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PIECEWISE LINEAR GROUPS AND TRANSFORMATION GROUPS

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The object of this paper is to show, via specific theorems, that the notions of topological group and transformation group become severely restricted when transposed to the piecewise linear category.

Let us understand a *piecewise linear* (*PL*) *group* to be a topological group G together with a piecewise linear structure on G (i.e., triangulation of G as a locally finite simplicial complex), in terms of which the group multiplication $G\times G\to G$ and inversion $G\to G$ are given by piecewise linear functions.

THEOREM A. *The only connected PL groups are the abelian Lie groups,* $(S^1)^m\times R^n$, *and in general the only PL groups are extensions of these by discrete groups.*

THEOREM B. *Two PL groups are PL isomorphic if and only if they are topologically isomorphic, and any topological isomorphism between them is automatically a PL isomorphism.*

Let G be a PL group, acting as a topological transformation group on the PL manifold M, via the map

$$F\colon G\times M\to M.$$

If F is a PL map, we say that G is a *PL transformation group* acting on M.

THEOREM C. *Let G be a PL transformation group acting on the PL manifold M. If*

(1) *the action is effective,*

(2) $\dim G\geqq 1$,

(3) *M is connected,*

then G has no fixed points (*in the sense that no point of M is left fixed by every element of G*).

Theorem C says, among other things, that equivariant suspension of PL actions of groups on spheres is generally not possible.

COROLLARY 1. *Let G be an effective PL transformation group acting on the*

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connected PL manifold M. Then the isotropy subgroup of each point of M is discrete. Hence all orbits have the same dimension.

COROLLARY 2. *Let G be an effective PL transformation group acting on the connected PL manifold M. Then* $\dim G \leqq \dim M$.

If $n \leqq m$, there is a *standard action* of R^n on R^m obtained by regarding R^n as a subgroup of R^m and letting it act on R^m by translation.

THEOREM D. *Let G be an effective PL transformation group acting on the connected PL manifold M. Then there are local coordinates about the identity in G, in terms of which the group operation is addition, and local coordinates about any preassigned point in M, in terms of which the action of G on M is standard.*

The proofs of these theorems, together with some subsidiary results, will be given in the following sections. I thank C. T. Yang for many helpful conversations.

1. **Proof of Theorem A.** To prove Theorem A, we first prove

THEOREM 1.1. *Every PL group is locally PL isomorphic to the Euclidean group of the same dimension.*

Let G be a PL group. Since G can be triangulated by a locally finite simplicial complex, G has at least some points which have neighborhoods PL homeomorphic to a Euclidean space. But then by homogeneity every point of G has such neighborhoods, so G is a locally Euclidean group.

Choose PL coordinates in R^n for some neighborhood of the identity in G, with the identity corresponding to $0 \in R^n$. We henceforth identify this neighborhood with, and call it, R^n. Next choose a neighborhood U of 0 in R^n such that $U^2 \subset R^n$, where U^2 denotes as usual the set of all products $u_1 * u_2$ with u_1 and u_2 in U. The multiplication $*$ in G is then given locally by some PL function

$$F\colon U \times U \to R^n, \qquad x * y = F(x, y).$$

Since F is PL on $U \times U$, there is some subdivision of $U \times U$, with $(0, 0)$ as a vertex, on each simplex of which F is affine. Let Δ^{2n} be a $2n$-simplex of $U \times U$ with one vertex at the origin. Then $F\colon \Delta^{2n} \to R^n$ is an affine function which takes $(0, 0)$ to 0, and is hence linear. Thus

$$x * y = F(x, y) = Ax + By$$

on Δ^{2n}, where A and B are linear maps of R^n to R^n.

Let (a, b) be a point of int Δ^{2n}, the interior of Δ^{2n}, and let V_a and V_b be neighborhoods of a and b in U such that $V_a \times V_b \subset \Delta^{2n}$. Choose a neighborhood W of 0 in U such that

$$a * W \subset V_a, \qquad W * b \subset V_b.$$

Define $f\colon W \to R^n$ by $f(w) = (a * w) - a$, so that for all $w \in W$, $a * w = a + f(w)$.

Since translation in the group G and subtraction in R^n are both PL homeomorphisms, f is a PL homeomorphism of W into R^n leaving 0 fixed. Similarly define $g\colon W\to R^n$ by $g(w)=(w*b)-b$, so that for all $w\in W$, $w*b=g(w)+b$. Then g is also a PL homeomorphism of W into R^n leaving 0 fixed.

Next we show that $Af=Bg$. For if w is any point of W, then on the one hand

$$(a*w)*b = (a+f(w))*b = A(a+f(w))+Bb = Aa+Af(w)+Bb.$$

On the other,

$$\imath*(w*b) = a*(g(w)+b) = Aa+B(g(w)+b) = Aa+Bg(w)+Bb.$$

Since these two are equal by associativity in G, we have $Af(w)=Bg(w)$ for all $w\in W$, as claimed.

Now let W_0 be a neighborhood of 0 in W such that $W_0^2\subset W$, and let $w_1, w_2\in W_0$. On the one hand

$$\begin{aligned}(a*(w_1*w_2))*b &= (a+f(w_1*w_2))*b\\ &= Aa+Af(w_1*w_2)+Bb,\end{aligned}$$

while on the other,

$$\begin{aligned}(a*w_1)*(w_2*b) &= (a+f(w_1))*(g(w_2)+b)\\ &= Aa+Af(w_1)+Bg(w_2)+Bb\\ &= Aa+Af(w_1)+Af(w_2)+Bb\end{aligned}$$

since $Af=Bg$. Again these two products are equal, so that

$$Af(w_1*w_2) = Af(w_1)+Af(w_2).$$

Now the linear map A is 1-1 on the open set $V_a\subset R^n$ by the cancellation law in G, hence globally 1-1. Then applying the linear map A^{-1} to both sides of the above equation, we get

$$f(w_1*w_2) = f(w_1)+f(w_2).$$

Hence f is a local PL isomorphism between G and the Euclidean group R^n, and the theorem is proved.

Theorem A now follows immediately. Since G is a locally Euclidean group, it is by [1] and [2] topologically isomorphic to some Lie group. If G is connected, then by Theorem 1.1 it must be abelian, so that Theorem A is obtained in this case. If G is not connected, the identity component G_0 of G is a PL group in its own right, and therefore abelian by the above conclusion. But G_0 is an open and closed normal subgroup of G, hence G/G_0 must be discrete. So Theorem A is proved in general.

2. **Proof of Theorem B.** Recall that a one-parameter subgroup of a topological group G is a continuous homomorphism $g\colon R^1\to G$. If G is a PL group and the

map g is a PL map, then we refer to g as a *one-parameter PL subgroup* of G. As an immediate corollary to Theorem 1.1, we have

LEMMA 2.1. *Every one-parameter subgroup of a PL group is itself PL.*

Let g be a one-parameter subgroup of the PL group G. By Theorem 1.1, G is locally PL isomorphic to some Euclidean group R^n. Since the local one-parameter subgroups of R^n are all linear, g must be PL on some neighborhood of $0 \in R^1$. Since translations in R^1 and in G are given by PL functions, g must be a PL map.

To prove Theorem B, we will prove the somewhat stronger

THEOREM 2.2. *If f: $G \to G'$ is a continuous homomorphism between the PL groups G and G', then f is automatically a PL homomorphism.*

Since translations in G and G' are given by PL functions, it will be sufficient to show that f is PL on some neighborhood of the identity in G.

If $g_1, g_2, \ldots, g_n$ are one-parameter subgroups of G, then they are PL maps by the above lemma, and hence the map φ: $R^n \to G$ defined by

$$\varphi(t_1, t_2, \ldots, t_n) = g(t_1) * g(t_2) * \cdots * g(t_n)$$

is a PL map of R^n into G. If $g_1', g_2', \ldots, g_n'$ are one-parameter subgroups of G', then the similarly defined map ψ: $R^n \to G'$ is also PL.

If $n = \dim G$, then it follows from Theorem 1.1 that the one-parameter subgroups g_i can be chosen so that φ is a PL homeomorphism (indeed, PL isomorphism) of a neighborhood of 0 in R^n onto a neighborhood of the identity in G. Do so, and then let $g_i' = fg_i$. Clearly $f\varphi = \psi$, so that by the choice of the g_i we can write $f = \psi\varphi^{-1}$ on some neighborhood of the identity in G, exhibiting f as a PL function on that neighborhood.

This completes the proof of Theorem 2.2, and with it, that of Theorem B.

Let us call a subgroup H of the PL group G a *PL subgroup* of G if it is a PL subspace of G, i.e., a subcomplex of some subdivision of G. Then a PL subgroup becomes a PL group in its own right.

THEOREM 2.3. *Every closed subgroup of a PL group is a PL subgroup, and every factor group of a PL group by a closed normal subgroup is itself a PL group in a natural way.*

This is easily seen to be true for the abelian Lie groups $(S^1)^m \times R^n$ with their natural PL structures, and for extensions of these by discrete groups. By Theorems A and B, there are no other PL groups.

3. **Proof of Theorem C.** Suppose now that G is an effective PL transformation group of dimension at least 1, acting on the connected PL manifold M. Choose local PL coordinates in R^n for G, with $0 \in R^n$ corresponding to the identity of G, in terms of which the group operation (by Theorem 1.1) is addition. Choose local

PL coordinates in R^m for M, with $0 \in R^m$ corresponding to a point of M supposedly left fixed by every element of G. Then the action is given locally by

$$R^n \times R^m \supset U \times V \xrightarrow{F} R^m,$$

where $U \times V$ is a neighborhood of $(0, 0)$ in $R^n \times R^m$ and F is a PL map. Note that if we set $f_x(y) = F(x, y)$, then by the choice of local coordinates in G we have

$$f_x f_{x'}(y) = f_{x+x'}(y).$$

Subdivide $U \times V$ so that F is affine on each simplex, so that $(0, 0)$ appears as a vertex and $U \times 0$ as a subcomplex. Let Δ^{n+m} be an $(n+m)$-simplex of this subdivision with one vertex at the origin and an n-face Δ^n in $U \times 0$. Then F: $\Delta^{n+m} \to R^m$ is an affine function which takes $(0, 0)$ to 0, hence is linear. Thus $F(x, y) = Ax + By$ for $(x, y) \in \Delta^{n+m}$, where A: $R^n \to R^m$ and B: $R^m \to R^m$ are linear maps. Since $0 \in R^m$ is a fixed point of G, we have

$$0 = F(x, 0) = Ax + B(0) = Ax$$

for $(x, 0) \in \Delta^n$. Since Δ^n is n-dimensional, we must have $A = 0$. Hence $F(x, y) = By$ on Δ^{n+m}.

Now choose any point $(a, b) \in \operatorname{int} \Delta^{n+m}$, and let U_a be a neighborhood of a in R^n such that $U_a \times b \subset \Delta^{n+m}$. Then let U' be a neighborhood of 0 in R^n such that $a + U' \subset U_a$. Now $f_a(b) = F(a, b) = Bb$, while for any $u' \in U'$,

$$f_a(f_{u'}(b)) = f_{a+u'}(b) = F(a+u', b) = Bb,$$

since $(a+u', b) \in \Delta^{n+m}$. (Note that in general $(u', b) \notin \Delta^{n+m}$.) Thus

$$f_a(b) = f_a(f_{u'}(b)),$$

and since f_a is 1-1, $f_{u'}(b) = b$ for all $u' \in U'$. Since the neighborhood U' of 0 in R^n leaves b fixed, the whole identity component G_0 of G must also leave b fixed.

It remains to be shown that the set of b's for which the above argument can be carried out is dense in some neighborhood of 0 in R^m. To see this, first pick an n-simplex Δ^n in $U \times 0$ with one vertex at the origin. Then pick a point $(a, 0) \wedge \in \cdot \operatorname{int} \Delta^n$. Then pick a neighborhood V' of 0 in R^m such that

$$a \times V' \subset \operatorname{int} (\operatorname{st} (\operatorname{int} \Delta^n, U \times V)),$$

the open star of $\operatorname{int} \Delta^n$ in the given subdivision of $U \times V$. For any $b \in V'$, the point (a, b) will then lie in the interior of some simplex Δ^p having Δ^n for a face. The above argument, which led to the conclusion that G_0 must leave b fixed, will be valid for this b provided that $p = n + m$.

Suppose then that $p < n+m$. Now Δ^n and $a \times V'$ span R^{n+m}, in the sense that

R^{n+m} is the smallest affine subspace of itself containing both of these. Since $\Delta^p \supset \Delta^n$, so do Δ^p and $a \times V'$ span R^{n+m}. Hence

$$\begin{aligned}\dim(\Delta^p \cap (a \times V')) &= \dim \Delta^p + \dim(a \times V') - (n+m) \\ &= p+m-(n+m) = p-n.\end{aligned}$$

Since $p < n+m$, we conclude that $\dim(\Delta^p \cap (a \times V')) < m$.

Thus the set of points b in V' for which (a, b) does *not* lie in the interior of an $(n+m)$-simplex having Δ^n as a face, is the union of finitely many pieces of planes of dimensions less than m, hence certainly nowhere dense in V'. For the remaining points of V' our earlier argument is valid, and hence G_0 leaves a dense subset of V' pointwise fixed. By continuity, G_0 must leave fixed the whole neighborhood V' of 0 in R^m.

Since the coordinate $0 \in R^m$ was assigned to an arbitrary fixed point, we conclude that G_0 leaves fixed a whole neighborhood of any point of M left fixed by G (or equally well by G_0). Thus the set of points of M left fixed by G_0 is open in M, and by continuity, closed in M. Now M is connected, so if G_0 leaves fixed one point of M, it must leave fixed every point of M. But $\dim G \geqq 1$ implies that G_0 contains more than just the identity, and then the action will fail to be effective. So indeed G can leave no point of M fixed, and the theorem is proved.

REMARK. Although Theorem C says that no point of M can be left fixed by every element of G, nevertheless certain points of M can be left fixed by certain elements of G, even if we also require G to be connected. Thus PL transformation groups do not in general act freely.

To see a simple example of this, consider the homeomorphisms φ and ψ of the plane R^2, given by

$$\varphi(x, y) = (x+1, y), \qquad \psi(x, y) = (-x, y+1).$$

The group of homeomorphisms generated by φ and ψ acts freely on the plane with the Klein bottle K^2 as orbit space.

Now let R^1 act on R^2 by defining $f_t(x, y) = (x, y+t)$. Since f_t commutes with both φ and ψ, we get an induced action of R^1 on K^2. Since $f_2 = \psi^2$, we can factor R^1 by the even integers and get an induced action of the circle group on K^2. This action is both effective and PL. Nevertheless, f_1 leaves two disjoint circles on K^2 pointwise fixed.

If we think of the Klein bottle as a twisted circle bundle over the circle, then the above action of the circle group is just rotation of the Klein bottle in the direction of the base space.

Proof of Corollary 1. Let G be an effective PL transformation group acting on the connected PL manifold M. If $y \in M$, the isotropy subgroup G_y of y is a closed subgroup of G, hence a PL subgroup by Theorem 2.3 and therefore a PL group in its own right. Since y is a fixed point of G_y, we must have $\dim G_y = 0$ by Theorem

C. Since some subdivision of G displays G_y as a simplicial complex, G_y must be discrete. Then all orbits have the same dimension as G, and Corollary 1 is proved.

Proof of Corollary 2. Again let G be an effective PL transformation group acting on the connected PL manifold M. Pick a point $y \in M$ and let G_y be the isotropy subgroup of y. By Corollary 1, G_y is discrete. Let U be a neighborhood of the identity in G which meets G_y only at the identity. Let V be a neighborhood of the identity in G such that $VV^{-1} \subset U$. If the action of G on M is given by

$$F\colon G \times M \to M,$$

set $f_x(y) = F(x, y)$. Then define $\varphi\colon V \to M$ by $\varphi(x) = f_x(y)$. If x_1 and x_2 lie in V and satisfy $\varphi(x_1) = \varphi(x_2)$, then we must have $f_{x_1 x_2^{-1}}(y) = y$. Hence $x_1 x_2^{-1} \in G_y \cap U$, which consists only of the identity of G. Thus $x_1 = x_2$, and so φ is 1-1 and hence an embedding of some neighborhood of the identity in G into M. Thus $\dim G \leqq \dim M$, and Corollary 2 is proved.

Remark. If M is not connected, Theorem C and Corollaries 1 and 2 are all false. For Corollary 2, there is an effective PL action of R^2 on the disjoint union $R^1_1 \cup R^1_2$ of two lines, given by the formula

$$\begin{aligned} f_{(s,t)}(x) &= x+s \quad \text{if } x \in R^1_1, \\ &= x+t \quad \text{if } x \in R^1_2. \end{aligned}$$

This also contradicts the first half of Corollary 1. For the second half of Corollary 1, one might consider a similar action of R^3 on the disjoint union $R^2 \cup R^1$. Theorem C is obviously false if M is not connected.

4. **Proof of Theorem D.** Let G be an effective PL transformation group acting on the connected PL manifold M. If $\dim G = n$ and $\dim M = m$, then by Corollary 2 we have $n \leqq m$. Choose local PL coordinates in R^n for G, with $0 \in R^n$ corresponding to the identity of G, in terms of which the group operation is addition. Choose local PL coordinates in R^m for M, with $0 \in R^m$ corresponding to an arbitrary preassigned point of M. Then the action of G on M is given locally in these coordinates by

$$R^n \times R^m \supset U \times V \xrightarrow{F} R^m,$$

where $U \times V$ is a neighborhood of $(0, 0)$ in $R^n \times R^m$ and F is a PL map. Our object is to change the local coordinates in M so that the action of G on M is in standard form in the new coordinates.

Subdivide $U \times V$ so that F is affine on each simplex, so that $(0, 0)$ is a vertex and $U \times 0$ a subcomplex. Let Δ^n be an n-simplex of $U \times 0$ with one vertex at the origin. Then F is linear on Δ^n.

Now $F(\Delta^n) \subset R^m$ is part of the orbit through $0 \in R^m$. If F/Δ^n were singular, we could conclude that the isotropy subgroup of $0 \in R^m$ was nondiscrete, in contradiction to Corollary 1. Hence F/Δ^n is nonsingular, and therefore $F(\Delta^n)$ is an n-simplex in R^m.

Let $(a, 0)$ be the barycenter of Δ^n. Let $L: R^{m-n} \to R^m$ be an affine embedding such that $L(0)=F(a, 0)$ and such that $L(R^{m-n})$ is perpendicular to $F(\Delta^n)$. Then define

$$H: R^m = R^n \times R^{m-n} \to R^m$$

by

$$H(y_1, y_2) = F(y_1 - a, L(y_2)).$$

CLAIM. *H is a PL homeomorphism between neighborhoods of* 0 *in* R^m.

H, as a composition of PL functions, is itself PL. Furthermore,

$$\begin{aligned} H(0,0) &= F(-a, L(0)) = F(-a, F(a, 0)) \\ &= F(0,0) = 0. \end{aligned}$$

Thus H takes the origin in R^m to itself. It remains to show that H is 1-1 on some neighborhood of the origin.

Since

$$H(y_1, y_2) = F(y_1 - a, L(y_2)) = F(-a, F(y_1, L(y_2)),$$

it will be sufficient to show that $F(y_1, L(y_2))$ is 1-1 on some neighborhood of the origin. Suppose then that

$$F(y_1, L(y_2)) = F(z_1, L(z_2)),$$

or equivalently that

$$F(y_1 - z_1, L(y_2)) = F(0, L(z_2)) = L(z_2)$$

for (y_1, y_2) and (z_1, z_2) near the origin in R^m.

Now the orbit through any point of R^m sufficiently near $F(a, 0)$ is, locally, a portion of an n-plane passing through that point and parallel to $F(\Delta^n)$. To see this, take any point (a', b') in $R^n \times R^m$ close to $(a, 0)$. Then (a', b') must lie in the interior of a simplex Δ^p of the given subdivision of $U \times V$, having Δ^n for a face. The orbit through $F(a', b')$ is, locally, the image under F of the n-plane $R^n \times b'$. Since Δ^p has Δ^n for a face, a portion of this n-plane lies in Δ^p. That is, there is an open neighborhood $U_{a'}$ of a' in R^n such that $U_{a'} \times b' \subset \Delta^p$. Since F is linear on Δ^p, $F(U_{a'} \times b')$ is a portion of an n-plane in R^m parallel to the n-simplex $F(\Delta^n)$.

Thus a small portion of the orbit through any point of R^m near $F(a, 0)$ is parallel to $F(\Delta^n)$ and therefore perpendicular to $L(R^{m-n})$. Hence it meets $L(R^{m-n})$ only once, so the equation

$$F(y_1 - z_1, L(y_2)) = L(z_2)$$

implies that

$$y_1 - z_1 = 0, \qquad L(y_2) = L(z_2).$$

Since L is 1-1, we conclude that $(y_1, y_2)=(z_1, z_2)$, so that H is indeed 1-1 on some neighborhood of the origin in R^m. This establishes the claim.

Recall that the standard action of R^n on R^m is obtained by regarding R^n as a subgroup of R^m and letting it act on R^m by translation. If we write $R^m = R^n \times R^{m-n}$ and $y \in R^m$ as $y=(y_1, y_2)$, then this standard action, $F_0: R^n \times R^m \to R^m$, is given by the formula

$$F_0(x, y) = F_0(x, (y_1, y_2)) = (x + y_1, y_2).$$

We see now that the following diagram is commutative.

$$\begin{array}{ccc} \text{Nbhd of } (0,0) \text{ in } R^n\times R^m & \xrightarrow{F_0} & \text{Nbhd of } 0 \text{ in } R^m \\ {\scriptstyle 1\times H}\big\downarrow & & \big\downarrow{\scriptstyle H} \\ \text{Nbhd of } (0,0) \text{ in } R^n\times R^m & \xrightarrow[F]{} & \text{Nbhd of } 0 \text{ in } R^m \end{array}$$

For

$$\begin{aligned} HF_0(x,(y_1,y_2)) &= H(x+y_1,y_2) = F(x+y_1-a, L(y_2)) \\ &= F(x, F(y_1-a, L(y_2)) = F(1\times H)(x,(y_1,y_2)). \end{aligned}$$

Thus $HF_0=F(1\times H)$, as claimed. But then using H to change local coordinates in M puts the action of G on M into standard form in the new local coordinates, completing the proof of Theorem D.

5. **General remarks.** Since the notion of PL transformation group is so restricted, one may consider actions of topological transformation groups on PL manifolds in which each individual homeomorphism is required to be PL, and nothing more. Unlike the differentiable case [3, Chapter V], this makes a difference even if the group is a PL group. Such actions can have fixed points, and suspension of actions on spheres is again possible. In return, their classification becomes harder.

For example, any two free topological actions of R^1 on R^1 are topologically equivalent. If we require the actions to be PL, then first of all one can deduce from Theorem C that, if effective, they must automatically be free, and then also that they must be PL equivalent. On the other hand, two free actions of R^1 on R^1, in which each individual homeomorphism is PL, need not be PL equivalent. The one-parameter group acting on R^1 via the formula $f_t(x)=x+t$ is *not* PL equivalent to the one-parameter group acting on $(0,\infty)$ via the formula $g_t(y)=e^ty$, even though the individual homeomorphisms are linear.

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