such that $\theta^*(S_H) \cong \tilde{M}$ as $\boldsymbol{Z}_2$-bundles over M.

DEFINITION 3.3. Let $\omega_1 \in H^1(P_H; \boldsymbol{Z}_2)$ be the generator of $H^1(P_H; \boldsymbol{Z}_2)$. ω_1 is called the universal 1st Stiefel-Whitney class. We define $\omega_1(M)$—the 1st Stiefel-Whitney class of the Fredholm manifold M—by

$$\omega_1(M) = \theta^*(\omega_1) \in H^1(M; \boldsymbol{Z}_2).$$

PROPOSITION. *M is orientable if and only if $\omega_1(M)=0$.*

Proof. Trivial.

REMARKS. (1) If M and $\overline{M}$ are two Fredholm manifolds having the same underlying manifold structure, it may be that $\omega_1(M) \neq \omega_1(\overline{M})$. See [6]. This is in contrast to the finite-dimensional situation where Stiefel-Whitney classes are topological invariants.

(2) The idea of orientation was originally introduced by A. J. Tromba using a Čech construction [14]. However the following singular cohomological description of $\omega_1(M)$ is easier to handle:

Definition of $\omega_1(M)$ as a singular cochain. Let $\lambda: I \to M$ be a singular 1-simplex. Cover $\lambda(I)$ by Fredholm charts $\{(U_i, \varphi_i)\}_{i\in 1,2,\ldots,m}$ such that:

(1) $\lambda(0) \in U_1$, $\lambda_1 \in U_m$;

(2) $U_i \cap U_j$ connected or empty;

(3) $U_i \cap U_{i+1} \neq \varnothing$.

Let

$$\sigma_i = +1 \quad \text{if } D(\varphi_i \circ \varphi_{i+1}^{-1}) \in GC^+(E),$$
$$= -1 \quad \text{otherwise.}$$

Define the cochain $W(M)$ by

$$W(\lambda) = \sum_{i=1}^{m} \sigma_i.$$

As in finite dimensions W is well defined, is a cocycle and its cohomology class represents $\omega_1(M)$.

We can now prove:

THEOREM 3.4. *Let $\{M_n\}_{n \geqq k}$ be a Fredholm filtration of a Fredholm manifold. Let $i_{n+1,n}: M_n \to M_{n+1}$ be the natural inclusion. Then*

(i) $i^*_{n+1,n}(\omega_1(M_{n+1})) = \omega_1(M_n)$;

(ii) $\{\omega_1(M_n)\}$ *represents an element in $H^1(M; \boldsymbol{Z}_2)$.*

This is precisely the 1st Stiefel-Whitney class of M.

Proof. Note, that $M_n \to M$ and $M_n \to M_{n+1}$ have trivial normal bundles. Then (i) follows from the Whitney sum formula.

(ii) From (i), $\{\omega_1(M_n)\}$ represents a class in proj lim $H^1(M_n; \boldsymbol{Z}_2)$. Now

$$\begin{aligned} H^1(M; \boldsymbol{Z}_2) &\cong \operatorname{Hom}(H_1(m); \boldsymbol{Z}_2) \\ &\cong \operatorname{Hom}(\operatorname{dir\,lim} H_1(M_n); \boldsymbol{Z}_2) \\ &\cong \operatorname{proj\,lim}(\operatorname{Hom}(H_1(M_n; \boldsymbol{Z}_2)) \\ &\cong \operatorname{proj\,lim} H^1(M_n; \boldsymbol{Z}_2). \end{aligned}$$

To show that $\{\omega_1(M_n)\}$ under this identification corresponds to $\omega_1(M_n)$, we note that any loop in M is homotopic to one in M_n (n large). Now if we use the singular cochain description the result follows. Q.E.D.

COROLLARY 1. *$RP^1 \subset RP^2 \subset \cdots \subset RP^N \subset \cdots \subset P_H$ is not a Fredholm filtration of P_H (with any Fredholm structure whatever).*

This leads to the following result:

THEOREM 3.5. *There is no C^∞, Φ_0-map $f: S_H \to H$ satisfying*

(1) $f(-x) = f(x)$.

(2) *There is a filtration of S_H by unit spheres $S^1 \subset S^2 \subset \cdots \subset S^n \subset \cdots$ and a flag $\{E^n\}$ such that $f(S^n) \subset E^n$.*

Proof. Consider the map

$$Ev: GL(H) \times S_H \to H$$

given by

$$Ev(\alpha, p) = \alpha(f(p)).$$

Then DEv is surjective at each point and hence an application of Quinn's transversality theorem [11], shows that there is an $\alpha \in GL(H)$ such that $\alpha \circ f: S_H \to H$ is transversal to each S^n. Then the existence of a map f as described would imply immediately, that $\{\boldsymbol{RP}^n\}_{n \geqq 1}$ constitutes a filtration of P_H. Q.E.D.

Added in proof. Several people who had seen the above in preprint form have requested more details concerning the proof of Theorem 2.3. Dan Burghelea and David Elworthy, in particular, have pointed out some obscurities and gaps. The following remarks, it is hoped, will clarify the situation.

Let $U \subset M$ be any coordinate set, homeomorphic to an open set in E. Then we can identify $TM \mid U \cong U \times E$. Let S, Ξ be the sprays considered in the proof of 2.3. We follow Robbin's treatment of finding solution curves: Consider

$$F, G \colon \boldsymbol{R} \times (U_0 \times E) \times C_0^1(I, U_0 \times E) \to C^0(I, U_0 \times E)$$

where $U_0 \subset U$ is open and has half the radius of U, defined by:

$$\begin{aligned} F(a, (x, v), \gamma)(t) &= (\dot{\gamma}_1(t), \dot{\gamma}_2(t)) - a(v+\gamma_2(t), 0), \\ G(a, (x, v), \gamma)(t) &= (\dot{\gamma}_1(t), \dot{\gamma}_2(t)) - a(v+\gamma_2(t), \xi(x+\gamma_1(t), v+\gamma_2(t))). \end{aligned}$$

Then $D_3(0, (x, v), 0)$ is a toplinear isomorphism and by the implicit function theorem there is an $\varepsilon > 0$, and a neighborhood $\mathscr{D}$ of the zero-section of $U_0 \times E$ such that there are functions:

$$H, K \colon (-2\varepsilon, 2\varepsilon) \times \mathscr{D} \to C_0^1(I, U_0 \times E)$$

satisfying

$$F(a, (x, v), H(a, (x, v))) = 0, \quad G(a, (x, v), K(a, (x, v))) = 0$$

From condition $SPR1$ (Lang [7, p. 69]), by suitably shrinking $\mathscr{D}$ we may suppose $\varepsilon = 1$. Then, the integral curves of the two sprays S and Ξ are:

$$\begin{aligned} \Gamma_{(x,v)}(t) &= H(1, (x, v))(t) + (x, v), \\ \alpha_{(x,v)}(t) &= K(1, (x, v))(t) + (x, v), \quad \text{respectively.} \end{aligned}$$

Now $D_1\xi(0, 0) = 0$, $D_2\xi(0, 0) = 0$ and hence, by choosing $\mathscr{D}$ suitably we can make Ξ and S C^1-close to one another. An easy computation then shows that, this brings $\Gamma, \alpha \colon D \to C^1(I, U_0 \times E)$ defined by:

$$\Gamma(x, v) = \Gamma_{(x,v)}, \ \alpha(x, v) = \alpha_{(x,v)}$$

C^1-close. Thus if $\mathscr{D}$ is suitably chosen exp, $\exp_\Xi \colon \mathscr{D} \to U$ will be C^1-close and hence $\exp_\Xi$ will be a diffeomorphism over each fibre.

Now if $t \colon M \to \boldsymbol{R}_+$ is any continuous function, we denote by tTM the set

$$\bigcup_{x \in M} \{v_x \in T_x M \mid \|v_x\|_x < t(x)\}$$

where we are using the Finsler on TM.

In particular, note that there is a $t \colon M \to \boldsymbol{R}_+$ such that $\exp_\Xi \colon tTM \to M$ is a diffeomorphism in each fibre. By making t smaller, if necessary, we may also assume that $\exp_\Xi (tTM|x) \subset V_x$ for any $x \in M_n$. (See steps 1 of 2.3.)

Furthermore there is a function $\gamma \colon M \to \boldsymbol{R}_+$ such that if $x \in M_n$, $v \in E^{\infty - n}$ and $\sigma(x, v) \in tTM$, $\sigma(x, v) \notin tTM/2$, $\rho(\exp_\Xi \circ \sigma(x, v), M_n) > \gamma(x)$.

Set, $\tilde{\lambda}(y)=\|D\psi_y^{-1}(\psi_y(y))\|$, where we use the norm in E and the Finsler in T_yM to compute this. Let $\lambda(y)=(\tilde{\lambda}(y))^{-1}$. Then $\exp_\Xi \circ \sigma\colon \mathscr{E}_{\lambda,n}\equiv M_n\times\lambda E^{\infty-n}\to M$ yields a tubular neighborhood Z_n of M_n in M. Let $V_n=\exp_\Xi\circ\sigma(\mathscr{E}_{\lambda/2,n})$ and $\tilde{U}_n=\bigcap_{m\geq n} V_m$.

Then int $\tilde{U}_n=U_n\subset Z_n$ are open sets and in view of the remark above satisfy the hypotheses of Lemma 2.4.

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University of California,
Los Angeles, California 90024