STRUCTURE SPACES OF SEMIGROUPS OF CONTINUOUS FUNCTIONS

BY
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Abstract. In a previous paper, we associated a topological space with each left ideal of a semigroup. Here, we determine this space when the semigroup under consideration is the semigroup of all continuous selfmaps of any space belonging to a fairly extensive class of topological spaces and the left ideal is taken to be the kernel of the semigroup.

1. Introduction and statement of main theorem. In [3], we associated with each left ideal \( Z \) of a semigroup \( T \), a structure space which we denoted by \( \mathcal{U}(T, Z) \). This space is formed as follows: a nonempty subset \( A \) of \( T \times Z \) is a bond if for any finite subset \( \{(t_i, z_i)\}_{i=1}^{N} \subseteq A \), the system of equations \( \{t_i x = z_i\}_{i=1}^{N} \) has a common solution \( x \) in \( Z \). An ultrabond is a bond which is not properly contained in any other bond. \( \mathcal{U}(T, Z) \) is then defined to be the family of all such ultrabonds. \( \mathcal{U}(T, Z) \neq \emptyset \) since the existence of a bond is immediate and by Zorn's Lemma, every bond is contained in ultrabond. We topologize \( \mathcal{U}(T, Z) \) in the following manner: for each \( (t, z) \in T \times Z \), let

\[
H(t, z) = \{A \in \mathcal{U}(T, Z) : (t, z) \in A\}
\]

and take \( \{H(t, z) : (t, z) \in T \times Z\} \) to be a subbasis for the closed subsets of \( \mathcal{U}(T, Z) \). For some general facts about such spaces, one should consult [3, pp. 319–324].

In the event \( Z \) is the kernel (minimal two-sided ideal) of \( T_1 \), we refer to \( \mathcal{U}(T, Z) \) as the \( \mathcal{X} \)-structure space of \( T \) and denote it simply by \( \mathcal{U}(T) \).

Before we can state the Main Theorem, we need to recall some facts about \( E \)-compact spaces which were introduced by Engleking and Mrówka in [1]. We use the terminology and notation adopted in [4] which gives a rather extensive account of the theory. Let \( E \) be any Hausdorff space. A space \( X \) is \( E \)-completely regular if it is homeomorphic to a subset of some cartesian product of copies of \( E \) and it is \( E \)-compact if it is homeomorphic to a closed subset of such a product. We will refer to an \( E \)-compact space \( Y \) which contains \( X \) as a dense subspace as an \( E \)-compactification of \( X \). An important fact about \( E \)-compactifications is that if \( E \) is compact in the usual sense, then each \( E \)-completely regular space \( X \) has a

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largest $E$-compactification $\beta_{E}X$ in the sense that any other $E$-compactification of $X$ is a continuous image of $\beta_{E}X$ under a map which keeps the points of $X$ fixed. This is an immediate consequence of Theorem 4.14 of [4, p. 177]. Before stating our Main Theorem, we introduce one additional concept.

**Definition.** Let $E$ be Hausdorff. A space $X$ is $E$-separated if it is Hausdorff and for each pair $H$ and $K$ of disjoint nonempty closed subsets of $X$, there exists a continuous function $f$ from $X$ into $E$ and two distinct points $p$ and $q$ of $E$ such that $f(x) = p$ for $x \in H$ and $f(x) = q$ for $x \in K$.

Evidently, a space $X$ is normal if and only if it is $\mathcal{I}$-separated where $\mathcal{I}$ denotes the closed unit interval. Furthermore, it follows rather quickly from Theorem 2.1 [4, p. 165] that every $E$-separated space is also $E$-completely regular. We are now in a position to state our main result which concerns the $\mathcal{X}$-structure space of the semigroup $S(X)$ of all continuous selfmaps of $X$ where the binary operation is ordinary composition.

**Main Theorem.** Suppose that $E$ is compact and that $X$ is $E$-separated and contains a copy of $E$. Then the $\mathcal{X}$-structure space of $S(X)$ is, in fact, $\beta_{E}X$ the largest $E$-compactification of $X$.

**2. Proof of the main theorem and two corollaries.** Since the proof will rely rather heavily in several instances upon Theorem (1.10) of [3, p. 322], we begin by discussing some concepts which are relevant to that result. Once again $T$ is a semigroup and $Z$ is a left ideal of $T$. If $T$ has a left identity, then for each $v \in Z$, $A_v = \{(t, tv) : t \in T\}$ is an ultrabond [3, Lemma (1.2), p. 320] and the set of all ultrabonds of this form is denoted by $\mathcal{A}(T, Z)$ and is referred to as the realization of $Z$. This is a subspace of $\mathcal{U}(T, Z)$ and may well be a proper subspace. Now to each element $a \in T$, one can associate in a very natural way an element $f_a$ in $S(\mathcal{A}(T, Z))$, the semigroup, under composition, of all continuous selfmaps of $\mathcal{A}(T, Z)$. The mapping $f_a$ is defined by $f_a(A_v) = A_{av}$ for each $A_v \in \mathcal{A}(T, Z)$. Theorem (1.8) of [3, p. 321] asserts, among other things, that $f_a$ is continuous. Now we are in a position to extract the portion of Theorem (1.10) of [3, p. 322] which we need here. It is as follows:

1. If the pair $(T, Z)$ is admissible and $T$ has a left identity then $\mathcal{U}(T, Z)$ is a Hausdorff compactification of $\mathcal{A}(T, Z)$ and each $f_a$ in $S(\mathcal{A}(T, Z))$ has a unique extension to a function $f_a^*$ in $S(\mathcal{U}(T, Z))$. The pair $(T, Z)$ is defined to be admissible [3, Definition (1.6), p. 320] if

- $(1)$ $A$ is an ultrabond and $A \in \mathcal{E}H(t_1, z_1)$ (the complement of $H(t_1, z_1)$ in $\mathcal{U}(T, Z)$) then there exist $(t_2, z_2)$ and $(t_3, z_3)$ in $T \times Z$ such that
  $$A \in \mathcal{E}H(t_2, z_2) \subseteq H(t_3, z_3) \subseteq \mathcal{E}H(t_1, z_1).$$

Now suppose we turn our attention to the semigroup $S(X)$ where $X$ satisfies the conditions stated in the Main Theorem. For any point $p \in X$, we denote by $\langle p \rangle$ the constant function which maps each point of $X$ into $p$. One easily verifies
that the kernel $K(X)$ of $S(X)$ is precisely the family of all constant function on $X$. Furthermore, it is not difficult to show that

(3) For any $f \in S(X)$ and $\langle x \rangle, \langle y \rangle$ in $K(X)$, $f \circ \langle y \rangle = \langle z \rangle$ if and only if $f(y) = z$.

(4) A subset $A$ of $S(X) \times K(X)$ is a bond if and only if \( \{ f^{-1}(z) : (f, \langle z \rangle) \in A \} \) has the finite intersection property.

(5) $A$ is an ultrabond if and only if $(f, \langle z \rangle) \notin A$ implies $f^{-1}(z) \cap g_1^{-1}(y_1) \cap \cdots \cap g_N^{-1}(y_N) = \emptyset$ for some finite subfamily $\{(g_i, \langle y_i \rangle)\}_{i=1}^N$ of $A$.

We next want to observe that the pair $(S(X), K(X))$ is admissible. A space $X$ is defined in [3, Definition (2.5), p. 327] to be a strong $S^*$-space if it is Hausdorff and for each pair of disjoint closed subsets $H$ and $K$ of $Y$ there exist distinct points $p$ and $q$ of $Y$ and a continuous selfmap $f$ of $Y$ such that $f(x) = p$ for $x \in K$. Since $X$ is $\tau$-separated and contains a copy of $E$, it follows readily that $X$ is a strong $S^*$-space. Consequently, by Theorem (2.7) [3, p. 328] the pair $(S(X), K(X))$ is admissible and (1) now applies. We will use (1) in proving that

(6) $\mathcal{U}(S(X))$ is $E$-compact.

By (1), $\mathcal{U}(S(X))$ is a Hausdorff space which is compact in the usual sense. Thus, if an embedding into a cartesian product of copies of $E$ exists, $\mathcal{U}(S(X))$ must necessarily be embedded as a closed subset. Consequently, we need only prove the existence of an embedding. According to Theorem (2.1) of [4, p. 165], it will be sufficient to show that for each closed subset $W$ of $\mathcal{U}(S(X))$ and each $A \in \mathcal{U}(S(X))$ with $A \notin W$, there exists a continuous function $f$ from $\mathcal{U}(S(X))$ into $E$ and a point $q \in E$ such that $f(B) = q$ for $B \in W$ and $f(A) \neq q$.

Since $\{H(g, \langle y \rangle) : g \in S(X), y \in X\}$ is a subbasis for the closed subsets of $\mathcal{U}(S(X))$, there exists a finite subfamily $\{(g_i, \langle y_i \rangle)\}_{i=1}^N$ of $S(X) \times K(X)$ such that

$$A \notin W^*, \quad W \subseteq W^*$$

where $W^* = H(g_1, \langle y_1 \rangle) \cup \cdots \cup H(g_N, \langle y_N \rangle)$. Since $A \notin W^*$, there exist by (5), finite subfamilies $\{(h_p, \langle v_p \rangle)\}_{p=1}^M$ of $A$ with the property that $g_i^{-1}(y_i) \cap V_i = \emptyset$ where

$$V_i = h_1^{-1}(v_1) \cap \cdots \cap h_N^{-1}(v_N)$$

for $1 \leq i \leq N$. Now let $V^* = \bigcap \{V_i\}_{i=1}^M$ and let $H = \bigcup \{g_i^{-1}(y_i)\}_{i=1}^N$. Then $H \cap V^* = \emptyset$ and since $X$ is $E$-separated and contains a copy $E^*$ of $E$, there exists a continuous function $f$ mapping $X$ into $E^*$ and two distinct points $p$ and $q$ of $E^*$ such that

$$f(x) = p \text{ for } x \in V^* \quad \text{and} \quad f(x) = q \text{ for } x \in H.$$

As we observed in the discussion preceding (1), the mapping $f_x$ defined by $f_x(A_{\langle x \rangle}) = A_{f_x(x)}$ is a continuous selfmap of $\mathcal{R}(S(X), K(X))$ which, by (1) has a unique extension to a continuous selfmap $f_x$ of $\mathcal{U}(S(X))$. Hereafter, we will denote the space $\mathcal{R}(S(X), K(X))$ more simply by $\mathcal{R}(S(X))$. Since $X$ is an $S^*$-space (in fact, a strong $S^*$-space), the canonical map $e$ which takes $x \in X$ into $A_{\langle x \rangle}$ in $\mathcal{R}(S(X))$
is a homeomorphism from $X$ onto $\mathcal{R}(S(X))$ [3, Theorem (2.3), p. 325]. We assert that

(9) \[ \mathcal{f} \text{ maps } \mathcal{U}(S(X)) \text{ into } e[E^*], \]

and

(10) \[ \mathcal{f}(A) = e(p) \quad \text{and} \quad \mathcal{f}(B) = e(q) \quad \text{for } B \in \mathcal{W}^*. \]

We recall first of all that $f$ maps all of $X$ into $E^*$. Hence, for any $A_{\langle x \rangle} \in \mathcal{R}(S(X))$,

\[ \mathcal{f}(A_{\langle x \rangle}) = A_{\langle f(x) \rangle} = e(f(x)) \in e[E^*]. \]

By (1), $\mathcal{R}(S(X))$ is dense in $\mathcal{U}(S(X))$ and since $E^*$ is compact, we get

\[ \mathcal{f}[\mathcal{U}(S(X))] = \mathcal{f}[\text{cl } \mathcal{R}(S(X))] \subseteq \text{cl } \mathcal{f}[\mathcal{R}(S(X))] \subseteq \text{cl } e[E^*] = e[E^*] \]

where $\text{cl}$ denotes closure. This verifies (9). Now we want to show that

(11) \[ A \in \text{cl } \{ A_{\langle x \rangle} : x \in \mathcal{V}^* \}. \]

Let $\mathcal{C}[H(k_1, \langle r_1 \rangle) \cup \cdots \cup H(k_M, \langle r_M \rangle)]$ be any basic open subset of $\mathcal{U}(S(X))$ which contains $A$. Then by (5), there exist finite subfamilies $\{\langle t_{1i}, \langle a_{ij} \rangle \rangle \}_{i=1}^M$ of $A$ such that $k_{i}^{-1}(r_i) \cap U_i = \emptyset$ where

\[ U_i = t_{1i}^{-1}(a_{i1}) \cap \cdots \cap t_{M}^{-1}(a_{M}). \]

By (4), there exists a point $x$ in $\mathcal{V}^* \cap U_1 \cap \cdots \cap U_M$. Thus, $x \notin k_{i}^{-1}(r_i)$, $i = 1, 2, \ldots, M$ from which it follows that $(k_{i}, \langle r_i \rangle) \notin A_{\langle x \rangle}$, $i = 1, 2, \ldots, M$. Therefore,

\[ A_{\langle x \rangle} \in \mathcal{C}[H(k_1, \langle r_1 \rangle) \cup \cdots \cup H(k_M, \langle r_M \rangle)] \]

and this proves (11). Now for any $x \in \mathcal{V}^*$, $f(x) = p$ and we have

\[ \mathcal{f}(A_{\langle x \rangle}) = A_{\langle f(x) \rangle} = e(f(x)) = e(p). \]

This fact, together with (11) implies that $\mathcal{f}(A) = e(p)$ which is the first half of (10).

In much the same way that we verified (11), one can show that if $B \in \mathcal{W}^*$, then $B \in \text{cl } \{ A_{\langle x \rangle} : x \in \mathcal{H} \}$ and since $f(x) = q$ for $x \in \mathcal{H}$, it follows that $\mathcal{f}(A_{\langle x \rangle}) = e(q)$ for each $x$ in $\mathcal{H}$. Therefore, $\mathcal{f}(B) = e(q)$ for each $B \in \mathcal{W}^*$ and this completes the proof of statement (10). In view of the discussion immediately following statement (6), it is a consequence of (9) and (10) that (6) is valid, that is, $\mathcal{U}(S(X))$ is $E$-compact.

Now we are in a position to show that $\mathcal{U}(S(X))$ is $\beta_c X$. Actually, we show that $\mathcal{U}(S(X)) = \beta_c \mathcal{R}(S(X))$ but since the canonical map $e$ maps $X$ homeomorphically onto $\mathcal{R}(S(X))$ we identify the two spaces. In order to conclude that $\mathcal{U}(S(X))$ is $\beta_c \mathcal{R}(S(X))$ it is sufficient, according to Theorem 4.14 of [4, p. 177] to show that $\mathcal{U}(S(X))$ is $E$-compact and also that every continuous function from $\mathcal{R}(S(X))$ into $E$ can be continuously extended to a function which maps $\mathcal{U}(S(X))$ into $E$. We have yet to verify the latter and for this, it will be sufficient to show that any continuous
function $f$ from $\mathcal{R}(S(X))$ into $\varepsilon[E^*]$ has a continuous extension to a function which maps $\mathcal{U}(S(X))$ into $\varepsilon[E^*]$. To get this extension we note that $g = e^{-1} \circ f \circ e$ belongs to $S(X)$ and hence $\hat{f}_e$ is a continuous selfmap of $\mathcal{U}(S(X))$ by (1). For any $A_{(x)} \in \mathcal{R}(S(X))$, we have

$$\hat{f}_e(A_{(x)}) = A_{g_{(x)}} = A_{(g(x))} = e(g(x)) = f(e(x)) = f(A_{(x)}).$$

Thus $\hat{f}_e$ is indeed an extension of $f$ which (since $\varepsilon[E^*]$ is compact and $\mathcal{R}(S(X))$ is dense in $\mathcal{U}(S(X))$) maps $\mathcal{U}(S(X))$ into $\varepsilon[E^*]$. This completes the proof of the Main Theorem.

If, in the Main Theorem, we take $E$ to be the closed unit interval, we immediately get the following result which first appeared in [3, p. 329] as Corollary (2.8).

**Corollary 1.** Suppose $X$ is normal, Hausdorff and contains an arc. Then the $\mathcal{X}$-structure space of $S(X)$ is the Stone-Čech compactification of $X$.

A partition of a space $X$ is any finite collection of mutually disjoint subsets of $X$ which are both closed and open and whose union is all of $X$. A 0-dimensional space here will mean a space whose Lebesgue dimension is zero, that is, one with the property that every open cover has a refinement by a partition of the space.

**Corollary 2.** Let $X$ be a normal 0-dimensional Hausdorff space. Then the $\mathcal{X}$-structure space of $S(X)$ is the Stone-Čech compactification of $X$.

**Proof.** Here again we apply the Main Theorem and in this case we take $E$ to be the two-point discrete space $\mathcal{D}$. The conclusion is immediate if $X$ has only one point so we assume that $X$ has more than one point and, consequently contains a copy of $\mathcal{D}$. To show that $X$ is $\mathcal{D}$-separated, let $H$ and $K$ be two disjoint closed subsets of $X$.

Then $\{CH, CK\}$ is a cover of $X$ and hence has a refinement by a partition $\{V_i\}_{i=1}^n$ of $X$.

Let $W = \bigcup \{V_i : V_i \subseteq CH\}$. Then $W$ is a subset of $X$ which is both closed and open. Furthermore, $H \subseteq CW$ and since $K \subseteq CH$ and $\{V_i\}_{i=1}^n$ is a refinement of $\{CH, CK\}$ it readily follows that $K \subseteq W$. Therefore, if $p$ and $q$ denote the two points of $\mathcal{D}$, the function which maps all of $W$ into $p$ and $CW$ into $q$ is continuous and we conclude that $X$ is $\mathcal{D}$-separated. Then by the Main Theorem, the $\mathcal{D}$-structure space of $S(X)$ is $\beta_{\mathcal{D}}X$, the largest $\mathcal{D}$-compactification of $X$. Now it is well known that the Stone-Čech compactification $\beta X$ of $X$ is the largest among all the compactifications of $X$. So, in order to conclude that $\beta_{\mathcal{D}}X = \beta X$, it is sufficient to observe that $\beta X$ is a $\mathcal{D}$-compactification of $X$. In [2, p. 243], a modified definition of Lebesgue dimension is used. However, the definition there agrees with the usual one for normal spaces. Consequently, it follows from Theorem 16.11 of [2, p. 245] that $\beta X$ is 0-dimensional. Then $\beta X$ is also 0-dimensional in the sense of [4], that is, it has a basis of sets which are both open and closed. But this implies that $\beta X$ is $\mathcal{D}$-compact [4, p. 176] and hence that $\beta X$ is a $\mathcal{D}$-compactification of $X$. 

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