TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 149, June 1970

A GENERALIZATION OF THE SIEGEL-WALFISZ THEOREM

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Abstract. The uniform prime number theorem for primes in arithmetic progressions is generalized to the setting of Hecke L-series.

1. **Introduction.** In the present paper, we will prove a generalization of the uniform prime number theorem of Siegel and Walfisz (Walfisz [13], Prachar [8, p. 144]) to the case of grössencharacters from an algebraic number field. Our Main Theorem was motivated by attempts to prove certain analogues of Artin's conjecture on primitive roots (Artin [1, p. viii]). These analogues of Artin's conjecture constitute an infinite-dimensional generalization of the Tchebotarev density theorem (Tchebotarev [11], Hasse [5, p. 133]), and will be the subject of a subsequent paper.

Throughout the present paper, let K be a normal algebraic number field of finite degree n and discriminant d. Let $\alpha \to \alpha^{(j)}$ $(1 \leqq j \leqq n)$ denote the embeddings of K into the complex field C, ordered so that the first r_1 are real and the jth and $(j+r_2)$th are complex-conjugate. Let

$$\begin{aligned} n_j &= 1, \qquad 1 \leqq j \leqq r_1, \\ &= 2, \qquad r_1+1 \leqq j \leqq r_1+r_2. \end{aligned}$$

For $\alpha \in K^* = K - \{0\}$, let $\alpha \equiv 1 \pmod{^* \mathfrak{A}}$ mean that α is multiplicatively congruent to 1 modulo the K-ideal $\mathfrak{A}$. Let χ be a grössencharacter of K defined modulo its conductor f_χ, such that for $\alpha \in K^*$, $\alpha \equiv 1 \pmod{^* f_\chi}$, we have

$$\chi((\alpha)) = \prod_{j=1}^{r_1+r_2} \left(\frac{\alpha^{(j)}}{|\alpha^{(j)}|}\right)^{m_j} |\alpha^{(j)}|^{in_j\varphi_j},$$

where $m_j \in \boldsymbol{Z}$, $\varphi_j \in R$ and (α) is the K-ideal generated by α.

For positive numbers A and x define

$$\mathscr{B}(A) = \{\chi \mid |\varphi_j| \leqq A,\ 1 \leqq j \leqq r_1+r_2\},$$

$$\pi(x, K, \chi) = \sum_{N\mathfrak{p} \leqq x;\, (\mathfrak{p}, f_\chi)=1} \chi(\mathfrak{p}),$$

Received by the editors December 11, 1968 and, in revised form, September 6, 1969.

AMS Subject Classifications. Primary 1041; Secondary 1250.

Key Words and Phrases. Number field, zeta function, grössencharacter, prime number theorem.

(1) Research partially supported by Air Force Office of Scientific Research Grant No. SAR/F-44620-67.

where $\mathfrak{p}$ runs over prime ideals of K. For all A, $\mathscr{B}(A)$ contains the ideal-class characters of K, since the $\varphi_j=0$ for such a character.

Our goal is the

MAIN THEOREM. *Let $A>0$, $\varepsilon>0$. Then there exists a positive constant $c=c(A, \varepsilon)$, not depending on K on χ, such that for $\chi \in \mathscr{B}(A)$, we have*

$$\pi(x, K, \chi) = E(\chi)li(x)+O(Dx\log^2 x \exp\{-cn(\log x)^{1/2}/D\}), \qquad x\to\infty,$$

where the O-term constant does not depend on K or χ, and

$$E(\chi) = 0, \qquad \chi \neq \chi_0 = \textit{the trivial grössencharacter},$$
$$= 1, \qquad \chi = \chi_0,$$
$$li(x) = \int_2^x \frac{dy}{\log y}.$$
$$D = n^3[|d|N(f_\chi)]^\varepsilon c^{-n}.$$

The proof of the Main Theorem is based on a study of the distribution of the zeros of Hecke L-functions. The methods used are generalizations of those used by Titchmarsh and Paige to study the zeros of Dirichlet's L-functions (Prachar [8, pp. 97–146]). Our results are improvements of some results of Fogels [3] and Mitsui [7]. Fogels, however, only considers ideal class L-functions whose infinite components are trivial. Further, his zero-free regions depend in an undetermined way on n. Some of Mitsui's results are proved for general grössencharacters, but they also depend in an undetermined way on the ground field K.

§2 will define notation and conventions. §3 will study the distribution of the zeros of zeta functions with grössencharacters. §4 will prove the Main Theorem.

The author owes a great debt to many people who have inspired and given helpful advice during the preparation of this paper. Especially, he would like to thank Dr. Oscar Goldman, in whose course on algebraic number theory the author first learned of Artin's conjecture. Thanks also go to Drs. Tamagawa, Furstenberg, Schacher, Randol and Steinberg, who patiently attended my seminar at Yale University and listened to the paper in a nascent state.

Our results were announced in [4].

2. **Conventions; Notation.** Throughout the paper, we will denote by $L(s, \chi)$ the L-function attached to χ:

$$L(s, \chi) = \sum_{(\mathfrak{A}, f_\chi)=1} \frac{\chi(\mathfrak{A})}{N\mathfrak{A}^s}, \qquad \operatorname{Re}(s) > 1,$$

where $\mathfrak{A}$ runs through integral ideals of K. It will sometimes be necessary to view χ as defined modulo some ideal f other than f_χ. Whenever χ is so considered, we associate the L-function

$$L(s, \chi, f) = \sum_{(\mathfrak{A}, f)=1} \frac{\chi(\mathfrak{A})}{N\mathfrak{A}^s}, \qquad \operatorname{Re}(s) > 1.$$

Define

$$L^*(s,\chi,f) = L(s,\chi,f), \qquad \chi \neq \chi_0,$$
$$= s(s-1)L(s,\chi,f), \qquad \chi = \chi_0,$$

and set $L^*(s,\chi)=L^*(s,\chi,f_\chi)$. It is well known that $L(s,\chi)$ has an analytic continuation as a meromorphic function having at most one pole. This pole is a simple pole located at $s=1$ and is present only when χ is trivial. Define

$$R(s,\chi) = B^{-s}L^*(s,\chi)\prod_{j=1}^{r_1+r_2}\Gamma\left(\frac{n_j(s+i\varphi_j)+|m_j|}{2}\right) \tag{2.1}$$

$$\mathscr{B} = [2^{-r_1}(2\pi)^{-n}|d|N(f_\chi)]^{1/2}. \tag{2.2}$$

Then $R(s,\chi)$ is an entire function of finite order and

$$R(s,\chi) = W(\chi)R(1-s,\bar{\chi}), \tag{2.3}$$

where $|W(\chi)|=1$ (Tate [10]).

We will denote by $\zeta_K(s)$ the Dedekind zeta function of K.

We will utilize the Landau O- and o-notations. We will agree that, in any formula containing an O-term or o-term, the constant implied by the O- or o-term will not depend on any quantity explicitly appearing in the formula, unless otherwise stated. The dependence of the O- and o-terms on other parameters will always be explicitly spelled out. We will have occasion to write down constants whose actual value is unimportant, but which are independent of all parameters entering into a discussion. Such absolute constants will be denoted by subscripted, lower-case c's, numbered consecutively in each section. Certain other constants will depend on the choice of A. These will be denoted by subscripted, lower-case a's, numbered consecutively in each paragraph.

3. **The zeros of Hecke L-functions.** In §3, we will find zero-free regions for $L(s,\chi)$ such that the dependence of the shape of the regions on the parameters on K and χ is explicit. The shape of the regions will depend on whether or not χ is real, that is, whether or not $\chi^2=\chi_0$. In subsection 3.1 we will derive certain results which are valid for both real and complex χ. §3.2 will derive the zero-free region when χ is complex. §3.3 will consider χ real.

3.1 *Some general lemmas.*

LEMMA 3.1.1. *Let $A>0$, σ and t be real, $0<\varepsilon\leqq\frac{1}{2}$. Then there exists a constant a_1, independent of K, ε, σ, t and $\chi\in\mathscr{B}(A)$ (but possibly depending on A) such that*

$$|L^*(\sigma+it,\chi,f)| \leqq a_1^n\varepsilon^{-n}N(f/f_\chi)[|d|N(f_\chi)(|t|+2)^n]^{(1+\varepsilon-\sigma)/2} \qquad (\chi\neq\chi_0),$$
$$|L^*(\sigma+it,\chi_0,f)| \leqq a_1^n\varepsilon^{-n}N(f/f_{\chi_0})[|d|N(f_\chi)]^{(1+\varepsilon-\sigma)/2}[|t|+2]^{(n+4)(1+\varepsilon-\sigma)/2}$$

uniformly for $-\varepsilon\leqq\sigma\leqq1+\varepsilon$.

Proof. One can immediately reduce the argument to the case $f=f_\chi$. The proof of the assertions for $\chi\neq\chi_0$ and $\chi=\chi_0$ are similar. Let us give only a sketch of the former. Trivially, we have

$$|L^*(1+\varepsilon+it,\chi)| \leqq (2/\varepsilon)^n. \tag{3.1}$$

By the functional equation (2.1) and Stirling's formula, we see that

$$|L^*(-\varepsilon+it,\chi)| \leqq a_1^n\varepsilon^{-n}A^n|t|^{n(1+\varepsilon)/2}. \tag{3.2}$$

A routine application of the Phragmén-Lindelöf theorem suffices to complete the proof.

LEMMA 3.1.2. *Let $\varepsilon>0$, $\sigma>1$. There exists a constant $c_1=c_1(\varepsilon)$, such that*

$$|\zeta_K'(\sigma)/\zeta_K(\sigma)| \leqq (1+\varepsilon)/(\sigma-1)$$

whenever $\sigma\leqq 1+c_1/n\log(2|d|)$.

Proof. We use the following result of Landau (Titchmarsh [12, p. 49]): If $f(s)$ is analytic in the disc $\{|s-s_0|\leqq r\}$, and if $|f(s)/f(s_0)|\leqq M$ for $|s-s_0|\leqq r$, then

$$\left|\frac{f'(s)}{f(s)}-\sum_\rho(s-\rho)^{-1}\right| \leqq \frac{c_2\log M}{r}, \qquad |s-s_0| \leqq r/4,$$

where c_2 does not depend on f, M, or r, and the sum runs over all zeros ρ of f in the disc $\{|s-s_0|\leqq r/2\}$. For our application, set $r=2$, $s_0=3/2$, $f(s)=(s-1)\zeta_K(s)$. Lemma 3.1.1 implies the existence of an absolute constant c_3 such that $|f(s)| \leqq c_3^n|d|=M$ in $\{|s-s_0|\leqq 2\}$. Then the above-cited result implies that

$$\left|\frac{\zeta_K'(\sigma)}{\zeta_K(\frac{1}{2})}+(\sigma-1)^{-1}-\sum_\rho(\sigma-\rho)^{-1}\right| \leqq c_4n\log(2|d|), \qquad |\sigma-3/2| \leqq \tfrac{1}{2}.$$

Therefore,

$$\frac{\zeta_K'(\sigma)}{\zeta_K(\sigma)} = -(\sigma-1)^{-1}+\operatorname{Re}\left[\sum_\rho(\sigma-\rho)^{-1}\right]+c_4\theta n\log(2|d|),$$

where $\theta=\theta(\sigma)$ satisfies $|\theta|\leqq 1$. For $\sigma>1$, we have $\zeta_K'(\sigma)/\zeta_K(\sigma)\leqq 0$. Also $\operatorname{Re}(\sigma-\rho)^{-1} \geqq 0$ since $\sigma\geqq 1$ and $\zeta_K(\sigma)\neq 0$ in this region. Therefore,

$$|\zeta_K'(\sigma)/\zeta_K(\sigma)| \leqq (\sigma-1)^{-1}+c_5n\log(2|d|).$$

Choosing $c_1<\varepsilon/c_5$, we see that $c_5n\log(2|d|)<\varepsilon/(\sigma-1)$ provided that $\sigma\leqq 1 +c_1/n\log(2|d|)$. This proves the lemma.

3.2 *L-functions with complex grössencharacters.* Throughout §3.2, let χ be a primitive, complex grössencharacter of K. Our object is to prove

THEOREM 3.2.1. *There exists a constant a_1 such that $L(\sigma+it,\chi)\neq 0$ whenever $\chi\in\mathscr{B}(A)$ and*

$$\sigma \geqq 1-a_1/\eta(t) \geqq 3/4,$$

where

$$\eta(t) = \eta(t,K,\chi) = n\log\{|d|N(f_\chi)(|t|+2)^n\}.$$

Proof. Let $\rho=\beta+it_1$ be a zero of $L(s,\chi)$ such that $7/8<\beta<1$. Since $L(s,\bar{\chi})^-=L(\bar{s},\chi)$, it suffices to consider the case $t_1\geqq 0$. Let $\beta=1-b/\eta(t_1)$, $b>0$. It suffices to show that there exists a constant a_2 such that $b>a_2$.

Set $\sigma_0=1+a/2\eta(t_1)$, $s_0=\sigma_0+it_1$, where $0<a\leqq\frac{1}{2}$ will be chosen later, subject only to the condition $a\leqq a_3$. It is clear that $L(s,\chi)\neq 0$ in the semidisc

$$\{|s-s_0| \leqq r = \tfrac{1}{2}, \operatorname{Re}(s-s_0) \geqq 0\}.$$

If $a\leqq a_3$, then ρ is contained in the disc $\{|s-s_0|\leqq r/2=\frac{1}{4}\}$, since $\beta\geqq 7/8$. A trivial computation shows that

$$|L(s_0,\chi)^{-1}| \leqq c_1[4a^{-1}\eta(t_1)]^n.$$

By Lemma 3.1.1, we see that for $s=\sigma+it$, $|s-s_0|\leqq\frac{1}{2}$, we have

$$|L(s,\chi)| \leqq a_4^n|d|N(f_\chi)(t_1+2)^{n/2}.$$

Thus, for $|s-s_0|\leqq\frac{1}{2}$,

$$|L(s,\chi)/L(s_0,\chi)| \leqq a_5^n|d|N(f_\chi)(t_1+2)^n a^{-n}\eta(t_1)^n = M,$$

and

$$\log M \leqq a_6\{\eta(t_1)+n\log(a^{-1})+n\log n\}.$$

From [8, pp. 384–85], we deduce that

$$\text{(3.3)}\qquad \begin{aligned}\operatorname{Re}\left\{\frac{L'(s_0,\chi)}{L(s_0,\chi)}\right\} &\geqq (\sigma_0-\beta)^{-1}-a_7\{\eta(t_1)+n\log(a^{-1})+n\log n\}\\ &= -a_7\{n\log(a^{-1})+n\log n\}+\{(a/2+b)^{-1}-a_7\}\eta(t_1).\end{aligned}$$

Applying identical reasoning to $L(s,\chi^2,f_\chi)$, with s_0 now equal to σ_0+2it_1, and applying [8, Theorem 4.5, p. 384], we see that

$$\text{(3.4)}\qquad \operatorname{Re}\left\{\frac{L'(\sigma_0+2it_1,\chi^2,f_\chi)}{L(\sigma_0+2it_1,\chi^2,f_\chi)}\right\} \geqq -a_8\{n^2+n\log(a^{-1})+\eta(t_1)\}.$$

Let $\varepsilon>0$ be given. By Lemma 3.1.2, there exists $c_2=c_2(\varepsilon)$ such that

$$|\zeta_K'(0_0)/\zeta_K(\sigma_0)| \leqq (1+\varepsilon)/(\sigma_0-1)$$

for $a\leqq c_2$. Throughout the remainder of the proof, assume that $a\leqq c_2$. Then

$$\text{(3.5)}\qquad \operatorname{Re}\{\zeta_K'(\sigma_0)/\zeta_K(\sigma_0)\} \geqq -2(1+\varepsilon)\eta(t_1)/a.$$

From equations (3.3)–(3.5) and Hadamard's classical lemma [14, p. 125], we derive

$$\text{(3.6)}\qquad \left\{-\frac{6(1+\varepsilon)}{a}+\frac{4}{a/2+b}-a_9\right\}\eta(t_1)-a_9\{n\log(a^{-1})+n^2\} \leqq 0.$$

Choose $\varepsilon<1/3$, so that

$$-6(1+\varepsilon)/a+8/a > 0.$$

Then, for $a \leqq a_{10} = a_{10}(A, \varepsilon)$,

$$\{-6(1+\varepsilon)/a + 8/a - a_9\}n^2 \log 2 - a_9\{n \log(a^{-1}) + n^2\} > 0.$$

Therefore, for $b \leqq a_{11}$,

$$\{-6(1+\varepsilon)/a + 4/(a/2+b)\} - a_9\{n \log(a^{-1}) + n^2\} > 0,$$

which contradicts (3.6). Therefore, $b > a_{11}$, which proves the theorem.

3.3 *L-functions with real grössencharacters.* Throughout §3.3, let χ be a real, primitive grössencharacter of K. In order to find uniform, zero-free regions for $L(s, \chi)$, it will be necessary to consider separately the zeros of $L(s, \chi)$ on and off the real axis. Our main results in this paragraph are:

THEOREM 3.3.1. *There exists a constant $a_1 = a_1(A)$ such that $L(\sigma + it, \chi) \neq 0$ whenever $\chi \in \mathscr{B}(A)$ and*

$$\sigma \geqq 1 - a_1/\eta(t) \geqq 3/4, \qquad t \neq 0,$$

where

$$\eta(t) = n \log\{|d| N(f_\chi)(|t|+2)^n\}.$$

THEOREM 3.3.2. *Let K/Q be normal and let $\varepsilon > 0$ be given. Let $\beta = 1 - \delta$ be a real zero of $L(s, \chi)$. Then there exists $a_2 = a_2(\varepsilon, A)$, such that*

$$\delta > a_2^n/[|d| N(f_\chi)]^\varepsilon, \qquad \beta \geqq 3/4.$$

Together, Theorems 3.2.1, 3.3.1 and 3.3.2 imply

THEOREM 3.3.3. *Let K/Q be normal and let χ be an arbitrary primitive grössencharacter of K. Let $\varepsilon > 0$ be given. Then there exists a constant $a_3 = a_3(A, \varepsilon)$ such that $L(\sigma + it, \chi) \neq 0$ whenever $\chi \in \mathscr{B}(A)$ and*

$$\sigma \geqq 1 - a_3^n/[|d| N(f_\chi)]^\varepsilon \eta(t) \geqq 3/4.$$

Our plan is to prove Theorems 3.3.1 and 3.3.2 first in the special case $\chi = \chi_0$. In this case, $L(s, \chi) = \zeta_K(s)$. Let $\rho = \beta + it_1$ be a nontrivial zero of $L(s, \chi)$. As in the proof of Theorem 3.2.1, set

$$\sigma_0 = 1 + a/2\eta(t_1), \qquad s_0 = \sigma_0 + it_1,$$

where a is a positive constant to be chosen later. Define $F(s) = \zeta_K(s)/\zeta(s)$. A well-known theorem of Brauer [6, p. 135] asserts that $F(s)$ is entire. We require two lemmas:

LEMMA 3.3.4. *Let $\nu(r, t\colon K)$ denote the number of zeros of $\zeta_K(s)$ in the disc $\{s \mid |s-1-it| \leqq r\}$. Then, for $n/\eta(t) \leqq r \leqq \frac{1}{2}$, we have $\nu(r, t\colon K) \leqq c_1 r \eta(t)$.*

Proof. Using the same reasoning as used in the proof of Lemma 3.1.2, we deduce that

$$\left| \frac{\zeta_K'}{\zeta_K}(s) + (s-1)^{-1} - \sum_\rho (s-\rho)^{-1} \right| \leqq c_2 \eta(t), \qquad |s-1-it| \leqq \tfrac{1}{2},$$

where the sum is over the zeros ρ of $\zeta_K(s)$ in the disc $\{s \mid |s-1-it| \leqq 1\}$. Thus,

$$\begin{aligned}\frac{2n}{r} &\geqq -n\frac{\zeta'(1+r)}{\zeta(1+r)} \\ &\geqq \left|\frac{\zeta_K'(1+r+it)}{\zeta_K(1+r+it)}\right| \\ &\geqq \operatorname{Re}\left\{\frac{\zeta_K'(1+r+it)}{\zeta_K(1+r+it)}\right\} \\ &\geqq -\operatorname{Re}(z-1)^{-1}+\operatorname{Re}\sum_\rho (z-\rho)^{-1}-c_2\eta(t),\end{aligned}$$

where $z=1+r+it$. But this latter expression is at least

$$\operatorname{Re}\sum_\rho (z-\rho)^{-1}-c_3\eta(t) \geqq (1/4r)\nu(r, t: K)-c_3\eta(t),$$

since $n/\eta(t) \leqq r \leqq \frac{1}{2}$. Thus the lemma follows.

LEMMA 3.3.5. *Let c be an absolute constant such that $0<|t_1|<cn/\eta(t_1)$. Given $\varepsilon>0$, there exists $a_0=a_0(\varepsilon, c)$ such that*

$$\operatorname{Re}\{F'(s_0)/F(s_0)\} \leqq (1+\varepsilon)/(\sigma_0-1)$$

for all $a \leqq a_0$, provided that c is sufficiently small.

Proof. Assume that the assertion is false. Then there exists $\varepsilon_0>0$ and a sequence of ordered pairs $(a_m, K_m)_{1\leqq m<\infty}$, with $a_m>0$, $a_m \to 0$ as $m\to\infty$, K_m a normal extension of Q such that

$$\operatorname{Re}\{F_{K_m}'(s_m)/F_{K_m}(s_m)\} \geqq (1+\varepsilon_0)/(\sigma_m-1),$$

where $s_m=\sigma_m+it_1$, $\sigma_m=1+a_m/2\eta(t_1)$, $F_K(s)=\zeta_K(s)/\zeta(s)$. Apply the lemma of Landau cited in the proof of Lemma 3.1.2 to the entire function $F_{K_m}(s)$ in the disc of radius 1, centered at $1+it_1$. In the disc $|s-1-it_1| \leqq \frac{1}{4}$,

$$\left|\frac{\zeta_{K_m}'(s)}{\zeta_{K_m}(s)}-\frac{\zeta'(s)}{\zeta(s)}-\sum_\rho (s-\rho)^{-1}\right| \leqq c_4\eta(t_1),$$

where the sum is over all zeros ρ of $F_{K_m}(s)$ in $|s-1-it_1| \leqq \frac{1}{2}$. Without loss of generality, assume that $\sigma_m \leqq 5/4$ for all m. Then

$$\operatorname{Re}\left\{\frac{\zeta_{K_m}'(s_m)}{\zeta_{K_m}(s_m)}\right\} = \operatorname{Re}\left\{\frac{\zeta'(s_m)}{\zeta(s_m)}+\sum_\rho (s_m-\rho)^{-1}\right\}+\theta c_4\eta(t_1),$$

where $|\theta| \leqq 1$. Since $a_m \to 0$, for all sufficiently large m we have $|\theta c_4\eta(t_1)| \leqq \varepsilon_0/2(\sigma_m-1)$. Thus, for all sufficiently large m,

$$\operatorname{Re}\sum_\rho (s_m-\rho)^{-1} \geqq (1+\varepsilon_0/2)/(\sigma_m-1).$$

From the proof of Lemma 3.1.2,

$$\frac{\zeta'_{K_m}(\sigma_m)}{\zeta_{K_m}(\sigma_m)} = -(\sigma_m-1)^{-1}+\mathrm{Re}\sum_{\tau}{}' (\sigma_m-\tau)^{-1}+\theta' c_5\eta(t_1),$$

where $|\theta'|\leqq 1$ and $\sum'$ is the sum over all zeros τ of $\zeta_{K_m}(s)$ in the disc $\{s \mid |s-\sigma_m|\leqq 1\}$. But the left-hand member of this last equation is nonpositive, so that for m sufficiently large,

$$\mathrm{Re}\sum_{\tau}{}' (\sigma_m-\tau)^{-1} \leqq (1+\varepsilon_0/4)/(\sigma_m-1).$$

Therefore,

$$\begin{aligned}|\sigma_m-s_m|\sum_{\rho}|s_m-\rho|^{-1}|\sigma_m-\rho|^{-1} &\geqq \mathrm{Re}\sum_{\rho}[(s_m-\rho)^{-1}-(\sigma_m-\rho)^{-1}]\\ &\geqq \varepsilon_0/2(\sigma_m-1).\end{aligned}$$

But since $|t_1|\leqq cn/\eta(t_1)$,

$$\begin{aligned}\sum_{\rho}|s_m-\rho|^{-2} &\geqq \varepsilon_0/2(\sigma_m-1)|t_1|\\ &\geqq c_6 n\eta(t_1)^2/ca_m,\end{aligned} \tag{3.7}$$

where $c_6=c_6(\varepsilon_0)$. On the other hand,

$$\sum_{\rho}|s_m-\rho|^{-2} \leqq \sum_{\rho}{}^{(1)}|s_m-\rho|^{-2}+\sum_{\rho}{}^{(2)}|s_m-\rho|^{-2}, \tag{3.8}$$

where $\sum^{(1)}$ and $\sum^{(2)}$ are, respectively, sums over the zeros ρ of $\zeta_{K_m}(s)$ in the disc $\{z \mid |1+it_1-z|\leqq n/\eta(t_1)\}$ and the annulus $\{z \mid n/\eta(t_1)\leqq|1+it_1-z|\leqq 1/2\}$. But the right-hand member of (3.8) is at most

$$\begin{aligned}&\sum_{\rho}{}^{(1)}|s_m-1-it_1|^{-2}+\sum_{\rho}{}^{(2)}|1+it_1-\rho|^{-2}\\ &\qquad\leqq \frac{4\eta(t_1)^2}{a_m^2}\nu(n/\eta(t_1),t_1\colon K_m)+\int_{n/\eta(t_1)}^{1/2}\frac{\nu(r,t_1\colon K_m)}{r^3}\,dr\\ &\qquad\leqq 8n\eta(t_1)^2/a_m^2,\end{aligned} \tag{3.9}$$

for m sufficiently large. Therefore, by (3.7) and (3.8), we derive $c_6/ca_m\leqq 8/a_m^2$, which is a contradiction if c is sufficiently small. This proves the lemma.

Proof of Theorem 3.3.1 in case $\chi=\chi_0$. In this case, $L(s,\chi)=\zeta_K(s)$. In the portion of the critical strip defined by $|t|\geqq 1$, we can apply the same reasoning as was used in the proof of Theorem 3.2.1, thereby proving the existence of a zero-free region of the required shape. Thus, we need consider a zero $\rho=\beta+it_1$ of $\zeta_K(s)$ such that $7/8<\beta<1$, $0<|t_1|\leqq 1$. Using Lemma 3.1.1 and the fact that $\zeta(s)\neq 0$ in the strip $\{\sigma+it \mid \sigma\geqq 0, 0\leqq t\leqq 1\}$, one can deduce the following estimate:

$$|F(s)/F(s_0)| \leqq c_1^n a^{-(n+1)}\eta(t_1)^n|d|^3(|t_1|+2)^{3n}, \qquad |s-s_0| \leqq \tfrac{1}{2}. \tag{3.10}$$

Reasoning as in the proof of Theorem 3.2.1, we derive

$$\text{(3.11)} \qquad \operatorname{Re}\left\{\frac{F'(\sigma_0+it_1)}{F(\sigma_0+it_1)}\right\} \geqq -c_2[n\log(a^{-1})+n\log n]+[(a/2+b)^{-1}-c_2]\eta(t_1),$$

$$\text{(3.12)} \qquad \operatorname{Re}\left\{\frac{F'(\sigma_0+2it_1)}{F(\sigma_0+2it_1)}\right\} \geqq -c_3[n^2+n\log(a^{-1})+\eta(t_1)].$$

Let c be an absolute constant to be fixed later. We distinguish two cases.

Case 1. $|t_1| \geqq c/\eta(t_1)$.

There exists a constant c_4 such that

$$\left|\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right| \leqq \frac{c_4}{|\sigma+it-1|}, \qquad 1 \leqq \sigma \leqq 2, \quad 0 < |t| \leqq 1.$$

Therefore, if $|t_1| < c/\eta(t_1)$,

$$|\zeta'(s_0)/\zeta(s_0)| \leqq c_5\eta(t_1).$$

Thus, from (3.10) and (3.11), we see that

$$\operatorname{Re}\left\{\frac{\zeta_K'(\sigma_0+it_1)}{\zeta_K(\sigma_0+it_1)}\right\} \geqq -c_6[n\log(a^{-1})+n\log n]+[(a/2+b)^{-1}-c_6]\eta(t_1),$$

$$\operatorname{Re}\left\{\frac{\zeta_K'(\sigma_0+2it_1)}{\zeta_K(\sigma_0+2it_1)}\right\} \geqq -c_7[n^2+n\log(a^{-1})]-c_7\eta(t_1).$$

From these two equations, we proceed exactly as in the proof of Theorem 3.2.1 to arrive at our result.

Case 2. $0<|t_1|<c/\eta(t_1)$.

Since χ is real, $\rho'=\beta-it_1$ is also a zero of $\zeta_K(s)$. Applying [8, Theorem 4.6, p. 385] to equation (3.10), we derive

$$\operatorname{Re}\left\{\frac{F'(s_0)}{F(s_0)}\right\} \geqq -c_8[n\log n+n\log(a^{-1})]+[(a/2+b)^{-1}-c_8]\eta(t_1)+\frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4t_1^2}.$$

Let $\varepsilon>0$ be arbitrary. By Lemma 3.3.5, we can choose $a_0=a_0(\varepsilon)$ so that

$$\operatorname{Re}\left\{\frac{F'(s_0)}{F(s_0)}\right\} \leqq \frac{1+\varepsilon}{\sigma_0-1} = \frac{2(1+\varepsilon)\eta(t_1)}{a}$$

for all $a \leqq a_0$. Thus, for $a \leqq a_0$,

$$\frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4t_1^2} \leqq c_9[n\log n+n\log(a^{-1})]+\left[\frac{2(1+\varepsilon)}{a}-(a/2-b)^{-1}+c_9\right]\eta(t_1).$$

But $n\log n \leqq c_{10}\eta(t_1)$, $n\log(a^{-1}) \leqq c_{11}\log(a^{-1})\eta(t_1)$, so that

$$\frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4t_1^2} \leqq \left[\frac{2(1+\varepsilon)}{a}-\frac{1}{a/2+b}+c_{11}\log(a^{-1})+c_{10}\right]\eta(t_1).$$

Choose $a=a(\varepsilon)$ so that $c_{10}\leqq\varepsilon/2a$, $c_{11}\log(a^{-1})\leqq\varepsilon/2a$. Then a simple calculation shows that

$$a/2+b \leqq \left[\frac{2+3\varepsilon}{a}-\frac{1}{a/2+b}\right][(a/2+b)^2+4c^2].$$

Set $\varepsilon=1/16$. If $b<a/16$, then the last inequality yields $a/2+b\leqq a/4$, which is a contradiction. Thus, $b\geqq a/16$, and ρ lies outside the region defined by

$$\sigma \geqq 1-a/32\eta(t) \geqq 3/4, \qquad t\neq 0.$$

Thus, the particular case of Theorem 3.3.1.

LEMMA 3.3.6 [6, p. 134]. *Let K/Q be normal and let $\varepsilon>0$ be arbitrary. Then there exists a constant $c_1=c_1(\varepsilon)$ such that* Re $s_{s=1}\zeta_K(s)\geqq c_1^n|d|^{-\varepsilon}$.

Proof of Theorem 3.3.2 in the special case $\chi=\chi_0$. Let $\varepsilon>0$ be arbitrary. Set $G(s)=(s-1)\zeta_K(s)$. By Lemma 3.1.1, there exists a constant c_1 such that

$$|G(s)| \leqq c_1^n\varepsilon^{-n}|d|^{\varepsilon/2}, \qquad |s-1| \leqq \varepsilon/2.$$

By Lemma 3.3.6, $|G(1)|\geqq c_2^n|d|^{-\varepsilon/2}$, $c_2=c_2(\varepsilon)$. Now for $|s-1|\leqq\varepsilon/4$, we have

$$|G'(s)| = \left|\frac{1}{2\pi i}\int_{|w-s|=\varepsilon/4}\frac{G(w)}{(w-s)^2}\,dw\right| \leqq c_3^n|d|^{\varepsilon/2}, \qquad c_3 = c_3(\varepsilon).$$

By the mean value theorem,

$$c_2^n|d|^{-\varepsilon/2} \leqq G(1)-G(\beta) = \delta G'(\sigma), \qquad \beta<\sigma<1, \quad \delta = 1-\beta.$$

We may assume $\beta\geqq 1-\varepsilon/4$, in which case,

$$\delta \geqq c_2^n|d|^{-\varepsilon/2}/G'(\sigma) \geqq c_4^n|d|^{-\varepsilon}, \qquad c_4=c_4(\varepsilon).$$

This proves Theorem 3.3.2 in the special case.

Proof of Theorems 3.3.1 and 3.3.2 in the general case. Let $\chi\neq\chi_0$ be a real grössencharacter of K. Let H be the group of all ideals of K which are prime to $f=f_\chi$. If $H_0\subset H$ is the subgroup consisting of all ideals on which χ is trivial, then $[H:H_0]=2$ and H_0 is a ray class group modulo f. Let L be the class field corresponding to H_0. Then L/K is a quadratic extension, so that L has absolute degree $N=2n$ and absolute discriminant $D=N(f)d^2$. Moreover, $\zeta_L(s)=\zeta_K(s)L(s,\chi)$. From the special case of Theorem 3.3.1 proven above, we deduce that $\zeta_L(\sigma+it)\neq 0$ for

$$\sigma \geqq 1-\frac{c_1}{N^2\log\{|D|(|t|+2)^N\}} \geqq \frac{3}{4}, \qquad t\neq 0.$$

Thus, $L(\sigma+it,\chi)\neq 0$ for

$$\sigma \geqq 1-\frac{c_2}{n^2\log\{|d|N(f)(|t|+2)^n\}} \geqq \frac{3}{4}, \qquad t\neq 0.$$

This is the general case of Theorem 3.3.1. The general case of Theorem 3.3.2 is proved similarly.

4. **The generalized Siegel-Walfisz theorem.** In the present section, we will utilize the uniform zero-free regions of §3 to prove the generalized Siegel-Walfisz theorem. Throughout §4, let $0<\varepsilon\leqq 1$ be fixed but arbitrary. Set

$$\eta(t) = \eta(t, K, \chi, \varepsilon) = a^{-n}n[|d|N(f_\chi)]^\varepsilon \log[|d|N(f_\chi)(|t|+2)^n],$$

where $a=a(A, \varepsilon)$ is the constant of Theorem 3.3.3, which we suppose to be at most 1.

LEMMA 4.1. *Let K/Q be normal. Then there exist constants $c_1=c_1(\varepsilon)$ and c_2 such that*

$$\left|\frac{L'(\sigma+it, \chi)}{L(\sigma+it, \chi)}-\frac{E(\chi)}{\sigma+it-1}\right| \leqq c_1 n\eta(t)$$

in the region $1-c_2/\eta(t)\leqq\sigma\leqq 2$.

Proof. If $1+c_2/\eta(t)\leqq\sigma\leqq 2$, we have

$$\left|\frac{L'(\sigma+it, \chi)}{L(\sigma+it, \chi)}\right| \leqq -\frac{\zeta_K'(\sigma)}{\zeta_K(\sigma)} \leqq -n\frac{\zeta'(\sigma)}{\zeta(\sigma)} \leqq c_3 n\eta(t),$$
$$|\sigma+it-1|^{-1} \leqq \eta(t).$$

Thus, the lemma is true if $1+c_2/\eta(t)\leqq\sigma\leqq 2$ for any c_2. In order to prove the desired estimate for some region of the form $1-c/\eta(t)\leqq\sigma\leqq 1+c/\eta(t)$, we will apply [8, Theorem 4.6, p. 385] with $F(s)=L^*(s,\chi)$, $r=\frac{1}{2}$, $s_0=\sigma_0+it_0$, $\sigma_0=1+c/\eta(t_0)$, with t_0 arbitrary and $0<c\leqq\frac{1}{2}$ chosen as follows: Let $c<1/16$ be chosen so that

$$|L^{*\prime}(s_0, \chi)/L^*(s_0, \chi)| \leqq c_4/(\sigma_0-1)$$

for $1<\sigma_0\leqq 1+a/\eta(t)$. This is possible by Lemma 3.1.2. Also, choose c so small that $L^*(s,\chi)\neq 0$ for $|s-s_0|\leqq r$, $\sigma\geqq 1-3c/\eta(t_0)$. Set $r_1=2c/\eta(t_0)<r/4$. Then $F(s)\neq 0$ for $|s-s_0|\leqq r/2$, $\mathrm{Re}\,(s-s_0)\geqq -2r_1$. By Lemma 3.1.1, for $|s-s_0|\leqq r$, $|F(s)/F(s_0)|\leqq M$, where

$$\log M \leqq c_5 n\eta(t_0). \tag{4.1}$$

Moreover,

$$\left|\frac{F'(s_0)}{F(s_0)}\right| \leqq \frac{c_6}{\sigma_0-1}+\frac{1}{|s_0-1|} \leqq \frac{c_7}{\sigma_0-1} \leqq c_8\eta(t_0).$$

All of the hypotheses of [8, Theorem 4.6, p. 385] are thus verified, and we derive

$$|F'(s)/F(s)| \leqq c_9 n\eta(t_0)$$

for $|s-s_0|\leqq r_1$. Thus, if

$$1-c/\eta(t_0) \leqq \sigma \leqq 1+c/\eta(t_0)$$

then

$$\left|\frac{L'(\sigma+it_0,\chi)}{L(\sigma+it_0,\chi)}+\frac{E(\chi)}{s+it_0-1}\right| \leqq c_9 n\eta(t_0).$$

Proof of the Main Theorem. Set

$$\psi(x, K, \chi) = \sum_{N\varphi m \leqq x} \chi(\varphi) \log N\varphi.$$

Applying [8, Theorem 3.1, p. 376] with $f(s)=-L'(s,\chi)/L(s,\chi)$, $\operatorname{Re}(s)>1$, $b=1+1/\log x$, $T>0$, $x=N+\frac{1}{2}$, $N=a$ positive integer, we see that

$$\psi(x, K, \chi) = \left(\frac{1}{2\pi i}\right)\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}\,ds + O\left(\frac{nx\log^2 x}{T}\right),$$

with O-term constant independent of K and χ. Let $c_1=c_1(\varepsilon)$ be a constant chosen so that $L(s,\chi)\neq 0$ and the conclusion of Lemma 4.1 both hold for $s=\sigma+it$ in the region $\sigma\geqq 1-c_1/\eta(t)\geqq 3/4$. Let P_1, P_2, Q_1, Q_2 be the points $1+1/\log x\pm iT$, $1-c_1/\eta(T)\pm iT$, respectively; let C_i $(i=1, 2, 4)$ be the directed line segments from P_1 to P_2, P_2 to Q_1, Q_2 to P_1, respectively; let C_3 be the curve $1-c_1/\eta(t)+it$, $-T\leqq t\leqq T$, running from Q_2 to Q_1. Applying Cauchy's theorem to the region bounded by the C_i $(i=1,\ldots,4)$, we see that

$$\psi(x, K, \chi) = E(\chi)x-\left(\frac{1}{2\pi i}\right)\int_{C_2-C_3+C_4} f(s)\frac{x^s}{s}\,ds + O\left(\frac{nx\log^2 x}{T}\right). \tag{4.2}$$

By Lemma 4.1,

$$\begin{aligned}\left|\int_{C_2+C_4} f(s)\frac{x^s}{s}\,ds\right| &\leqq \frac{c_2 n\eta(T)}{T}\int_{1-c_1/\eta(T)}^{1+1/\log x} x^\sigma\,d\sigma\\ &= O\left(\frac{xn\eta(T)}{T}\right).\end{aligned} \tag{4.3}$$

For s on C_3, $\sigma\geqq 3/4$, and $|s-1|^{-1}\leqq\eta(T)$, so that

$$\begin{aligned}\left|\int_{C3} f(s)\,\frac{x^s}{s}\,ds\right| &\leqq c_3 n\int_0^T \eta(t)x^{1-c_1/\eta(t)}\frac{dt}{t+3/4}+c_4\eta(T)\int_0^T x^{1-c_1/\eta(t)}\frac{dt}{t+3/4}\\ &\leqq c_5 n\eta(T)x^{1-c_1/\eta(T)}\log T.\end{aligned} \tag{4.4}$$

Upon setting $T+2=\exp((\log x)^{1/2})$, and possibly increasing the constant $a=a(\varepsilon, A)$ in the definition of $\eta(t)$, we derive

$$\eta(T) \leqq a^{-n}n^2[|d|N(f_\chi)]^\varepsilon(\log x)^{1/2}. \tag{4.5}$$

Therefore, from (4.2)–(4.5),

$$\psi(x, K, \chi) = E(\chi)x+O(Dx\log^2 x\exp\{-n(\log x)^{1/2}/D\}).$$

A standard argument using partial summation to express $\pi(x, K, \chi)$ in terms of $\psi(x, K, \chi)$ now suffices to prove the Main Theorem.

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