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EXACT DYNAMIC SYSTEMS ARE TREE-LIKE AND VICE VERSA[1]

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Abstract. This paper gives an analytic characterization of those dynamic systems whose graph of trajectories is a tree.

Let X be a C^∞ dynamical system on a connected manifold $\mathcal{M}$ of dimension $n \geqq 2$. Let A be the algebra of real- (or complex-) valued C^∞ functions on $\mathcal{M}$. The usual differentiation d maps A to the A-module M of C^∞ 1-forms. We shall regard X as the usual first order differential operator of the type $X = \sum a^i(x)\partial/\partial x^i$ in local coordinates. If w is a 1-form, denote by $\langle X \mid w\rangle$ the contraction such that, if $w = \sum b_i(x)\, dx^i$, then $\langle X \mid w\rangle = \sum a^i(x)b_i(x)$ in local coordinates. Denote by M_0 the submodule of M that consists of all $w \in M$, such that $\langle X \mid w\rangle = 0$ or, equivalently, such that the integral $\int w$ vanishes along any arc of trajectory of X. Use d_X to denote the derivation which is the composition

$$A \xrightarrow{d} M \longrightarrow M/M_0 = M_X.$$

We propose to study the dynamical system X through the algebra A equipped with the derivation d_X.

From the point of view of an algebraic homotopy theory suggested in [1] and [2], the "simplest" kind of derivations d_X should be those such that the sequence

$$0 \longrightarrow K \longrightarrow A \xrightarrow{d_X} M_X \longrightarrow 0$$

is exact, where K denotes the field of real- (or complex-) numbers. The precise meaning of the exact sequence is that, given any C^∞ 1-form w, the first order partial differential equation $Xu = \langle X \mid w\rangle$ has a unique C^∞-solution u on the manifold $\mathcal{M}$ up to a difference of a constant. We weaken this condition by allowing C^0-solutions and call a dynamical system satisfying the weakened condition an exact dynamical system. The precise definition of the derivative Xu will be explained in the sequel. It can be shown to be the derivative in the sense of distribution.

Throughout the paper, we shall assume that X has only a finite number of critical

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points, each of which is hyperbolic, and also that each trajectory $\gamma(t)$ of X is defined for all t, $-\infty < t < \infty$. (A critical point p is hyperbolic if each of the eigenvalues of the linear part of X at p has nonzero real part.)

Our main result is the next geometrical characterization of exact dynamical systems.

MAIN THEOREM. *A dynamical system X is exact if and only if the following conditions hold for either X or $-X$*: (a) *the manifold $\mathcal{M}$ is the union of the stable manifolds of the critical points*, (b) *each of the stable manifolds is embedded in $\mathcal{M}$*, (c) *the graph consisting of critical points and separatrices is a tree.*

Note that, if X is exact, then the manifold is necessarily noncompact. Another corollary of the theorem is that each of the stable manifolds must be of dimension either n or $n-1$.

1. Denote by $T(t)$ the group of diffeomorphisms of $\mathcal{M}$ generated by X. In other words, if $p \in \mathcal{M}$, then $\gamma(t) = T(t)p$ is the trajectory of X such that $\gamma(0) = p$. For any real- (or complex-) valued function u on $\mathcal{M}$, define Xu to be the function on $\mathcal{M}$ such that

$$Xu(p) = (d/dt)_{t=0}u(T(t)p).$$

In proving the necessity part of the main theorem, we shall capitalize on the idea of X-positive 1-forms. A C^∞ 1-form w is said to be X-positive about a point $p \in \mathcal{M}$, if $\langle X \mid w \rangle \geqq 0$ everywhere in $\mathcal{M}$ and >0 at p. If p is not a critical point, we can always construct an X-positive 1-form about p having support in an arbitrarily small neighborhood of p.

Observe that, if w is an X-positive 1-form then, the integral $\int w$ is nonnegative along any arc of trajectory of X.

We shall more than once make use of the observation that, if all trajectories (of X) passing through a given neighborhood of p have both empty α- and ω-limits, then there exists a nonconstant C^∞ first integral, i.e., a nonconstant C^∞ function f on $\mathcal{M}$ such that $Xf = 0$.

A separatrix will be meant to be a trajectory, whose α- and ω-limits are both critical points. The 1-dimensional complex consisting of critical points and separatrices will be called the separatrix graph (of X).

We shall not distinguish between a trajectory $\gamma: R \to \mathcal{M}$ and the set $\gamma(R)$, when there is no ambiguity.

The proof of the necessity part of the main theorem will be divided into several propositions, some of which can be stated under a weaker hypothesis.

2. PROPOSITION. *If X is exact, then the ω- (α-) limit of each trajectory either consists of a critical point or is empty.*

Proof. Suppose that p is a noncritical point of the ω-limit of a trajectory γ.

Let w be an X-positive 1-form about p. There exists $\varepsilon > 0$ such that, if the set $\gamma([a, b])$, $-\infty < a < b < \infty$, lies sufficiently close to the point p, then

$$\int_{\gamma|[a,b]} w \geqq \varepsilon(b-a).$$

Since p belongs to the ω-limit of γ, we conclude that

$$\lim_{t\to\infty} \int_{\gamma|[0,t]} w = \infty.$$

On the other hand, since X is exact, there exists a continuous function u such that $Xu = \langle X \mid w\rangle$. This implies that

$$u(\gamma(t)) = u(\gamma(0)) + \int_{\gamma|[0,t]} w \to \infty$$

as $t \to \infty$, which is absurd.

Investigating further in this direction, let us define the notion of ω- (α-) related limit.

DEFINITION. A point $q \in \mathcal{M}$ is an ω- (α-) related limit point of a trajectory γ if, given any neighborhood of $\gamma(0)$, there exists a trajectory which passes through a point of the neighborhood and whose ω- (α-) limit contains q.

PROPOSITION. *If X is exact, and if a trajectory γ has empty ω- (α-) limit, then γ has no ω- (α-) related limit point.*

Proof. Suppose that q is a critical point which is an ω- (α-) related limit point of γ. Then there exists a sequence $\{p_m\}$ of points of $\mathcal{M}$ convergent to $\gamma(0)$ such that

$$\lim_{m\to\infty} T(t_m)p_m = q$$

for some sequence of real numbers $\{t_m\}$ that tends to ∞.

Let V be a neighborhood of q whose closure is disjoint to the set $\gamma(R)$. Let U_i be a neighborhood of $\gamma(i)$, $i = 1, 2, \ldots$, disjoint from V such that the family of open sets $\{U_1, U_2, \ldots\}$ is locally finite. For each i, construct an X-positive 1-form w_i about $\gamma(i)$ having support in U_i. We normalize w_i so that $\int_{\gamma|[i-1,i]} w_i = 1$. Set $w = w_1 + w_2 + \cdots$.

Let γ_m be the trajectory such that $\gamma_m(t) = T(t)p_m$. Then

$$\int_{\gamma_m|[0,t_m]} w = \sum_i \int_{\gamma_m|[0,t_m]} w_i \to \infty$$

as $m \to \infty$.

On the other hand, there exists a continuous function u with $Xu = \langle X \mid w\rangle$. This leads to the contradiction that, as $m \to \infty$,

$$u(T(t_m)p_m) = U(\gamma_m(t_m)) = u(p_m) + \int_{\gamma_m|[0,t_m]} w \to \infty.$$

COROLLARY. *If X is exact, then every trajectory has either an ω-limit point or an α-limit point.*

Proof. Let γ be a trajectory whose ω- and α-limits are both empty. Then $p=\gamma(0)$ has a neighborhood such that every trajectory that passes through it has neither ω- nor α-limit point. Hence there exists a nonconstant C^∞ first integral, which is absurd.

We reach the conclusion that, if X is exact, then $\mathscr{M}$ is the union of stable and unstable manifolds of the critical points.

3. PROPOSITION. *If X is exact, then both the stable and unstable manifolds of each critical point are embedded in $\mathscr{M}$.*

Proof. Let $\phi\colon R^m \to \mathscr{M}$ be the immersion of the unstable manifold of a critical point p such that $\phi(0)=p$. There exists a neighborhood V of 0 in R^m such that every trajectory, after leaving ϕV, will never return to ϕV. We may further demand that the closure $\overline{V}$ is compact and that the restriction of ϕ to a neighborhood of $\overline{V}$ is an embedding.

Suppose that ϕ is not an embedding. Then there exists a sequence $\{x_i\}$ in R^m bounded away from $y \in R^m$ such that $\{\phi x_i\}$ converges to ϕy in $\mathscr{M}$. We may assume that $y \in V$. Otherwise we may replace y and x_i respectively by $\phi^{-1}T(-\tau)\phi y$ and $\phi^{-1}T(-\tau)\phi x_i$ for some sufficiently large τ. Since ϕ restricted to V is an embedding, only a finite number of ϕx_i may lie in ϕV. Set

$$a_i = \sup\{t : T(t)\phi x_i \in \phi V\}.$$

The sequence $\{x_i\}$ may be chosen such that $\{\phi^{-1}T(a_i)\phi x_i\}$ converges to z, which lies on the boundary of V. Let γ be the trajectory given by $\gamma(t)=T(t)\phi y$. Then the set $\gamma((-\infty, 0])$ does not contain ϕz. Set $\gamma_i(t)=T(t)\phi x_i$.

Let w be an X-positive 1-form about ϕz with support disjoint from $\gamma((-\infty, 0])$. Let u be a continuous function on $\mathscr{M}$ such that $Xu=\langle X \mid w\rangle$. Then

$$u(\phi y)-u(p) = \int_{\gamma|(-\infty,0]} w = 0$$

and

$$u(\phi x_i)-u(p) = \int_{\gamma_i|(-\infty,0]} w$$

which are bounded away from 0. Since $\{\phi x_i\}$ converges to ϕy this is absurd. Hence the proposition is proved.

4. If X is exact, then there can be no simple closed separatrix polygon. Since there are only a finite number of critical points, there can be only a finite number of separatrices.

PROPOSITION. *If X is exact, then one of the following two statements holds*:

(a) *$\mathscr{M}$ is the union of the stable manifolds of X.*

(b) *$\mathscr{M}$ is the union of the unstable manifolds of X.*

Proof. Denote by W^+ (W^-) the union of the n-dimensional unstable (stable) manifolds of X. Observe that W^+ and W^- are disjoint. Otherwise there would be infinitely many separatrices. Moreover, the union $W^+ \cup W^-$ is open and dense in $\mathscr{M}$.

Let $\Gamma = \overline{W}^+ \cap \overline{W}^-$ be the common boundary. Every trajectory γ lying in Γ has at least one α-related limit point as well as one ω-related limit point. It follows from §2 that γ can be either a separatrix or a critical point. Consequently Γ consists of a finite number of connected components, each of which is a finite tree. Since it is impossible for Γ to separate $\mathscr{M}$, one of W^+ and W^-, say W^+, must be empty. This implies that every trajectory must have at least one ω-related limit point and therefore a ω-limit point. Hence $\mathscr{M}$ is the union of the stable manifolds of X.

REMARK. If $\mathscr{M}$ is the union of stable manifolds, then the unstable manifold of each critical point can be only of dimension 0 or 1. Otherwise, there would be infinitely many separatrices. Therefore $\mathscr{M}$ is actually the union of stable manifolds of dimensions n and $n-1$.

5. We conclude the necessity part of the proof of the main theorem by the next proposition.

PROPOSITION. *If X is exact, the separatrix graph is a tree.*

Proof. Assume that $\mathscr{M}$ is the union of the stable manifolds of X. It suffices to show that the separatrix graph Λ is connected. Let Λ' be the union of a number of connected components of Λ, and Λ'', the union of the remaining connected components of Λ. Denote by W' (W'') the union of the n-dimensional stable manifolds whose critical point belongs to Λ' (Λ''). Set $\Gamma = \overline{W}' \cap \overline{W}''$.

Suppose that both Λ' and Λ'' are nonempty. Then Γ is nonempty. If a trajectory is contained in Γ, its ω-limit point also belongs to Γ. Since there is no simple closed separatrix polygon in Γ, there must be a critical point p in Γ such that there is no separatrix from p that is contained in Γ. This means that the 1-dimensional unstable manifold of p is entirely contained in, say $\overline{W}'$, and intersects $\overline{W}''$ at the point p only. On the other hand, since p belongs to Γ, there are trajectories lying in W'', which pass by the hyperbolic critical point p arbitrarily closely. It is clear that one of the two separatrices must lie in $\overline{W}''$. This is a contradiction. Hence Λ is connected.

6. For the purpose of proving the sufficiency part of the main theorem, we give the next lemma.

LEMMA. *Let a critical point p of X be a sink, and w, a C^∞ 1-form, then the function u such that $u(q)$ is equal to the integral $\int w$ along the arc of trajectory from the sink p to q, is continuous about p.*

Proof. Let $x = (x^1, \ldots, x^n)$ be a system of local coordinates about p with $x(p) = 0$.

Write $g(x) = \langle X \mid w \rangle$. Then $g(x) = O(\|x\|)$, and, there exist $\mu > 0$ and $C > 0$ such that, for points q about p,

$$\|x(T(t)q)\| \leqq C\|x(q)\|e^{-\mu t}, \qquad t \geqq 0.$$

The function u is explicitly given by the integral

$$u(q) = \int_{\infty}^{0} g(x(T(t)q))\,dt,$$

whose existence and continuity in q are guaranteed by the above inequality.

Proof of the sufficiency part of the main theorem. We shall assume that $\mathcal{M}$ is the union of the stable manifolds of the critical points of X. Since the separatrix graph is a tree, the unstable manifold of each critical point can be of dimension at most 1. Moreover there must be critical points which are sinks.

If f is a first integral, then f must be constant on each stable manifold and along the separatrix tree. Consequently f must be a constant.

Assign to each critical point and each separatrix a weight in the following manner:

(a) The weight of a sink is 1.

(b) Each separatrix has the weight of its ω-limit point.

(c) The weight of a critical point, which is not a sink, is equal to the sum of the weights of the two separatrices, which make up its 1-dimensional unstable manifold.

Let w be a C^∞ 1-form. Choose a sink p, and define a function u on the separatrix tree by

$$u(q) = \int_{p}^{q} w$$

integrating along the separatrix tree. Since the value of u is given at each critical point, we may extend u to the stable manifold of the critical point. Therefore u is well defined on $\mathcal{M}$. It follows from the lemma, that u is continuous about any critical point which is a sink.

The function u is continuous at a point q, if it is continuous at $T(t)q$ for some value of t. It is not difficult to verify that u is continuous at q if u is continuous at the critical point that is the ω-limit of the trajectory passing through q. Therefore we only have to verify the continuity of u at each critical point. This can be done by induction on the weight of critical points. Hence the main theorem is proved.

7. We now return to the notion of an exact dynamical system in the strong sense, i.e., a dynamical system X such that, given any C^∞ 1-form, the partial differential equation

$$(1) \qquad Xu = \langle X \mid w \rangle$$

has a C^∞ solution u which is unique up to a difference of a constant.

THEOREM. *A dynamical system X is exact in the strong sense if and only if $\mathcal{M}$*

is either the stable or the unstable manifold of a single critical point, about which the partial differential equation admits locally a C^∞ solution for any given C^∞ 1-form w.

We shall only give a brief sketch of the proof: The sufficiency part is clear. For the necessity part, let us suppose that $\mathscr{M}$ is the union of stable manifolds of critical points. One only has to verify that there is no critical point that is not a sink. Suppose the contrary. Let the critical points be given weights as before. There must be a critical point p of weight 2. The unstable manifold of p consists of p and two separatrices whose respective ω-limit points p_1 and p_2 must be sinks. Choose a point q on the first separatrix and an X-positive 1-form w with support lying in a sufficiently small neighborhood of q. Let u be a C^∞ solution of (1). About p, the $n-1$-dimensional stable manifold is the common boundary of the respective n-dimensional stable manifolds of p_1 and p_2. The function u is constant in the latter but not in the former. Construct w such that u is not differentiable at p. Thus we obtain a contradiction.

REMARK. An example of a sink, about which the partial differential equation (1) admits locally a C^∞ solution for any given C^∞ 1-form w, is the origin of $X = -a \sum_{i=1}^{n} x^i \partial/\partial x^i$ with $a > 0$. Is any sink satisfying the above condition C^∞ equivalent [3] to this one?

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