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ON PROPERTIES OF SUBSPACES OF $l_p, 0<p<1$

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Abstract. The material presented in this paper deals with some questions concerning projections, quotient spaces, and linear dimension in l_p spaces, and also includes a remark about weak Schauder bases in l_p spaces and an example of an infinite-dimensional closed subspace of l_p which is not isomorphic to l_p.

1. **Introduction.** For $p>0$, let l_p be the set of all real sequences, (a_n), such that $\sum |a_n|^p<\infty$. It is well known that l_p is a complete linear metric space with paranorm given by $\sum |a_n|^p$ when $0<p<1$, and that l_p is a Banach space with norm given by

$$\left(\sum |a_n|^p\right)^{1/p} \quad \text{when } p \geqq 1.$$

A great deal is known about the structure of l_p spaces for $p\geqq 1$. Perhaps not quite so much is known about these spaces for $0<p<1$. We plan to discuss here properties of subspaces of the latter spaces and show that in most cases these properties are quite different from those of the normed spaces.

The paper will be divided into five sections. §1 will contain some comments and definitions which might be helpful to the reader as well as a summary of results. §2 will deal mainly with the concept of complemented subspaces of l_p, $0<p<1$. We will show in this section that each l_p space is isomorphic to all of its subspaces of finite codimension, that each l_p space contains a subspace isometrically isomorphic to l_p no infinite-dimensional subspace of which is complemented in l_p, and that if l_p is isometrically isomorphic to one of its subspaces which has the Hahn-Banach extension property, then this subspace is complemented in l_p. §3 will contain an example of a subspace of l_p which is not isomorphic to l_p. §4 will contain some results about subspaces of l_p which are obtained as kernels of linear mappings. In particular, we will show that each l_p, $0<p<1$, contains a closed proper subspace such that any continuous linear functional on l_p which vanishes on this subspace vanishes on all of l_p. We will also show that l_p contains a closed subspace which is not contained in any proper complemented subspace of l_p. Finally, §5 will contain some results on linear dimension, complementing those known for l_p, $p\geqq 1$.

Most of our terminology is standard; however, a few remarks probably are in order. We use the word norm to denote the l_p paranorm when $0<p<1$, and we

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use the standard notation for a norm to denote this paranorm—even though it is p-homogeneous ($\|ta\| = |t|^p \|a\|$), and not homogeneous in the usual sense. In cases of possible ambiguity, we will denote the p-norm by $\| \quad \|_p$. We will say that a subspace, X, of l_p has the *Hahn-Banach extension property* if each continuous linear functional on X can be extended to a continuous linear functional on all of l_p. We will let $\{e_i\}$ denote the unit vector bases in l_p, i.e., $e_i = (0, \ldots, 0, 1, 0, \ldots)$, we will write $X \approx Y$ to denote that X is linearly isomorphic to Y, and we will use the known properties of the space $(X_1 \oplus X_2 \oplus \cdots)_X$ given in [9]. Finally, we will say that two bases, $\{x_k\}$ and $\{y_i\}$, are *equivalent bases* if $\sum \alpha_i y_i$ converges if and only if $\sum \alpha_i x_i$ converges.

2. **Complemented subspaces.** It is known [9] that each infinite-dimensional closed subspace of l_p, $p \geqq 1$, contains an infinite-dimensional subspace which is complemented in l_p, or in the terminology of Whitley [12], l_p, $p \geqq 1$, is subprojective. That the situation is considerably different when $0 < p < 1$ will be shown in Theorem 2.3. We begin by proving a theorem which is basically not new (see [3]) but whose proof contains estimates which are essential to our later work.

THEOREM 2.1. *If X is a closed infinite-dimensional subspace of l_p, $0 < p < 1$, then X contains a subspace isomorphic to l_p.*

Proof. Since X is infinite dimensional, X contains a sequence $\{b_n\}$ such that $\|b_n\| = 1$ and each b_n is of the form

$$b_n = (0, \ldots, 0, b^n_{k_n}, b^n_{k_n+1}, \ldots),$$

where k_n can be chosen arbitrarily large. Select b_n such that

$$\sum_{k=k_{n+1}}^{\infty} |b^n_k|^p < \frac{1}{2^{n+1}},$$

and define the sequence $\{C_n\}$ such that

$$C_n = (0, \ldots, 0, b^k_{k_n}, \ldots, b^n_{k_{(n+1)}-1}, 0, \ldots).$$

We note the $\{C_n\}$ is a basic sequence equivalent to the unit vector basis in l_p. Indeed, this follows immediately from the following:

$$\left\| \sum_{n=1}^{\infty} \lambda_n C_n \right\| = \sum_{n=1}^{\infty} |\lambda_n|^p \sum_{k=k_n}^{k_{(n+1)}-1} |b^n_k|^p \leqq \sum_{n=1}^{\infty} |\lambda_n|^p,$$

while

$$\left\| \sum_{n=1}^{\infty} \lambda_n C_n \right\| \geqq \tfrac{1}{2} \sum_{n=1}^{\infty} |\lambda_n|^p.$$

We also note that $\{b_n\}$ is a basis equivalent to $\{C_n\}$. This follows from the following:

$$\left\|\sum_{n=1}^{m} \lambda_n(b_n-C_n)\right\| = \left\|\sum_{n=1}^{m} \lambda_n(0,\ldots,0,b^n_{k_{n+1}},\ldots)\right\|$$
$$\leqq \sum_{n=1}^{m} |\lambda_n|^p \frac{1}{2^{n+1}} \leqq \tfrac{1}{2}\sum_{n=1}^{m}|\lambda_n|^p \sum_{k=k_n}^{k(n+1)-1} |b^n_k|^p$$
$$= \tfrac{1}{2}\left\|\sum_{n=1}^{m}\lambda_n C_n\right\|.$$

Hence $\{b_n\}$ is a basis for a subspace of X which is isomorphic to l_p.

COROLLARY 2.2. *The space l_p, $0<p<1$, contains no infinite-dimensional subspace isomorphic to a Banach space.*

Proof. If B were a Banach space isomorphic to a subspace of l_p, then by the previous theorem, l_p would be isomorphic to a subspace of B. Since l_p contains no bounded convex neighbourhood, this is impossible.

THEOREM 2.3. *For each p, $0<p<1$, l_p contains a subspace, Y, isometrically isomorphic to l_p such that no infinite-dimensional subspace of Y is complemented in l_p.*

Proof. Let Y be the subspace of l_p whose basis elements, b_n, are given by

$$\begin{aligned} b_1 &= (1,0,\ldots)\\ b_2 &= (0,1/2^{1/p},1/2^{1/p},0,\ldots)\\ b_3 &= (0,0,0,1/3^{1/p},1/3^{1/p},1/3^{1/p},0,\ldots)\\ &\ldots\ldots\ldots\ldots\end{aligned}$$

We will show that no infinite-dimensional subspace of Y has the Hahn-Banach extension property, and this will prove the theorem.

Let z be an infinite-dimensional subspace of Y. We note that the sequence

$$(\pm 1, \pm 2^{1/p-1}, \pm 2^{1/p-1}, \pm 3^{1/p-1}, \pm 3^{1/p-1}, \pm 3^{1/p-1},\ldots)$$

represents a continuous linear functional on Y and hence on Z for any choice of signs. We denote this functional, when the signs are all positive, by f, and for notational convenience, we let f be denoted by the sequence $(f_1,f_2,\ldots)$, i.e., $f_1=1, f_2=2^{1/p-1}$, etc. We choose a sequence, $\{z_n\}$, of unit vectors in Z as follows.

$$\begin{aligned} z_1 &= (\alpha^1_1,\alpha^1_2,\alpha^1_3,\alpha^1_4,\alpha^1_5,\alpha^1_6,\ldots,\alpha^1_{n_2},\ldots,\alpha^1_{n_2+n_2'},\ldots)\\ z_2 &= (0,\ldots,0,\alpha^2_{n_3},\ldots,\alpha^2_{n_3+n_3'},\ldots,\alpha^2_{n_4},\ldots,\alpha^2_{n_4+n_4'},\ldots)\\ &\ldots\ldots\ldots\ldots\\ z_k &= (0,\ldots,0,\alpha^k_{n_{2k-1}},\ldots,\alpha^k_{n_{2k-1}+n_{2k-1}'},\ldots,\alpha^k_{n_{2k}},\ldots,\alpha^k_{n_{2k}+n_{2k}'},\ldots)\\ &\ldots\ldots\ldots\ldots\end{aligned}$$

where we have chosen our notation to indicate that a "block" begins at n_{2k} and ends at $n_{2k}+n'_{2k}$. Choose z_1 arbitrarily and then choose n_2 such that

$$\sum_{j=1}^{n_2+n_2'} |\alpha_j^1|\,|f_j| - \sum_{j=n_2+n_2'+1}^{\infty} |\alpha_j^1|\,|f_j| \geqq \frac{1}{2}\sum_{j=1}^{\infty} |\alpha_j^1|.$$

Having chosen z_{k-1} and n_{2k-2}, choose z_k so that $n_{2k-1} > n_{2k-2}+n'_{2k-2}$. Then choose n_{2k} so that

$$\sum_{j=1}^{n_{2k}+n_{2k}'} |\alpha_j^k|\,|f_j| - \sum_{j=n_{2k}+n_{2k}'+1}^{\infty} |\alpha_j^k|\,|f_j| \geqq \frac{K^{1/p-1}}{2}\sum_{j=1}^{\infty} |\alpha_j^k|.$$

The above choice of n_{2k} is clearly possible because of the forms of the linear functionals involved and because of the fact that our choice of n_{2k-1} has required us to "skip" at least one block at each step of the process.

We now let "sign" denote the function such that sign $r = r/|r|$ if $r \neq 0$ and sign $0 = 0$, and we let g be the continuous linear functional on Z whose representation is given by

$$(f_1 \operatorname{sign} \alpha_1^1, \ldots, f_{n_3-1} \operatorname{sign} \alpha_{n_3-1}^1, f_{n_3} \operatorname{sign} \alpha_{n_3}^2, \ldots, f_{n_5-1} \operatorname{sign} \alpha_{n_5-1}^2, f_{n_5} \operatorname{sign} \alpha_{n_5}^3, \ldots).$$

Suppose that g has a continuous extension to l_p. Then there is a linear functional, h, on l_p which agrees with g on the sequence $\{z_n\}$. Let $(m_1, m_2, \ldots)$ be a bounded sequence which is the representation of h, and suppose that $\sup |m_j| \leqq M$. Then

$$g(z_k) \geqq \frac{K^{1/p-1}}{2}\sum_{j=1}^{\infty} |\alpha_j^k| \geqq \frac{K^{1/p-1}}{2M}\,|h(z_k)|,$$

and since $0<p<1$, this is clearly impossible.

PROPOSITION 2.4. *Suppose that X is a closed subspace of l_p, $0<p<1$, such that X contains no subspace which is both complemented and isomorphic to l_p. Then given any $\varepsilon>0$ there exists an integer N such that $n \geqq N$ and $a=(0,\ldots,0,a_n,a_{n+1},\ldots) \in X$ with $\|a\| \leqq 1$ implies $\sum_{j=n}^{\infty} |a_j| \leqq \varepsilon$.*

Proof. Suppose that there exists some $\varepsilon>0$ such that for any N we can find a vector $a=(0,\ldots,0,a_n,a_{n+1},\ldots) \in X$ with $\|a\|_p=1$, $\|a\|_1 \geqq \varepsilon$ and $n \geqq N$. We construct a sequence $\{a_n\}$ of p-unit vectors in X of the form

$$\begin{aligned}
a_1 &= (a_1^1, a_2^1, \ldots, a_{n_2}^1, \ldots)\\
a_2 &= (0, \ldots, 0, a_{n_3}^2, a_{n_3+1}^2, \ldots, a_{n_4}^2, \ldots)\\
&\cdots\cdots\cdots\cdots\\
a_k &= (0, \ldots, 0, a_{n_{2k-1}}^k, \ldots, a_{n_{2k}}^k, \ldots)\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

in the following manner. Choose a_1 arbitrarily. Then choose n_2 such that

$$\sum_{j=1}^{n_2} |a_j^1| \geqq \frac{\varepsilon}{2}, \quad\text{and}\quad \sum_{j=n_2+1}^{\infty} |a_j^1|^p < \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^p.$$

Having chosen a_{k-1} and n_{2k-2}, choose a_k such that $n_{2k-1}>n_{2k-2}$. Then select n_{2k} such that

$$\sum_{j=n_{2k-1}}^{n_{2k}} |a_j^k| \geqq \frac{\varepsilon}{2}, \quad \text{and} \quad \sum_{j=n_{2k}+1}^{\infty} |a_j^k|^p < \frac{1}{2^{k+1}}\left(\frac{\varepsilon}{2}\right)^p.$$

Since $\varepsilon \leqq 1$, the last inequality implies that

$$\sum_{j=n_{2k}+1}^{\infty} |a_j^k|^p < \frac{1}{2^{k+1}}.$$

Hence, according to the calculations contained in the proof of Theorem 2.1, the sequence $\{b_n\}$ given by

$$\begin{aligned} b_1 &= (a_1^1, \ldots, a_{n_2}^1, 0, \ldots) \\ b_2 &= (0, \ldots, 0, a_{n_3}^2, \ldots, a_{n_4}^2, 0, \ldots) \\ &\cdots \\ b_k &= (0, \ldots, 0, a_{n_{2k-1}}^k, \ldots, a_{n_{2k}}^k, 0, \ldots) \\ &\cdots \end{aligned}$$

is a basic sequence equivalent to the unit vector basis in l_p, and, furthermore, the sequence $\{a_n\}$ is a basic sequence in X equivalent to the unit vector basis in l_p. If Y denotes the subspace spanned by $\{b_k\}$, it is easy to see that the mapping P defined by

$$P(x) = \sum_{j=1}^{\infty} f_j(x) b_j,$$

where f_k is the linear functional corresponding to the sequence

$$(0, \ldots, 0, m_{n_{2k-1}}, \ldots, m_{n_{2k}}, 0, \ldots),$$

where each m_j^i may be chosen so that $\sup_{i,j} |m_j^i| \leqq 2/\varepsilon$ and $f_k(b_k)=1$, is a continuous projection of l_p onto Y.

Let A be the linear mapping defined by

$$A(x) = x - P(x) + \sum_{j=1}^{\infty} f_j(P(x)) a_j.$$

A is a well-defined continuous mapping since $\{a_j\}$ and $\{b_j\}$ are equivalent bases. We will show that A is a one-to-one mapping of l_p onto itself, and this will imply that A is bicontinuous by the open mapping theorem. A minor calculation shows that

$$A(x) = x + \sum_{k=1}^{\infty} f_k(x)(0, \ldots, 0, a_{n_{2k}+1}^k, \ldots),$$

and from this it is easy to deduce that A is one-to-one. To see that A maps onto l_p, let y be any arbitrary element such that $\|y\|_p=1$. We will determine an x in l_p

such that $Ax=y$. Let $x_j=y_j$ for $j=1,\dots,n_2$. Then, let

$$x^{n_2} = (x_1, x_2, \dots, x_{n_2}, 0, \dots),$$

and let $x_j=y_j-f_1(x^{n_2})a_j^1$ for $j=n_2+1,\dots,n_4$. Having chosen $x_1, x_2, \dots, x_{n_{2k}}$, let

$$x^{n_{2k}} = (0, \dots, 0, x_{n_{2k-2}+1}, \dots, x_{n_{2k}}, 0, \dots),$$

and let

$$x_j = y_j - \sum_{r=1}^{k} f_r(x^{n_{2r}})a_j^r$$

for $j=n_{2k}+1,\dots,n_{2k+2}$. We will show that $x=(x_1, x_2, \dots)$ is in l_p and this will complete the proof that A is an isomorphism.

Note that $\|x^{n_2}\|_p \leqq 1$. Hence $|f_1(x^{n_2})| \leqq (2/\varepsilon)\|x^{n_2}\|_1 \leqq 2/\varepsilon$. This implies that

$$\begin{aligned}\|x^{n_4}\|_p &\leqq 1+(2/\varepsilon)^p\|(0,\dots,0,a^1_{n_2+1},\dots,a^1_{n_4},0,\dots)\|_p \\ &\leqq 1+(2/\varepsilon)^p(\varepsilon/2)^p\tfrac{1}{2} \leqq 1+\tfrac{1}{2}<2.\end{aligned}$$

Hence $\|x^{n_4}\|_1 \leqq 2^{1/p}$, and this implies that $|f_2(x^{n_4})| \leqq (2/\varepsilon)2^{1/p}$. Assuming that $|f_2(x^{n_{2j}})| \leqq (2/\varepsilon)2^{1/p}$ for $j=2,\dots,k$, we see that

$$\begin{aligned}\|x^{n_{2k+2}}\|_p &\leqq 1+\sum_{r=1}^{k} |f_r(x^{n_{2r}})|^p \, \|(0,\dots,0,a_{n_{2r}+1},\dots,a_{n_{2r+2}},0,\dots)\|_p \\ &\leqq 1+\left(\frac{2}{\varepsilon}\right)^p\left(\frac{\varepsilon}{2}\right)^p\frac{1}{2}+\sum_{r=2}^{k}\left(\frac{2}{\varepsilon}\right)^{p_2}\frac{1}{2^{r+1}}\left(\frac{\varepsilon}{2}\right)^p \leqq 2,\end{aligned}$$

and this implies that $\|x^{n_{2k+2}}\|_1 \leqq 2^{1/p}$ which in turn implies that $|f_{k+1}(x^{n_{2k+2}})| \leqq (2/\varepsilon)2^{1/p}$. Thus, this last inequality holds for all $k \geqq 1$. Using this fact, one can show very easily that $\|x\|_p \leqq 2$.

Since the isomorphism, A, maps the space spanned by $\{b_k\}$ onto the space spanned by $\{a_k\}$, the mapping Q given by $Q=APA^{-1}$ is a continuous projection of l_p onto X.

LEMMA 2.5. *If X is a complemented subspace of l_p, $0<p<1$, and X contains a subspace Y which is both complemented in X and isomorphic to l_p, then X is isomorphic to l_p.*

Proof. See Proposition 4 of [9].

THEOREM 2.6. *For each p, $0<p<1$, l_p is isomorphic to all of its subspaces of finite codimension.*

Proof. It suffices to show that l_p is isomorphic to all of its hyperplanes. Let X be a hyperplane in l_p. Since X is complemented in l_p, if X is not isomorphic to l_p, X contains no complemented subspace isomorphic to l_p by Lemma 2.5. Thus Proposition 2.4 applies. Suppose that $l_p=Rx \oplus X$ where Rx is the space spanned by the vector $x=(x_1, x_2, \ldots)$. Then there are vectors of the form $(0, \ldots, 0, a_n, 0, \ldots)$ in $x+X$ for each n with $a_n \neq 0$. Thus, we can find two vectors of the form

$$y = (-x_1, \ldots, -x_{n-1}, -x_n+a_n, -x_{n+1}, \ldots)$$

and

$$z = (-x_1, \ldots, -x_n, -x_{n+1}+a_{n+1}, -x_{n+2}, \ldots)$$

in X for each choice of n. Since $y-z=(0, \ldots, 0, a_n, -a_{n+1}, 0, \ldots) \in X$, we can find p-unit vectors in X with l_1 norms greater than $2^{1-1/p}$ in direct contradiction to Proposition 2.4.

LEMMA 2.7. *If* $0<p<2$ *and* ξ *and* η *are real numbers, then* $|\xi+\eta|^p+|\xi-\eta|^p \leqq 2(|\xi|^p+|\eta|^p)$ *and equality holds only when* ξ *or* η *is zero.*

Proof. See [11].

THEOREM 2.8. *If a subspace* X *of* l_p, $0<p<1$, *is isometrically isomorphic to* l_p *and has the H-B-extension property, then* X *is complemented in* l_p.

Proof. Let T be the isometry, and let $Te_i=f_i$. It is easy to see from Lemma 2.6 that f_i and f_j have disjoint supports when $i \neq j$. Since $\{f_i\}$ is a basis for X equivalent to $\{e_i\}$, one can define a continuous linear functional h on X such that $h(f_i)=1$. If $\|f_{i_k}\|_1 \to 0$ for some subsequence, h cannot be extended to l_p. Hence $\|f_i\|_1 \geqq \varepsilon$ for all i for some $\varepsilon>0$. We can now define a projection of l_p onto X as we did in the proof of Proposition 2.4.

REMARK. It is easy to see that if $X \oplus Y=l_p$, $0<p<1$, then $X^c \oplus Y^c=l_1$ where X^c, Y^c are the closures, in l_1, of X and Y. It also follows that X^* and Y^* are isomorphic to m since X^* and Y^* are isomorphic to complemented subspaces of m which must be isomorphic to m [7]. It is not true, however, that the conjugate of every closed infinite-dimensional subspace of l_p is isomorphic to m. This will be shown in the proof of Theorem 3.1. We note in closing this section that it is easy to show that every finite-dimensional subspace of l_p is complemented.

3. A subspace not isomorphic to l_p.

For each p, $1 \leqq p<2$, l_p contains an infinite-dimensional subspace which is not isomorphic to l_p (see [5] and [9]). Whether this situation persists for $p>2$ is still unknown; however, it does persist for $0<p<1$. We show this in the following theorem.

THEOREM 3.1. *For each* p, $0<p<1$, l_p *contains an infinite-dimensional subspace,* X, *which is not isomorphic to* l_p.

Proof. Let $\varphi_n(t)=\operatorname{sign}\sin(2^n\Pi t)$, $n=0, 1, \ldots$, be the Rademacher functions for t in $[0, 1]$. By Khinchine's inequality (see Paley [8]), given any $p>0$, there exist constants B and C such that

$$B\left(\sum |a_n|^2\right)^{p/2} \leqq \int_0^1 \left|\sum a_n\varphi_n(t)\right|^p dt \leqq C\left(\sum |a_n|^2\right)^{p/2}.$$

For a given integer n, $n \geqq 0$, divide the unit interval into 2^n equal intervals and then subdivide one of these intervals into infinitely may subintervals of length $2^{-(n+1)}$, $2^{-(n+2)}, \ldots$. We now embed l_p in L_p in the usual manner by constructing the appropriate scalar multiple of the characteristic function on each of the intervals. The embedding, T_n, of l_2^n into L_p given by

$$T_n(a_1, \ldots, a_n) = \sum_{k=1}^{n} a_k \varphi_k$$

is also an embedding of l_2^n into l_p because of the way we have embedded l_p in L_p. Let M_n denote the image of l_2^n in l_p under T_n. Khinchine's inequality yields

$$\|T_n x\|_p \leqq \|x\|_2^p C \quad \text{and} \quad B\|x\|_2^p \leqq \|T_n x\|_p.$$

Therefore

$$\|T_n x\|_p^p \leqq \|x\|_2^{p^2} C^p \quad \text{and} \quad B^p \|x\|_2^{p^2} \leqq \|T_n x\|_p^p.$$

These inequalities imply that the mapping of

$$R = (l_2^1 \oplus l_2^2 \oplus l_2^3 \oplus \cdots)_{l_{p2}}$$

onto

$$(m_1 \oplus m_2 \oplus \cdots)_{l_p}$$

is an isomorphism of R into $(l_p \oplus l_p \oplus \cdots)_{l_p}$ and this last space is isomorphic to l_p. If R is isomorphic to l_p, then R^* must be isomorphic to m, the space of all bounded sequences. However R^* is isomorphic to $(l_2^1 \oplus l_2^2 \oplus \cdots)_m$ and it has been shown by Lindenstrauss in [5] that this space is not isomorphic to m.

4. **Subspaces which are kernels of mappings.** It is well known that any separable Banach space is the image of l_1 under a continuous linear mapping. This statement has its analogue for $0<p<1$, and the kernel of this mapping is a subspace of l_p which has some interesting properties. We say that a linear topological space is *locally bounded* if it has a bounded neighborhood of zero.

THEOREM 4.1. *Every separable locally bounded F-space is isomorphic to a quotient space of l_p for some p in* (0, 1)(1).

Proof. Aoki [1] and Rolewicz [10] have shown that a p-homogeneous norm, $\| \ \|_p$, can be defined in a locally bounded F-space for every p satisfying $0<p<\log_{C(X)}2$ where $C(X)$ is the modulus of concavity of the space X. Let $\{x_n\}$ be any countable collection of points in X which is dense in the unit sphere,

$$S_X = \{x \in X : \|x\|_p = 1\},$$

of X, and define a mapping T of l_p into X as follows. Let $\{e_i\}$ be the unit vector basis in l_p, and let $Te_i = x_i$. Extend T linearly to the span of $\{e_i\}$. Since

$$\left\|T\left(\sum_{k=1}^{n} \lambda_k e_k\right)\right\|_p \leqq \sum_{k=1}^{n} |\lambda_k|^p \|x_k\|_p \leqq \left\|\sum_{k=1}^{n} \lambda_k e_k\right\|,$$

(1) After submitting this paper for publication, the author discovered that Theorem 4.1 was contained in J. H. Shapiro's doctoral dissertation (University of Michigan, 1969).

T is continuous and can be extended continuously to all of l_p. Given any point x in S_X, one can construct a series in l_p which converges to a point whose image is x. Hence the mapping T is onto X.

COROLLARY 4.2. *For each p, $0<p<1$, and for each q, $q \geqq p$, l_q and L_q are isomorphic to quotient spaces of l_p.*

Proof. This follows immediately from the proof of the preceding theorem because of the fact that

$$\left\| T\left(\sum_{k=1}^{n} \lambda_k e_k\right)\right\|_q \leqq \sum_{k=1}^{n} |\lambda_k|^q \leqq \left(\sum_{k=1}^{n} |\lambda_k|^p\right)^{q/p},$$

if $p \leqq q < 1$, and

$$\left\| T\left(\sum_{k=1}^{n} \lambda_k e_k\right)\right\|_q \leqq \sum_{k=1}^{n} |\lambda_k| \leqq \left(\sum_{k=1}^{n} |\lambda_k|^p\right)^{1/p},$$

if $q \geqq 1$.

THEOREM 4.3. *For each p, $0<p<1$, l_p contains a closed proper subspace, X, such that any continuous linear functional in l_p which vanishes on S vanishes on all of l_p.*

Proof. Choose X such that l_p/X is isomorphic to L_p. Since L_p contains no nonzero continuous linear functionals, there can be no nonzero linear functional in l_p which vanishes on X.

COROLLARY 4.4. *For each p, $0<p<1$, l_p contains a closed proper subspace which is weakly dense in l_p.*

COROLLARY 4.5. *For each p, $0<p<1$, l_p contains a weak Schauder basis which is not a basis.*

Proof. Let X be the subspace given in Corollary 4.3. It is easy to see that X is a dense subspace of l_1. Hence, by a well-known theorem of Krein, Milman, and Rutman, l_1 has a basis $\{b_n\}$ where b_n is in X for each n. This last condition implies that $\{b_n\}$ cannot be a basis for l_p while $\{b_n\}$ is clearly a weak Schauder basis for l_p.

THEOREM 4.6. *For each p, $0<p<1$, l_p contains a closed subspace, X, which is not contained in any proper complemented subspace.*

Proof. Let X be the kernel of a continuous linear mapping of l_p onto L_p, and suppose $X \subset Y$ where Y is a subspace complemented in l_p. If $Y \oplus Y_0 = l_p$, then $Y/X \oplus (X+Y_0)/X \approx L_p$. But $(X+Y_0)/X \approx Y_0$ and Y_0, being a subspace of l_p, has nonzero continuous linear functionals. This means that L_p also has nonzero continuous linear functionals which is not the case.

THEOREM 4.7. *For each p, $0<p<1$, l_p contains a subspace, X, such that $X \oplus Y$ fails to be complete for all infinite-dimensional subspaces Y of l_p.*

Proof. Let X be the kernel of any continuous mapping of l_p onto l_1. If $X \oplus Y$ is complete then Y is isomorphic to a subspace of l_1, and this is impossible.

5. **Linear dimension.** If X and Y are linear topological spaces, then $\dim_l X \leqq \dim_l Y$ if and only if X is isomorphic to a subspace of Y. If this is the case, we will say that X has linear dimension less than that of Y. If this is not the case, we will write $\dim_l X \nleqq \dim_l Y$. If neither X can be embedded in Y nor Y embedded in X, we will say that they are incomparable. These ideas date back to Banach [2], and the problem of linear dimension for l_p and L_p, $p \geqq 1$, spaces has now been completely solved (see [6] and the references cited there). We will examine the case when $p > 0$.

If $f \in L_p$, let $S_\varepsilon^p = \{t \in [0, 1] : |f(t)| \geqq \varepsilon \|f\|_p\}$ and say that $f \in M_\varepsilon^p$ if and only if $|S_\varepsilon^p| \geqq \varepsilon$ where $|S_\varepsilon^p|$ denotes the Lebesgue measure of S_ε^p. It has been shown by Kadec and Pełczyński in [4] that if X is a subspace of L_p, $1 \leqq p < \infty$, and $X \not\subset M_\varepsilon^p$ for any $\varepsilon > 0$, then X contains a subspace isomorphic to l_p. We use these ideas in the following.

Theorem 5.1. *Suppose that $p, q > 0$. If $p \neq q$, then* $\dim_l l_p$ *and* $\dim_l l_q$ *are incomparable*; $\dim_l l_2 \leqq \dim_l L_p$; $\dim_l L_p \leqq \dim_l l_q$ *implies $p = q = 2$*; $\dim_l l_p \nleqq \dim_l L_q$ *if $p < q$*; $\dim_l L_p \leqq \dim_l L_q$ *implies $p \geqq q$*; *and* $\dim_l L_p \leqq \dim_l L_q$ *if $q \leqq p$ and* $1 < p \leqq 2$.

Proof. Suppose that $0 < p < q < 1$ and that $\dim_l l_p \leqq \dim_l l_q$. Then l_q contains a bounded basic sequence $\{f_n\}$ equivalent to the unit vector basis in l_p. Since the sequence $\{f_n\}$ is bounded, the series $\sum \alpha_n f_n$ converges for all sequences (α_n) in l_q. This is a contradiction which implies $\dim_l l_p \nleqq \dim_l l_q$. If $\dim_l l_q \leqq \dim_l l_p$, then by Theorem 2.1, l_q contains a subspace isomorphic to l_p which was just shown to be impossible.

$\dim_l l_2 \leqq \dim_l L_p$ follows immediately from Khinchine's inequality given in the proof of Theorem 3.1.

If $0 < p$, $q < 1$, then clearly $\dim_l L_p \nleqq \dim_l l_q$ since L_p has no continuous linear functionals. The other cases are either well known or are obvious.

If $0 < p < q < 1$, then $\dim_l l_p \nleqq \dim_l L_q$ follows from the argument given in the first part of this proof. Since l_p can be isometrically embedded in L_p, this implies that $\dim_l L_p \nleqq \dim_l L_q$ when $p < q$.

If $1 < p < 2$, select a number r such that $1 < r < p$ and $q < r$. Then L_p is isomorphic to a subspace X of L_r (see [6]), and since l_r is not isomorphic to a subspace of L_p, $X \subset M_\varepsilon^r$ for some $\varepsilon > 0$. This implies that X is a closed subspace of L_q and so $\dim_l L_p \leqq \dim_l L_q$.

We are unable to settle the remaining case which we leave as a problem: If $0 < p < 1$ and $q > p$ is $\dim_l L_q \leqq \dim_l L_p$ or is $\dim_l l_q \leqq \dim_l L_p$?

We note in conclusion that a stronger result than that contained in the last part of the preceding theorem can be obtained. This is done in the following.

Theorem 5.2. *For each p, $1 < p \leqq 2$, L_p is isomorphic to a subspace of M, the space of all measurable functions on* [0, 1] *with topology given by convergence in measure.*

Proof. For a given p, we choose an r as in the last part of the proof of the preceding theorem. Then if $L_p \approx X$ where X is a subspace of L_r, $X \subset M_\varepsilon^r$ for some $\varepsilon > 0$. The natural mapping of X (as a subspace of L_r) into M is continuous and one-to-one and the inverse of this mapping is also continuous since $X \subset M_\varepsilon^r$. This shows that X, as a subspace of M, is isomorphic to L_p.

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