A CHARACTERIZATION OF INTEGRAL CURRENTS(1)

BY

JOHN E. BROTHERS

1. Introduction. In [6] J. Kráľ defined for each \( r > 0 \) a function \( v_\lambda^r \) which for \( z \in \mathbb{R}^n \) and \( r = \infty \) gives a measure of the generalized solid angle under which the boundary \( A \) of a bounded Borel set \( C \) is visible from \( z \). For smooth \( A \), \( v_\lambda^r(z) \) is the integral over the space of lines \( \lambda \) passing through \( z \) of the number of points in \( A \cap \lambda \cap \{ x : 0 < |x - z| < r \} \). This function is used to give geometric conditions on an open set \( C \) which ensure the existence of solutions to the Neumann boundary value problem in \( C \). In particular, Kráľ shows that \( C \) has finite perimeter \( P(C) \) (a definition is in [6, 2.8]) if and only if \( v_\lambda^\infty(z) < \infty \) for \( z \notin A \).

On the other hand, \( v_\lambda^\infty \) is related to the formulas of integral geometry. We show in 4.5 that there exists an absolute constant \( \gamma \) such that if \( P(C) < \infty \), then

\[
\int_{\mathbb{R}^n} v_\lambda^\infty \ dH^n = \gamma r P(C).
\]

In the present paper we seek to extend the results of Kráľ relating the geometry of the boundary of \( C \) with properties of \( v_\lambda^\infty \) to analogous propositions concerning boundaries of \( k+1 \) dimensional objects in \( \mathbb{R}^n \), \( 0 < k < n \). For smooth boundaries \( A \) the appropriate analogue of \( v_\lambda^\infty(z) \) is the integral \( V^A(z) \) over the space of \( l \) planes \( B \) containing \( z \), \( k+l \geq n \), of the \( k+l-n \) dimensional Hausdorff measure of \( A \cap B \sim \{ z \} \). We use the slicing theory developed by H. Federer in [3] to extend \( V^A \) to the case where \( A \) is a \( k \) dimensional flat chain \( T \). (If \( S \) is the \( n \) current obtained by integration over \( C \), then \( \partial S \) is flat; \( P(C) \) equals the mass \( M(\partial S) \) of \( \partial S \), and \( V^\partial S = v_\lambda^\infty \). Rectifiable currents are flat.)

In §5 we derive for \( V^T \) generalizations of the properties of \( v_\lambda^\infty \) obtained by Kráľ in [6, §§1,2]. We also obtain an absolute continuity result for \( T \) which extends [5, 8.5].

In §6 we show that \( M(\partial S) < \infty \) if \( V^\partial S(z) < \infty \) for sufficiently many points \( z \in \mathbb{R}^n \). Combining this with results in §5 we obtain the extension of Kráľ's characterization of sets of finite perimeter:

\[
M(\partial S) < \infty \text{ if and only if } V^\partial S(z) < \infty \text{ for } z \notin \text{spt } \partial S.
\]

From this we obtain our characterization of integral currents by recalling from [5, 8.14] that if \( S \) is rectifiable and \( M(\partial S) < \infty \), then \( S \) is an integral current.

Received by the editors June 4, 1969 and, in revised form, December 8, 1969.

(1) Research partially supported by National Science Foundation grants GP-7505 and GP-11603.

Copyright © 1970, American Mathematical Society
Now suppose $S$ is rectifiable and $\partial S = T + U$ with $M(T) < \infty$. In [3, 2.2] Federer proved that if the support of $U$ has Hausdorff measure zero in dimension $k$, then $S$ is an integral current, thus extending [5, 8.14]. We further extend this in §6 by showing that $S$ is an integral current if for sufficiently many $z \in \mathbb{R}^n$,

$$(\text{spt } U) \cap B \sim \{z\} = \emptyset$$

for almost all $l$ planes $B$ containing $z$.

Finally, we show in §7 that if $T$ is obtained by integration over an oriented $k$ dimensional submanifold $T$ of class 1 and $z \in T$, then $VT(z) < \infty$ for $k + l > n$, and also for $k + l = n$ if the tangent spaces of $T$ satisfy a Hölder condition. (For $k = n - 1$ such surfaces are the Lyapunov surfaces of classical potential theory.)

### 2. Preliminaries

The purpose of this section is to fix basic notation and terminology. We readopt the notation and terminology of [1] and [5].

Fix $n > 1$, and let $k < n$ and $l < n$ be positive integers such that $k + l \leq n$. Denote

$$B = \mathbb{R}^n \cap \{x : x_1 = \cdots = x_{n-l} = 0\} \sim \{0\}$$

and $\psi_B = dx^{n-l+1} \wedge \cdots \wedge dx^n$, and orient $B$ so that $B^\sim = e_{n-l+1} \wedge \cdots \wedge e_n$.

The map

$$\begin{array}{ccc} f: \text{SO}(n) \times B & \rightarrow & \mathbb{R}^n \times \text{SO}(n) \\ (g, b) & \mapsto & (g(b), g), \end{array}$$

embeds $\text{SO}(n) \times B$ as a proper submanifold $\Phi$ of $\mathbb{R}^n \times \text{SO}(n)$ of dimension $m + l$, where $m = \frac{1}{2}n(n-1)$ is the dimension of $\text{SO}(n)$. Defining

$$p: \mathbb{R}^n \times \text{SO}(n) \rightarrow \mathbb{R}^n, \quad p(x, g) = x,$$

$$q: \mathbb{R}^n \times \text{SO}(n) \rightarrow \text{SO}(n), \quad q(x, g) = g,$$

we observe that $q^{-1}(g) \cap \Phi = g(B) \times \{g\}$ and, for $0 \neq x \in \mathbb{R}^n$,

$$F_x = p^{-1}\{x\} \cap \Phi = \{(x, g) : g^{-1}(x) \in B\}$$

is a compact submanifold of $\Phi$ of dimension $m + l - n$ which is orientable since $\Phi$ is orientable.

Fix $o \in B \cap S^{n-1} = B \cap \{x : |x| = 1\}$. Define

$$\begin{array}{ccc} \pi: \text{SO}(n) & \rightarrow & S^{n-1} \\ g & \mapsto & g(o). \end{array}$$

Set $I = \text{SO}(n) \cap \{g : g(o) = o\}$; $I$ has dimension $\mu = \frac{1}{2}(n-1)(n-2)$. Assign a bi-invariant metric to $\text{SO}(n)$ so that $\pi_\#(g)$ is an orthogonal projection for $g \in \text{SO}(n)$. Orient $\text{SO}(n)$ and denote by $\omega$ the unit positively oriented $m$ form on $\text{SO}(n)$. $e \in \text{SO}(n)$ is the identity map of $\mathbb{R}^n$.

Denote

$$F = \text{SO}(n) \cap \{g : g^{-1}(o) \in B\} = q(F_o) = \pi^{-1}(B)^{-1},$$

and observe that if $0 \neq x = |x|g_x(o)$, then

$$F_x = \{x\} \times g_x(F);$$

where
in particular, $F_x$ is connected for $l > 1$. Furthermore,
\[ H^{n+1-n}(F_x) = H^n(I)H^{l-1}(S^{l-1}) \]
\[ = \left[ \sum_{i=2}^{n-1} ia(i) \right] la(l). \]

Whenever $z \in \mathbb{R}^n$ define $r_z(y) = |z - y|$ for each $y \in \mathbb{R}^n$; set $r_0 = r$. We also define
\[ \rho_z: \mathbb{R}^n \sim \{ z \} \rightarrow S^{n-1}, \quad \rho_z(x) = (x - z)/|x - z|; \]
set $\rho_0 = \rho$.

$\mathbb{R}^n \times SO(n)$ acts on $\mathbb{R}^n$ as the group of orientation preserving isometries of $\mathbb{R}^n$ through association of $(z, g) \in \mathbb{R}^n \times SO(n)$ with the isometry, also denoted by $(z, g)$, which maps $x$ to $z + g(x)$. If $T \in E_k(\mathbb{R}^n)$, we denote
\[ (z, g)\#T = z + g#T. \]

If $v \in \bigwedge^k (\mathbb{R}^n)$, $w \in \bigwedge^l (\mathbb{R}^n)$, $*v \wedge *w \neq 0$, $\omega \in \bigwedge^k (\mathbb{R}^n)$, $\zeta \in \bigwedge^l (\mathbb{R}^n)$, define
\[ v \cap w = (-1)^l \| *v \wedge *w \|^{-1} \ast (*v \wedge *w) \in \bigwedge^k (\mathbb{R}^n) \]
and
\[ \omega \cap \zeta = (-1)^l \ast (*\omega \wedge *\zeta), \quad t = (k + l)(n + 1). \]

Recalling the integral geometric constants $\gamma(n, k, l)$ and $\gamma^2(n, k, l)$ defined in [1, 2.13], we also define
\[ \delta(n, k, l) = \gamma^2(n - 1, k, l - 1)k(l - n)la(l)(2k + l - n)^{-1}. \]

Let $X$ be an open subset of $\mathbb{R}^n$. $F_k(X)$ is the closure of $N_k(X)$ in $E_k(X)$ with respect to the flat seminorm $F$ [4, 4.1.12], which is complete relative to $F$. Elements of $F_k(X)$ are called $k$ dimensional flat chains. Rectifiable currents are clearly flat; consequently, since the restriction of $H^n$ to a bounded, open subset of $\mathbb{R}^n$ corresponds to a rectifiable $n$ current, the restriction of $H^n$ to a bounded Borel subset of $\mathbb{R}^n$ corresponds to a flat $n$ current. Finally, we let
\[ F_h^{\infty}(X) \]
be the subset of $D_h(X)$ consisting of all $T$ such that
\[ T \wedge \gamma \in F_k(X) \quad \text{for every } \gamma \in D^0(U). \]

Suppose $z \in \mathbb{R}^n$ and consider $T \in E_k(\mathbb{R}^n)$. We shall denote by $T_z \in D_k(\mathbb{R}^n \sim \{ z \})$ the restriction of $T$ to $D_k(\mathbb{R}^n \sim \{ z \})$. We recall from [4, 4.1.21] that if $T \in F_k(\mathbb{R}^n)$ and $M(T) < \infty$, then $M(T_z) = \| T \|(\mathbb{R}^n \sim \{ z \}) = M(T)$. 

3. Lifts of currents. Let $I$ act to the left on $F$ by means of left translation. Since $I$ is connected, this action will preserve an orientation of $F$. Since $F = \pi^{-1}(B)^{-1}$,
\[ T_{\pi}\left( F \right) = \pi_{\#}(T)\left( B \cap S^{n-1} \right); \]
thus, if $0 \neq w \in \bigwedge^{m+1-n}_n \{T_e(F)\}$, and $v \in \bigwedge^{n-1}_n \{T_e(SO(n))\}$ is such that $\pi_\#(e)(v) = e_1 \wedge \cdots \wedge e_{n-1}$, then $v \wedge w \neq 0$. Choose $w$ so that $\langle v \wedge w, \omega(e) \rangle > 0$, and for $l > 1$ orient $F$ so that $w$ is positively oriented. If $l = 1$, then $F$ has two components. Let $\sigma \in SO(n)$ be such that $\sigma(e_j) = e_j$ for $j < n-1$, and $\sigma(e_n) = -e_n$ for $j = n-1$, $n$. Right translation by $\sigma$ leaves $F$ invariant and permutes the components of $F$; orient $F$ so that $w$ is positively oriented, and so that right translation by $\sigma$ reverses orientation.

3.1. **Lemma.** $\Phi$ is the bundle space of a fibre bundle $\mathcal{B}$ with fibre $F$, structure group $I$, projection $p : \Phi$ and base space $R^n \sim \{0\}$.

**Proof.** Let $U$ be an open subset of $R^n \sim \{0\}$ such that on $U_1 = \{x/|x| : x \in U\}$ there is a cross-section $\sigma_U : U_1 \to SO(n)$ for $\pi$. Define a coordinate function $\varphi_U : U \times F \to \Phi$ by

$$\varphi_U(x, g) = (x, \sigma_U(x/|x|)g);$$

it is easy to verify that the set of $\varphi_U$ provides $\Phi$ with the desired bundle structure.

3.2. **Remarks.** One proceeds as in [1, §3] to define the lifting map

$$L_\mathcal{B} : E_\#(R^n \sim \{0\}) \cap \{T : M(T) < \infty\} \to E_\#(\Phi).$$

Inasmuch as $H^{m+1-n}(F) \neq 1$, suitable modifications must be made in the assertions of [1, §3] concerning the properties of $L_\mathcal{B}$.

Furthermore, [1, 3.5] can be used to show that if $U$ is an open subset of $R^n \sim \{0\}$ having compact closure, then there exists $c$ such that $F[L_\mathcal{B}(T)] \leq cF(T)$ whenever $T \in E_\#(U)$ and $M(T) < \infty$. One uses this and the $F$ completeness of $E_\#(\Phi)$ to extend $L_\mathcal{B}$ to an $F$ continuous linear map of $F_\#(R^n \sim \{0\})$ into $F_\#(\Phi)$; it is clear that [1, 3.3] remains valid. Moreover, since $\delta$ is $F$ continuous, it follows from [1, 3.5(4)] that $\delta \circ L_\mathcal{B} = L_\mathcal{B} \circ \delta$.

Whenever $\gamma \in SO(n)$ we define

$$\gamma_\# : \Phi \to R^n \times SO(n), \quad \gamma_\#(x, g) = (\gamma(x), \gamma g).$$

$\gamma_\#$ is an isometry of $\Phi$ onto $\Phi$, and it is easy to verify that $\gamma_\#$ induces a bundle map of $\mathcal{B}$ onto $\mathcal{B}$, hence we conclude from [1, 3.3 and 3.5(2)] that if $T \in E_\#(R^n \sim \{0\})$ and $M(T) < \infty$, then

$$\gamma_\# [L_\mathcal{B}(T)] = L_\mathcal{B} [\gamma_\#(T)], \quad \gamma_\# [L_\mathcal{B}(T)'^{-1}] = L_\mathcal{B} [\gamma_\#(T)'^{-1}].$$

In case $l = 1$ we define

$$\Sigma : \Phi \to R^n \times SO(n), \quad \Sigma(x, g) = (x, \sigma g).$$

$\Sigma$ is an isometry of $\Phi$ onto $\Phi$. Suppose $T \in E_\#(R^n \sim \{0\})$, $M(T) < \infty$, and $\text{spt } T$ lies in a coordinate neighborhood $U$ with associated coordinate function $\varphi_U : U \times F \to \Phi$ defined as in the proof of 3.1. Observing that $\Sigma \circ \varphi_U = \varphi_U \circ (e \times \sigma)$
and \( L_{\mathbf{a}}(T) = \varphi_{\mathbf{a}}(T \times F) \) by [1, 3.3], we use [1, 3.3] in conjunction with a suitable partition of unity to conclude that for \( T \) with arbitrary support,

\[
\Sigma_{\mathbf{a}}[L_{\mathbf{a}}(T)] = -L_{\mathbf{a}}(T), \quad \Sigma_{\mathbf{a}}[L_{\mathbf{a}}(T)]^* = -L_{\mathbf{a}}(T)^*.
\]

3.3 Definition. For \( (x, g) \in \Phi \), \( u(x, g) \) is the linear right inverse of \( p|_{\Phi}(x, g) \) whose range is orthogonal to \( T_{(x, g)}(F_x) \).

Let \( v \) be a \( k \) vector field on \( \mathbb{R}^n \), \( \nu^v \) is the function on \( \Phi \) such that

\[
\nu^v(x, g) = \|u(x, g)[v(x)]\|^{-1}.
\]

3.4. Lemma. Suppose \( T \in \mathcal{E}_b(\mathbb{R}^n, \varnothing) \), \( M(T) < \infty \).

(i) If \( h \) is a bounded Baire function on \( \Phi \), then

\[
\|L_{\mathbf{a}}(T\|(hv^v) = \int_{\mathbb{R}^n} \int_{F_x} h(x, g) dH^{n+1-n}(x, g) d\|T\| x.
\]

(ii) If \( \psi \) is a bounded Baire \( k + l - n \) form on \( \Phi \), then

\[
(-1)^t[L_{\mathbf{a}}(T) \wedge q^\# \omega](\psi)
\]

\[
= \int \langle ([*T^-(x) \wedge g^\# *B^-], 0), \psi(x, g) \rangle v^{T^-}(x, g) x^{-n} d[L_{\mathbf{a}}(T)](x, g),
\]

where \( t = (k + l - n)(m + l + 1) \).

Proof. Using [1, 3.6(2)] and a suitable partition of unity we see that we can assume \( \text{spt } T \) to lie in a coordinate neighborhood \( U \) with associated coordinate function \( \varphi_U : U \times F \to \Phi \) defined as in the proof of 3.1. From [1, 3.5(5)] we infer that, for \( \|L_{\mathbf{a}}(T)\| \) almost all \((x, g) \in \varnothing \),

\[
(1) \quad L_{\mathbf{a}}(T)^-(x, g) = \nu^{T^-}(x, g) u(x, g)[T^-](x) \wedge F_x^-(x, g).
\]

(i) Denoting the right member of our conclusion by \( \lambda(h) \), we see that \( \lambda = \varphi_{U^\#}(\|T\| \times \|F\|) \). On the other hand, (1) implies that

\[
L_{\mathbf{a}}(T)^-(x, g) = [\nu^{T^-} \varphi_{U^\#}(T^-, 0) \wedge F_x^-](x, g)
\]

for \( \|L_{\mathbf{a}}(T)\| \) almost all \((x, g) \in \varnothing \), hence by [1, 3.3 and 3.1],

\[
\nu^{T^-} L_{\mathbf{a}}(T) = \nu^{T^-} \varphi_{U^\#}(T \times F) = [\nu^{T^-} \varphi_{U^\#}(T^-, 0) \wedge F_x^-] \lambda = L_{\mathbf{a}}(T)^- \lambda.
\]

(ii) We must show that for \( \|L_{\mathbf{a}}(T)\| \) almost all \((x, g) \in \varnothing \),

\[
(2) \quad (-1)^t L_{\mathbf{a}}(T)^-, q^\# \omega \wedge \psi(x, g)
\]

\[
= \langle ([*T^- (x) \wedge g^\# *B^-], 0), \psi(x, g) \rangle \nu^{T^-}(x, g) x^{-n}.
\]

From [1, 3.5] it follows that for \( \|L_{\mathbf{a}}(T)\| \) almost all \((x, g) \in \varnothing \),

\[
(3) \quad \|L_{\mathbf{a}}(T)^-(x, g)\| = \|T^-(x)\| = 1,
\]

hence we can use the maps \( \gamma_{\Phi} \) (and \( \Sigma \) for \( l = 1 \)) which were introduced in 3.2 to infer that we need only verify (2) for \((x, g) \) such that \( x = |x|_o, g^\#(B) = B \), and (1) and (3) hold.
We compute \( i = i(x, g) \) as follows: Choose the cross-section \( \sigma_U \) used in defining \( \varphi_U \) so that
\[
\sigma_U \#(o) = \text{transpose } \pi_\#(e).
\]
Define \( I \) on \( SO(n) \) by \( I(\gamma) = \gamma^{-1} \). Since \( F_x = \{x\} \times F = \{x\} \times \pi^{-1}(B)^{-1} \), we have
\[
T_{(x, g)}(F_x) = \{0\} \oplus I_\#(g^{-1})[T_{x^{-1}}(\pi^{-1}(B))]
\]
\[
= \{0\} \oplus I_\#(g^{-1}) \circ L_{g^{-1}}(e)[T_e[\pi^{-1}(B)]]
\]
\[
= \{0\} \oplus R_{g^{-1}}(e)[T_e[\pi^{-1}(B)]].
\]
On the other hand, it is easy to verify that for \( 1 \leq j \leq n - l \),
\[
v_j = (e_j, |x|^{-1} R_{g^{-1}}(e) \sigma_U(o)(e_j)) = \varphi_U(o)(x, g)(e_j, 0);
\]
\( v_j \) is clearly orthogonal to \( T_{(x, g)}(F_x) \), hence
\[
i(e_j) = v_j, \quad 1 \leq j \leq n - l.
\]
Finally, since \( T_{(x, g)}(q^{-1}(g) \cap \Phi) = (B \cup \{0\}) \oplus \{0\} \),
\[
i(e_j) = (e_j, 0) \quad \text{for } n - l + 1 \leq j \leq n.
\]
We next observe that
\[
\alpha = T^{-}(x) - (-1)^{(n+1)} B^{-} \wedge \star T^{-}(x) \wedge \star B^{-}
\]
has the property \( \star \alpha \wedge \star B^{-} = 0 \), hence use (1) to conclude
\[
\langle L_\#(T^{-}(x, g), q_\# \omega \wedge \psi(x, g) \rangle
\]
\[
= v^{-}(x, g) \langle [(-1)^{(n+1)} i(\star B^{-}) \wedge (\star T^{-}(x) \wedge \star B^{-}), 0] + i(\alpha) \wedge F_{x^{-}}(x, g), q_\# \omega \wedge \psi(x, g) \rangle
\]
\[
= (-1)^{(n+1)} \{ (k+1-n)(m+1-n) \} v^{-}(x, g) \langle i(\star B^{-}) \wedge F_{x^{-}}(x, g), q_\# \omega(x, g) \rangle
\]
\[
\times \langle (\star T^{-}(x) \wedge \star B^{-}), 0 \rangle, \psi(x, g) \rangle.
\]
Finally,
\[
\langle i(\star B^{-}) \wedge F_{x^{-}}(x, g), q_\# \omega(x, g) \rangle
\]
\[
= \langle q_\#(x, g)[i(\star B^{-})] \wedge F^{-}(g), \omega(g) \rangle
\]
\[
= (-1)^{(n+1)} |x|^{-n} \langle R_{g^{-1}}(e) \sigma_U(o)(e_1 \wedge \cdots \wedge e_{n-1}) \wedge F^{-}(g), \omega(e) \rangle
\]
\[
= (-1)^{(n+1)} |x|^{-n} \langle \sigma_U(o)(e_1 \wedge \cdots \wedge e_{n-1}) \wedge F^{-}(e), \omega(e) \rangle
\]
\[
= (-1)^{(n+1)} |x|^{-n},
\]
whence follows (2).

3.5. Definitions. Orient \( R^n \times SO(n) \) so that \( p_\#(dx^1 \wedge \cdots \wedge dx^n) \wedge q_\# \omega \) is positively oriented.

Let \( \Omega = (0, SO(n)^-) \).
Let $P: SO(n) \times B \to B$ be the projection. Referring to the basis of $T_{(x, g)}(\Phi)$ constructed in the proof of 3.4 for $x = |x|, g \neq B = B$, one infers that the intersection of $T_{(x, g)}(\Phi)$ with the orthogonal complement $V_{(x, g)}$ of $T_{(x, g)}[g(B) \times \{g\}]$ in $R^n \oplus \{0\}$ is $\{0\}$. It follows from 3.2 that this remains true for arbitrary $(x, g) \in \Phi$. There therefore exists

$$\Psi \in E'[\{R^n \sim \{0\}\} \times SO(n)]$$

such that

$$f^\#(\Psi) = P^\#(\psi_B)$$

and, for $(x, g) \in \Phi$,

$$\langle v \wedge w, \Psi(x, g) \rangle = 0$$

whenever $v \in V_{(x, g)}$ and $w \in \bigwedge_{k+1-n} (T_{(x, g)}[R^n \times SO(n)])$.

3.6. **Lemma.** Let $\varphi$ be a k form on $R^n$ and $(x, g) \in \Phi$. If

$$v \in \bigwedge_{k+1-n} (T_{(x, g)}[g(B) \times \{g\}])$$

then

$$(-1)^{(n-1)} \langle v \wedge \Omega(x, g), p^\#(p^\# \varphi \wedge g^{-1}\# \psi_B)(x, g) \rangle$$

$$= \langle v \wedge \Omega(x, g), \ast((p^\# \varphi) \wedge \ast(\Omega' \wedge q^\# \omega))(x, g) \rangle.$$

**Proof.** One uses [1, 4.2(2)] to verify that the left member of our assertion is equal to

$$(-1)^{m(n-1)} \langle v \wedge \Omega(x, g), p^\#(p^\# \varphi \wedge g^{-1}\# \psi_B)(x, g) \rangle.$$

In order to complete the proof we need only compute

$$s = \langle w, (p^\# g^{-1}\# \psi_B)(x, g) \rangle$$

for $w \in \bigwedge_{n-l} [T_{(x, g)}[R^n \times \{g\}]]$. Denote by $i$ the transpose of $p^\#(x, g)$ and by $w_0$ the orthogonal projection of $i[p^\#(x, g)(w)]$ on $T_{(x, g)}[g(B) \times \{g\}]$. Then, identifying $g(B) \times \{g\}$ with $g(B)$ and setting $y = g^{-1}(x)$, we have

$$f^\#(g, y)(0, g^{-1}(w_0)) = w_0$$

and by [1, 4.2(1)],

$$(-1)^{(n-1)} s = \langle g^\# \ast p^\#(x, g)(w), \psi_B \rangle$$

$$= \langle g^\# \ast p^\#(w_0), \psi_B \rangle$$

$$= \langle P^\#(g, y)f^\#(g, y)^{-1}(w_0), \psi_B \rangle$$

$$= \langle w_0, \Psi(x, g) \rangle$$

$$= \langle i[p^\#(x, g)(w)], \Psi(x, g) \rangle$$

$$= \langle i[p^\#(x, g)(w)] \wedge \Omega(x, g), (\Omega' \wedge q^\# \omega)(x, g) \rangle$$

$$= \langle w, (\Omega' \wedge q^\# \omega)(x, g) \rangle$$

$$= (-1)^{(l+m(n-1)} \langle w, (\Omega' \wedge q^\# \omega)(x, g) \rangle.$$
4. Plane intersections of currents. If \( S \in F_\ell(\Phi) \) and \( j \geq m \), then for \( H^m \) almost all \( g \in \text{SO} (n) \),

\[
\langle S, q|\Phi, g \rangle \in F_{j-n}(\Phi)
\]

is the slice of \( S \) by \( q|\Phi \) over \( g \) as characterized in [4, §4.3].

4.1. Definition. Suppose \( T \in F_k(R^n \sim \{z\}) \). Whenever

\[
\langle L_\Phi(-z+T), q|\Phi, g \rangle \in F_{k+1-n}(\Phi)
\]

we define

\[
T \cap (z+gB) = (-1)^t[z+p_\beta\langle L_\Phi(-z+T), q|\Phi, g \rangle],
\]

where \( t = (k+l-n)(m+l+1)+(k+l)(n+1) \).

4.2. Theorem. (1) If \( T \cap (z+gB) \) exists, then

\[
\text{spt } [T \cap (z+gB)] \subset (\text{spt } T) \cap (z+gB).
\]

(2) If \( U \) is an open subset of \( R^n \sim \{z\} \) having compact closure in \( R^n \sim \{z\} \), then there exists \( c < \infty \) such that whenever \( T \in F_k(U) \),

\[
\int_{\text{Spt \text{H}^m g}} F[T \cap (z+gB)]dH^m g \leq cF(T).
\]

(3) If \( M(T) < \infty \), then for \( H^m \) almost all \( g \in \text{SO} (n) \) the following are true:

(i) \( M[T \cap (z+gB)] < \infty \).

(ii) For \( \|T \cap (z+gB)\| \) almost all \( x \in R^n \),

\[
[T \cap (z+gB)]^-(x) = T^-(x) \cap gB^-.
\]

(iii) If \( T \) is an oriented proper submanifold of class 1, then \( T \cap (z+gB) = A \cup Z \) where \( A \) is an orientable, proper \( k+l-n \) submanifold of class 1 and \( H^{k+1-n}(Z) = 0 \); if \( A \) is oriented according to (ii), then \( A = T \cap (z+gB) \).

(4) Whenever \( \varphi \in E'(R^n) \), \( i \leq k+l-n \), and \( T \cap (z+gB) \) exists, then so does

\[
(T \wedge \varphi) \cap (z+gB) = [T \cap (z+gB)] \wedge \varphi.
\]

(5) If \( k+l>n \) and \( T \cap (z+gB) \) exists, then

\[
\partial [T \cap (z+gB)] = (-1)^{n-1}\partial T \cap (z+gB).
\]

Furthermore, if \( T \) is normal, then \( T \cap (z+gB) \) is normal for \( H^m \) almost all \( g \in \text{SO} (n) \).

Proof. (1) follows from [1, 3.5(3)] and [4, 4.3.1].

(2) follows from 3.2, [4, 4.3.1 and 4.3.2(5)].

(3)(i) follows from [1, 3.5(2)] and [4, 4.3.2(2)].

(3)(ii) follows from 3.4(ii) and [4, 4.3.2(1) and (2)].

(3)(iii) follows from the coarea formula [4, 3.2.12], [1, 3.5(6)], [4, 4.3.8] and (ii).

(4) follows from [4, 4.3.1] and the second statement of [1, 3.6(2)].

(5) follows from 3.2, [4, 4.3.1] and (3)(i).
4.3. Remark. 4.2(2) and (3) and [4, 4.1.23] show that the intersections \( T \cap (z + g_\# B) \) are intrinsically determined by the action of \( \text{SO}(n) \) on \( \mathbb{R}^n \). In particular, if \( w, z \in \mathbb{R}^n \) and \( h \in \text{SO}(n) \), then

\[
(w + h_\#)[T \cap (z + g_\# B)] = (w + h_\#T) \cap [w + h(z) + hg_\#B]
\]

for \( H^m \) almost all \( g \in \text{SO}(n) \).

That our intersections are usually the same as those defined in [1, §4] will follow in §5.

Suppose \( z \in \mathbb{R}^n \) and consider \( T \in F_k(\mathbb{R}^n) \). Let \( \mathcal{U} \) be a partition of unity subordinate to a locally finite open cover of \( \mathbb{R}^n \sim \{z\} \). Then for \( H^m \) almost all \( g \in \text{SO}(n) \),

\[
(T \wedge u) \cap (z + g_\# B) \in F_{k+1-n}(\mathbb{R}^n)
\]

for each \( u \in \mathcal{U} \); for such \( g \) we define

\[
T \cap (z + g_\# B) = \sum_{u \in \mathcal{U}} (T \wedge u) \cap (z + g_\# B) \in F_{k+1-n}(\mathbb{R}^n \sim \{z\}).
\]

One uses 4.2(4) to verify that this definition does not depend on the choice of \( \mathcal{U} \), and that if \( z \notin \text{spt} \, T \), then the definition reduces to the one in 4.1. With appropriate modifications the assertions of 4.2 hold for \( T \).

4.4. Definition. If \( T \in F_k(\mathbb{R}^n) \) and \( z \in \mathbb{R}^n \), then

\[
V^T(z) = H^m(I)^{-1} \int_{\text{SO}(n)} M[T \cap (z + g_\# B)] \, dH^m g.
\]

If \( k = n-1 \), \( l = 1 \) and \( S \) corresponds to the restriction of \( H^n \) to a bounded Borel set \( C \), then \( \frac{1}{l} V^S \) coincides with the function \( v_\alpha^C \) introduced in [6].

4.5. Remark. If \( T \) is a rectifiable \( k \) current, then it follows from [5, 8.16], 4.2(3)(iii) and [1, 5.8 and 5.9] that for \( r > 0 \) and \( T_{z,r} = T \cap \{x : |x-z| < r\} \),

\[
\int_{\mathbb{R}^n} V^{T_{z,r}}(z) \, dH^n z = \gamma(n, k, l) \alpha(l) r^l M(T).
\]

If \( S \) is the \( n \) current corresponding to the restriction of \( H^n \) to a bounded open set \( C \), and if \( M(\partial S) = P(C) < \infty \), then \( S \) is rectifiable, hence \( \partial S \) is rectifiable by [5, 8.14]. If \( l = 1 \), then \( \frac{1}{l} V^{(S)_r}(z) \) is equal to \( \nu_\alpha^C(z) \) as defined in [6], whence follows the integral geometric formula mentioned in §1.

5. Integral geometric formulas. Whenever \( 0 \neq x \in \mathbb{R}^n \) define

\[
I_x = \text{SO}(n) \cap \{g : g(x) = x\}
\]

and choose \( g_x \) so that \( x = |x| g_x(o) \). Since

\[
I_x = g_x I g_x^{-1}, \quad H^n(I_x) = H^n(I) = H^{n-1}(S^{n-1})^{-1} H^m[\text{SO}(n)].
\]
5.1. Lemma. Suppose $T \in E_k(\mathbb{R}^n)$, $M(T) < \infty$, $\varphi$ is a bounded Baire $k$ form with support in $\mathbb{R}^n \sim \{0\}$, and $h$ is the bounded Baire function on $\Phi$ such that
\[ h(x, g) = \langle *T^\sim(x) \wedge g \mathring{\varphi} B, *\varphi(x) \wedge g^{-1} \# \psi_B \rangle. \]
Then
\[ H^m(I)^{-1} \int_{\mathbb{R}^n} \int_{I_{xg}} h(x, g) \, dH^m g \, d\|T\| x \]
\[ = \gamma^2(n-1, k, l-1) T(\varphi) \quad \text{for } \varphi = f_0 \rho \# \varphi_0, \]
\[ = \gamma^2(n-1, k-1, l-1) T(\varphi) \quad \text{for } \varphi = (f_1 \rho \# \varphi_1) \wedge dr, k+l > n. \]

Proof. Suppose $\varphi = f_0 \rho \# \varphi_0$ and $\|T^\sim(x)\| = 1$. Observing that we can assume $T^\sim(x)$ to lie in the orthogonal complement $X$ of $Rx$, we apply [1, 6.2, 9.3 and 2.13] to obtain
\[ H^m(I)^{-1} \int_{I_{xg}} h(x, gg_\alpha) \, dH^m g = \gamma^2(n-1, k, l-1) \langle T^\sim(x), \varphi(x) \rangle. \]
Now suppose $\varphi = (f_1 \rho \# \varphi_1) \wedge \rho$ and $\|T^\sim(x)\| = 1$. Observing that we can assume $T^\sim(x) = u \wedge x$, where $u$ lies in $X$, we again apply [1, 6.2] in $X$ to obtain our conclusion.

5.2. Lemma. Suppose $h$ is a bounded Baire function on $\Phi$ such that if $g_0 \in SO(n)$ and $g_0 B = B$, then $A(x, gg_0) = A(x, g)$ for $(x, g) \in \Phi$. Then
\[ 1a(I) \int_{I_{xg}} h(x, g) \, dH^m g = \int_{F_x} h(x, g) \, dH^{m+1-n}(x, g). \]
Proof. First observe that
\[ H^m(I) \int_{F_x} h(x, g) \, dH^{m+1-n}(x, g) = \int_{I_{xg}} \int_{F_x} h(x, g) \, dH^{m+1-n}(x, g) \, dH^m g. \]
\[ = \int_{F_x} \int_{I_{xg}} h(x, g) \, dH^m g \, dH^{m+1-n}(x, g). \]
Fix $(x, g) \in F_x$ and denote $\gamma^{-1}(x) = y \in B$. Choose $g_0$ so that $g_0(y) = |y| o, g_0 B = B$, and set $g_x = \gamma g_0^{-1}$. Then for some $g_1 \in I_x$ we have $g_x = g^{-1} g_x$ and
\[ \int_{I_{xg}} h(x, g) \, dH^m g = \int_{I_{xg}} h(x, gg_x g_0) \, dH^m g \]
\[ = \int_{I_{xg}} h(x, g g_x) \, dH^m g = \int_{I_{xg}} h(x, g) \, dH^m g. \]

5.3. Theorem. If $T \in E_k(\mathbb{R}^n)$ and $\varphi \in E_k(\mathbb{R}^n \sim \{z\})$, then
\[ H^m(I)^{-1} \int_{B(0,o)} (z + g \mathring{\varphi}(r_{z-1}^\varphi \cap g^{-1} \# \psi_0) \, dH^m g \]
\[ = 1a(I) \gamma^2(n-1, k, l-1) T(\varphi) \quad \text{for } \varphi = f_0 \rho \# \varphi_0 \]
\[ = 1a(I) \gamma^2(n-1, k-1, l-1) T(\varphi) \quad \text{for } \varphi = (f_1 \rho \# \varphi_1) \wedge dr, k+l > n. \]
Proof. With the assistance of 4.2(2) we see that we can assume that \( z = 0 \),\n
0 \notin \text{spt} T, and \( M(T) < \infty \). Referring to 3.5 we let \( \Xi \) be the \( k + l - n \) form on \( \mathbb{R}^n \times \text{SO}(n) \) such that whenever
\[ v \in \bigwedge_{k+l-n} [T_{(x,g)}(\mathbb{R}^n \times \text{SO}(n))], \]
\[ (-1)^{m(n-l)} \langle v, \Xi(x,g) \rangle = \langle v \wedge \Omega(x,g), *[(p^* \varphi) \wedge *([\Psi^* \wedge q^* \omega])(x,g) \rangle. \]

Suppose \( \varphi = f_0 \rho^* \varphi_0 \). We apply 5.1, 5.2, 3.4(i), 3.6, 3.4(ii), \[4, 4.3.2(1)], 3.6 and \[1, 2.5\] to verify that
\[ H^{m+l-n}(F) \gamma^2(n-1, k, l-1) T(\varphi) \]
\[ = \int_{\mathbb{R}^n} \langle *T^- (x) \wedge g^* \gamma B \rangle, 0 \rangle, \Xi(x,g) \rangle \frac{d}{\mathbb{L}^m(T)(x,g)} \]
\[ = (-1)^{k+l} \int_{\text{SO}(n)} \langle L_\varphi(T), q, g \rangle (r^{n-1} \circ \rho \Xi) \frac{dH^{mg}}{\mathbb{S}^O(n)} \]
\[ = (-1)^{k+l} \int_{\text{SO}(n)} (T \cap g_* B[r^{n-1} \varphi \wedge g^{-1} \rho \psi_t]) \frac{dH^{mg}}{\mathbb{S}^O(n)} \]
\[ = \int_{\text{SO}(n)} T \cap g_* B[r^{n-1} \varphi \wedge g^{-1} \rho \psi_t] \frac{dH^{mg}}{\mathbb{S}^O(n)}. \]

The proof for \( \varphi = (f_1 \rho^* \varphi_1) \wedge dr \) is analogous.

5.4. Lemma. (1) If \( z \in \mathbb{R}^n \), then
\[ \Sigma_2 = \rho_{z}^*[E_k(S^{n-1})] \cup \rho_{z}^*[E^{k-1}(S^{n-1})] \wedge dr_z \]
spans \( E^k(\mathbb{R}^n \sim \{z\}) \) over \( E^0(\mathbb{R}^n \sim \{z\}) \).

(2) If \( z_1, \ldots, z_{k+1} \in \mathbb{R}^n \) determine a \( k \)-plane \( \Pi \), then
\[ \Sigma = \bigcup_{i=1}^{k+1} \rho_{z_i}^*[E^k(S^{n-1})] \]
spans \( E^k(\mathbb{R}^n \sim \Pi) \) over \( E^0(\mathbb{R}^n \sim \Pi) \).

Proof. (1) It is clear that for each \( z \neq x \in \mathbb{R}^n \), \( \Sigma_2(x) \) spans \( \wedge^k(\mathbb{R}^n) \). Thus choose \( \psi_1 \in \Sigma_2, i = 1, \ldots, v = (\Xi) \), such that \( \psi_1(x), \ldots, \psi_v(x) \) is a basis of \( \wedge^k(\mathbb{R}^n) \). Then \( \psi_1(y), \ldots, \psi_v(y) \) are linearly independent for \( y \) in some neighborhood \( N \) of \( x \), hence \( \psi_1 | N, \ldots, \psi_v | N \) span \( E^k(N) \) over \( E^0(N) \), and our conclusion follows with use of a suitable partition of unity.

(2) Fix \( x \in \mathbb{R}^n \sim \Pi \). Then \( \wedge^k(\mathbb{R}^n) \) is spanned by \( \Sigma(x) \). In fact, if this were not true there would exist \( 0 \neq w \in \wedge^k(\mathbb{R}^n) \) such that for \( i = 1, \ldots, k+1, \)
\[ \langle w, \rho_{z_i}^*[E^k(S^{n-1})](x) \rangle = \{0\}, \]
or equivalently, \( \rho_{z_i}^*(x)(w) = 0 \). But for each \( i, \)
\[ \dim \ker \left[ \rho_{z_i}^*(x) | \wedge^k(\mathbb{R}^n) \right] = \left( \frac{n-1}{k-1} \right). \]
hence we would have \( w = (x - z_1) \land w_i \), and the linear independence of the \((x - z_i), i = 1, \ldots, k + 1\), would allow us to conclude that
\[
w = \alpha(x - z_1) \land \cdots \land (x - z_k) \notin \ker \rho_{z_{k+1}}(x).
\]

Proceeding as in (1) we conclude that for some neighborhood \( N \) of \( x \), \( \Sigma |N \) spans \( E^k(N) \) over \( E^0(N) \), and our conclusion follows with use of a suitable partition of unity.

5.5. **Definition.** If \( z \in \mathbb{R}^n \), then \( D^k(z) \) is the linear subspace of \( D^k(\mathbb{R}^n \sim \{z\}) \) consisting of sums of the form \( \sum_{i=1}^{k+1} f_i \rho_{z_i}^\psi \), with \( f_i \in D^0(\mathbb{R}^n \sim \{z\}) = D^0(z) \) and \( \varphi_i \in D^k(S^{n-1}) \).

5.6. **Lemma.** If \( T \in F_k(\mathbb{R}^n) \) and \( M(T) < \infty \), then
\[
H^*(I)V_T(0) = \lambda(I) \int_{\mathbb{R}^n - \{0\}} \int_{I_{x \neq y}} \| T^* - (x) \land g_{\#}B^- \| |x|^{-n} dH^ng \; d\|T\|_x.
\]

**Proof.** Choosing \( f_i \in D^0(0), i = 1, 2, \ldots \), such that \( 0 \leq f_1 \leq f_2 \leq \cdots \) and
\[
\lim_{i \to \infty} f_i(y) = 1 \quad \text{for} \ y \neq 0,
\]
we apply 4.4, 4.1, [4, 4.3.2(2)], 3.4(ii), 3.4(i) and 5.2 to obtain
\[
H^*(I)V_T^{\wedge f_i}(0) = M[L_{\#}(T \land f_i) \land q_{\#} \omega] = \int_0 \| T^* - (x) \land g_{\#}B^- \| |x|^{-n} d\|L_{\#}(T \land f_i)\|_x(g, f_i)
\]
\[
= \lambda(I) \int_{\mathbb{R}^n} \int_{I_{x \neq y}} \| T^* - (x) \land g_{\#}B^- \| dH^ng(x) |x|^{-n} f_i(x) \; d\|T\|_x.
\]
Consequently, we infer from 4.2(4) that
\[
\lambda(I) \int_{\mathbb{R}^n - \{0\}} \int_{I_{x \neq y}} \| T^* - (x) \land g_{\#}B^- \| dH^ng |x|^{-n} d\|T\|_x
\]
\[
= H^*(I) \lim_{i \to \infty} V_T^{\wedge f_i}(0)
\]
\[
\geq \int_{SO(n)} \liminf_{i \to \infty} M[(T \land g_{\#}B) \land f_i] \; dH^ng
\]
\[
\geq \int_{SO(n)} M(T \land g_{\#}B) \; dH^ng
\]
\[
= H^*(I)V_T(0),
\]
which implies our assertion for the case \( V_T(0) = \infty \). On the other hand, if \( V_T(0) < \infty \), then
\[
H^*(I)V_T^{\wedge f_i}(0) = \int_{SO(n)} M[(T \land g_{\#}B) \land f_i] \; dH^ng
\]
\[
= \int_{SO(n)} \| T \land g_{\#}B \| (f_i) \; dH^ng,
\]

hence
\[
\lim_{i \to \infty} V_T^{\wedge f_i}(0) = V_T(0),
\]

which implies our assertion for such \( T \).
5.7. Lemma. Suppose \( r + s \leq n \), \( v \in \wedge r (\mathbb{R}^n) \), and \( w \in \wedge s (\mathbb{R}^n) \).

(1) If \( r \neq n/2 \) or \( s \neq n/2 \), then

\[
H^m[\text{SO}(n)]\gamma^2(n, n-r, n-s)\|v\| \|w\| \leq \left( \frac{r+s}{r} \right) \int_{\text{SO}(n)} \|v \land g \# w\| \, dH^m g.
\]

If \( v \) or \( w \) is simple, then this is true for arbitrary \( r \) and \( s \), and the factor \( \left( \frac{r+s}{r} \right) \) can be omitted.

(2) \[
\int_{\text{SO}(n)} \|v \land g \# w\| \, dH^m g \leq H^m[\text{SO}(n)]\gamma(n, n-r, n-s)\|v\| \|w\|.
\]

If \( v \) and \( w \) are simple, then equality holds. If \( v \) is not simple and the smallest linear subspace \( L \) of \( \mathbb{R}^n \) containing \( v \) has dimension \( \lambda \leq r + s + 1 \), or if an analogous condition holds for \( w \), then the inequality is strict.

Proof. (1) Consider \( \varphi \in \wedge r (\mathbb{R}^n) \) and \( \psi \in \wedge s (\mathbb{R}^n) \) such that \( \|\varphi\| \leq 1 \) and \( \|\psi\| \leq 1 \). Then by [1, 6.2 and 9.3]

\[
H^m[\text{SO}(n)]\gamma^2(n, n-r, n-s)\langle v, \varphi \rangle \langle w, \psi \rangle \leq \left( \frac{r+s}{r} \right) \int_{\text{SO}(n)} \|v \land g \# w\| \, dH^m g
\]

if \( r \neq n/2 \) or \( s \neq n/2 \), whence follows the first statement. If \( w \) is simple, let \( \psi \) be the metric dual of \( w/|w| \). Then \( \langle w, \psi \rangle = |w| \) and

\[
H^m[\text{SO}(n)]\gamma^2(n, n-r, n-s)\langle v, \varphi \rangle |w| \leq \int_{\text{SO}(n)} \|v \land g \# w\| \, dH^m g.
\]

(2) If \( v \) and \( w \) are simple, equality follows from [1, 5.6 and 5.9]. On the other hand, suppose \( v \) is not simple and use [5, 2.2] to obtain simple \( r \) vectors \( \alpha_1, \ldots, \alpha_M \) and simple \( s \) vectors \( \beta_1, \ldots, \beta_N \) such that \( \alpha_1 \) and \( \alpha_2 \) are linearly independent,

\[
v = \sum_{i=1}^{M} \alpha_i, \quad \|v\| = \sum_{i=1}^{M} |\alpha_i|,
\]

and

\[
w = \sum_{i=1}^{N} \beta_i, \quad \|w\| = \sum_{i=1}^{N} |\beta_i|.
\]

Inasmuch as

\[
\int_{\text{SO}(n)} \|v \land g \# w\| \, dH^m g \leq \sum_{i=1}^{N} \sum_{i=1}^{M} \int_{\text{SO}(n)} |\alpha_i \land g \# \beta_j| \, dH^m g
\]

\[
= H^m[\text{SO}(n)]\gamma(n, n-r, n-s) \sum_{i=1}^{N} \sum_{i=1}^{M} |\alpha_i| |\beta_j|
\]

\[
= H^m[\text{SO}(n)]\gamma(n, n-r, n-s)\|v\| \|w\|,
\]

the remainder of (2) will follow if we can find a simple \( s \) vector \( \beta \neq 0 \) such that

\[(*) \quad \|\alpha_1 + \alpha_2\land \beta\| < |\alpha_1 \land \beta| + |\alpha_2 \land \beta|.\]
First assume \( r + s = n \). Corresponding to \( j = 1, 2 \) let \( P_j \) be the subset of
\[ W = \bigwedge_s (\mathbb{R}^n) \cap \{ \beta : \beta \text{ is simple and } |\beta| = 1 \} \]
consisting of \( \beta \) for which \( \alpha_j \wedge \beta \) is positively oriented, \( Q_j = -P_j \), and \( R_j = W \sim (P_j \cup Q_j) \). Then \( P_j \) and \( Q_j \) are open in \( W \), \( R_j \) has no interior, and \( R_j \) is the boundary of \( P_j \) and of \( Q_j \). Further, \( R_1 \neq R_2 \). In fact, if \( A_j \) is the \( r \) dimensional linear subspace of \( \mathbb{R}^n \) containing \( \alpha_j \), \( C_j \) is the orthogonal complement of \( A_1 \cap A_2 \) in \( A_j \), and \( D \) is the orthogonal complement of \( A_1 + A_2 \) in \( \mathbb{R}^n \), then
\[ \dim (D + C_j) = s. \]

If \( 0 \neq \beta_0 \in \bigwedge_s (D + C_1) \), then
\[ \alpha_1 \wedge \beta_0 = 0, \quad \alpha_2 \wedge \beta_0 \neq 0. \]

Therefore, \( P_1 \cap Q_2 \neq \emptyset \); any \( \beta \in P_1 \cap Q_2 \) will satisfy (*).

In case \( \lambda \leq r + s \) we apply the result of the last paragraph with \( \mathbb{R}^n \) replaced by \( L \) to obtain a simple \( \beta_0 \in \bigwedge_{\lambda - r} (L) \) such that
\[ \| (\alpha_1 + \alpha_2) \wedge \beta_0 \| = |(\alpha_1 + \alpha_2) \wedge \beta_0| < |\alpha_1 \wedge \beta_0| + |\alpha_2 \wedge \beta_0|. \]

Thus if \( \beta_1 \neq 0 \) is a simple \( r + s - \lambda \) vector lying in the orthogonal complement of \( L \), \( \beta = \beta_0 ^\perp \) will satisfy (*).

In case \( \lambda = r + s + 1 \) we choose \( \beta_0 \) as above, choose a simple \( s \) vector \( \beta \) and \( \beta_2 \in L \) such that \( \beta_0 = \beta \wedge \beta_2 \), and conclude that
\[ \| (\alpha_1 + \alpha_2) \wedge \beta \| = |(\alpha_1 + \alpha_2) \wedge \beta| < |\alpha_1 \wedge \beta| + |\alpha_2 \wedge \beta|. \]

5.8. Example. Let \( n = 6, r = 3, s = 1 \), and
\[ v = \frac{1}{2} [e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6]. \]

One uses a procedure similar to the one used on [7, p. 54] to show that
\[ \|v\| = \frac{1}{2} |e_1 \wedge e_2 \wedge e_3| + \frac{1}{2} |e_4 \wedge e_5 \wedge e_6| = 1. \]

Let \( w_1 \) be a linear combination of \( e_1, e_2, e_3 \), and \( w_2 \) be a linear combination of \( e_4, e_5, e_6 \) such that \( |w_1| = |w_2| = 1 \). Then if \( w = \alpha w_1 + \beta w_2 \), one uses the fact that \( \|e_1 \wedge e_2 + e_3 \wedge e_4\| = 2 \) to verify that
\[ \|v \wedge w\| = \frac{1}{2} (|\alpha| + |\beta|) = \frac{1}{2} |e_1 \wedge e_2 \wedge e_3 \wedge w| + \frac{1}{2} |e_4 \wedge e_5 \wedge e_6 \wedge w|; \]

consequently,
\[ \int_{SO(6)} \|v \wedge g \sigma w_1\| \, dH^{15}g = H^{15}[SO (6)]_3 (6, 3, 5). \]

5.9. Theorem. Assume \( T \in F_k (\mathbb{R}^n) \) and \( z \in \mathbb{R}^n \).

(1) \[ \delta(n, k, l) M(T_z \wedge r_z^{l-n}) \leq V^T(z) \leq l\alpha(l) M(T_z \wedge r_z^{l-n}). \]
Consequently,

\[ V^T(z) \leq \delta(n, k, l) M(T_z) \text{dist}(z, \text{spt } T) \]  

\[ \delta(n, k, l) M(T_z) \leq V^T(z) \sup \{|x-z| : x \in \text{spt } T\}^{n-1}. \]

(2) Suppose \( r > 0 \) and \( T \) lies in the image of \( E_k(z + rS^{n-1}) \) under the map induced by the inclusion of \( z + rS^{n-1} \) in \( \mathbb{R}^n \). Then

\[ \delta(n, k, l) \gamma(n-1, k, l-1) r^{n-1} M(T) \leq V^T(z) \leq \delta(n, k, l) \gamma(n-1, k, l-1) r^{n-1} M(T). \]

Suppose \( M(T) < \infty \). The second inequality becomes equality when \( T^- \) is simple. If for \( x \) in a set of positive \( \|T\| \) measure it is true that \( T^-(x) \) is not simple and the smallest linear subspace of \( \mathbb{R}^n \) containing \( *T^-(x) \) has dimension not greater than \( 2n-k-l+1 \), then the second inequality is strict.

(3) Suppose \( l = n-k \) and \( T \) is rectifiable with \( k \) dimensional density \( \Theta^k \). Then

\[ V^T(z) = (n-k) \alpha(n-k) \gamma(n-1, k, n-k-1) \int_{S^{n-1}} \sum_{r_{z}^{-1}(\theta)} \Theta^k dH^k \theta. \]

(4) Suppose \( l = n-k \) and \( M(T) < \infty \). Whenever \( z \neq x \in \mathbb{R}^n \) and \( \|T^-(x)\| = 1 \) let \( \tau(x) \) be the orthogonal projection of \( T^-(x) \) on \( T_x(z + |x-z|S^{n-1}) \). Define

\[ \gamma_s(x) = H^k(I)^{-1} \|\tau(x)\|^{-1} \int_{I_{x-z}} |T^-(x) \wedge gg_{x-z}B^*| \ dH^k g \]

whenever \( \|T^-(x)\| = 1 \). Then

\[ (n-k) \alpha(n-k) \gamma(n-1, k, n-k-1) \sup \{T(r_{z}^{-k} \varphi) : \varphi \in D^k(z), M(\varphi) \leq 1\} \]

\[ \leq V^T(z) = (n-k) \alpha(n-k) \gamma(n-1, k, n-k-1) \sup \{T(\gamma_s r_{z}^{-k} \varphi) : \varphi \in D^k(z), M(\varphi) \leq 1\} \]

\[ \leq (n-k) \alpha(n-k) \gamma(n-1, k, n-k-1) \sup \{T(r_{z}^{-k} \varphi) : \varphi \in D^k(z), M(\varphi) \leq 1\}. \]

The second inequality becomes equality if \( \rho_{\#}[T^-(x)] \) is simple for \( \|T\| \) almost all \( x \in \mathbb{R}^n \sim \{z\} \), in which case

\[ \gamma_s = \gamma(n-1, k, n-k-1). \]

(5) If \( l = n-k \), \( M(T) < \infty \), and \( \gamma_s(x) \) is a continuous function of \( z \) for \( \|T\| \) almost all \( x \), then \( V^T \) is lower semicontinuous.

Proof. We can assume \( z = 0 \). Choose \( f_i \in D^0(0), \ i = 1, 2, \ldots \), such that \( 0 \leq f_1 \leq f_2 \leq \cdots \leq 1 \) and whenever \( K \) is a compact subset of \( \mathbb{R}^n \sim \{0\} \) there exists \( i \) for which \( f_i|_K = 1 \).

If \( M(T) < \infty \), then the second inequality of (1) follows from 5.6. In general, we use 4.2(4) to infer that

\[ V^T(0) \leq \lim \inf_{i \to \infty} V^{T \wedge f_i}(0) \]

\[ \leq \delta(n, k, l) \lim \inf_{i \to \infty} M(T_z \wedge f_i) \wedge r_{z}^{-n-1} \]

\[ \leq \delta(n, k, l) M(T_z) \wedge r_{z}^{-n-1}. \]
Since $\delta(n, k, n-k) = 0$, we assume $k+l > n$. Fix $\varphi \in E^k(R^n \sim \{0\})$ such that $M(\varphi) \leq 1$, and use 5.4 to write $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in D^k(0)$ and $\varphi_2 \in D^{k-1}(0) \wedge \text{dr}$. Noting that $M(\varphi_1) \leq 1$ and $M(\varphi_2) \leq 1$, we obtain from 5.3

$$l_\alpha(l) H^p(I) T(r^{-n} \varphi) = \gamma^2(n-1, k, l-1) - 1 \int T \cap g_\# B(\varphi_1 \cap g^{-1}_\# \psi_\varphi) dH^n g$$

$$+ \gamma^2(n-1, k-1, l-1) - 1 \int T \cap g_\# B(\varphi_2 \cap g^{-1}_\# \psi_\varphi) dH^n g$$

$$\leq l_\alpha(l) \delta(n, k, l) - 1 H^p(I) V^T(0).$$

(2) follows from 5.6 together with 5.7 with $R^n$ replaced by $T_x(r S^{n-1})$.

Suppose $l = n-k$. Consider $0 \neq x \in R^n$. Inasmuch as

$$T^{-}(x) \wedge g_\# B^{-} = \tau(x) \wedge g_\# B^{-}$$

for $g \in I_x g_x$, 5.7 implies that

(*)

$$\gamma^2(n-1, k, n-k-1) \leq \gamma_0(x) \leq \gamma(n-1, k, n-k-1),$$

with the second inequality becoming equality if $\tau(x)$ is simple. Moreover,

$$|x|^{-k} \tau(x) = \rho_\varphi(x)[T^{-}(x)],$$

and if $T$ is rectifiable we infer from 5.6 and [5, 8.16] that

$$V^T(0) = (n-k) \alpha(n-k) \gamma(n-1, k, n-k-1) \int_{R^n \sim \{0\}} |x|^{-k} |\tau(x)| d\|T\| x$$

(**)

$$= (n-k) \alpha(n-k) \gamma(n-1, k, n-k-1) \int_{A} |\rho_\varphi(x)[T^{-}(x)]| \Theta^k(x) dH^k x,$$

where $A$ is the union of a countable family of proper $k$ dimensional submanifolds of class 1 of $R^n$ and $T^{-}(x)$ is tangent to one of these submanifolds for $\|T\|$ almost all $x$. (3) now follows from application of the area formula [4, 3.2.3] to $\rho|A$.

We next observe that

$$\sup \{T(r^{-k} \varphi) : \varphi \in D^k(0), M(\varphi) \leq 1\}$$

$$= \sup \left\{ \int r^{-k} \langle \tau, \varphi \rangle d\|T\| : \varphi \in D^k(0), M(\varphi) \leq 1 \right\}$$

$$= \sup \left\{ \int r^{-k} \langle \tau, \psi \rangle d\|T\| : \psi \in D^k(R^n \sim \{0\}), M(\psi) \leq 1 \right\}$$

$$= \int r^{-k} \gamma_0 \|\tau\| d\|T\|,$$

and similarly,

$$\int r^{-k} \gamma_0 \|\tau\| d\|T\| = \sup \{T(\gamma_0 r^{-k} \varphi) : \varphi \in D^k(0), M(\varphi) \leq 1\}.$$
In order to prove (5) it suffices to prove lower semicontinuity at 0. We can assume \( V^T(0) > 0 \). Fix \( 0 < \varepsilon < V^T(0) \) and choose \( \varphi \in D^0(0) \) with
\[
\varepsilon < (n-k)\alpha(n-k)T(\gamma_0 r^{-k}\varphi) \leq V^T(0).
\]
Inasmuch as \( \varphi_y = (-y, e)^\Theta \varphi \in D^k(y) \) for \( y \in \mathbb{R}^n \),
\[
\lim_{y \to 0} \gamma_y(x)r^{-k}(x)\varphi_y(x) = \gamma_0(x)r^{-k}(x)\varphi(x)
\]
for \( \|T\| \) almost all \( x \in \mathbb{R}^n \), and \( \|\gamma_y r^{-k}\varphi_y\| \leq M(r^{-k}\varphi) \), we can use Lebesgue's dominated convergence theorem to conclude that
\[
\liminf_{y \to 0} V^T(y) \geq \liminf_{y \to 0} (n-k)\alpha(n-k)T(\gamma_y r^{-k}\varphi_y)
\]
\[
= (n-k)\alpha(n-k)T(\gamma_0 r^{-k}\varphi) > \varepsilon,
\]
whence
\[
\liminf_{y \to 0} V^T(y) \geq V^T(0).
\]

5.10. Remark. Suppose \( S \) is the restriction of \( H^n \) to a bounded Borel set \( C \) and \( \varphi_0 \) is the unit \( n-1 \) form on \( S^{n-1} \) such that \( dr \wedge \varphi_0 \) is positively oriented. Then for \( h \in \mathcal{D}^0(z) \), \( \partial S(h\rho_0^\Theta \varphi_0) \) is the double layer potential \( W_h(z) \) considered by Král in [6]. In this case 5.3 (with \( T = \partial S \)) reduces to the formula derived in [6, 2.5]. Furthermore, if \( M(\partial S) = P(C) < \infty \), then 5.9(4) reduces to the formula given for \( v^*_m \) in [6, 1.6].

5.11. Corollary. Consider \( T \in F_k(\mathbb{R}^n) \).

(1) If there exist \( z_1, \ldots, z_{k+2} \in \mathbb{R}^n \) which do not lie on a \( k \)-plane, such that for \( i = 1, \ldots, k+2 \),
\[
(spt T) \cap (z_i + gB) = \emptyset
\]
for \( H^m \) almost all \( g \in SO(n) \), then
\[
T = 0.
\]

(2) If \( k + l > n \) and there exists \( z \in \mathbb{R}^n \) such that for \( H^m \) almost all \( g \in SO(n) \),
\[
(spt T) \cap (z + gB)
\]
has measure 0 with respect to \( k + l - n \) dimensional integralgeometric measure in \( z + g(B) \) (in particular, if this set is \( \emptyset \), then \( T = 0 \)).

Proof. Consider (1) and assume \( l = n - k \). By 4.2(1), \( V^T(z_i) = 0 \) for \( i = 1, \ldots, k+2 \). Let \( \Pi_j \) denote the \( k \)-plane containing \( \{z_k : k \neq j\} \), and fix \( \varphi \in D^k(\mathbb{R}^n - \Pi_j) \). Recalling 5.4(2) and applying 5.3 we conclude that \( T(\varphi) = 0 \). Inasmuch as
\[
\bigcup \{ \mathbb{R}^n \sim \Pi_j : j = 1, \ldots, k+2 \} = \mathbb{R}^n,
\]
the proof is complete.

Consider (2). From [4, 4.1.20] it follows that \( T \cap (z + gB) = 0 \) for \( H^m \) almost all \( g \in SO(n) \), hence \( V^T(z) = 0 \), and therefore \( T = 0 \) by 5.9(1).
5.12. Corollary. Suppose $T \in F_k(\mathbb{R}^n)$, $A \subset \mathbb{R}^n$, and the hypothesis of either 5.10(1) or (2) is satisfied with "spt $T"$ replaced by "(spt $T$) $\cap$ A".

1. If $A$ is open, then $\text{spt} \ T \subset \mathbb{R}^n - A$.
2. If $A$ is a Borel set and $M(T) < \infty$, then $\|T\|(A) = 0$.

Proof. One argues as in the proof of [4, 4.1.21], using 5.11 in place of [4, 4.1.20].

5.13. Remark. Theorem 6.6 in [1] is erroneous. However, the following is true:

Let $X, G$ and $\Psi$ be as in [1, §6]. Consider $S \in N_k(X)$ and $T \in N_l(X)$.

1. If $k > n/2$ or $l > n/2$, then

$$
\left(2n-k-l \over n-k\right)^{-1} \gamma(n, k, l) M(S) M(T) \leq \int G M(S \cap g \# T) d\Psi g.
$$

If $S^{-}$ or $T^{-}$ is simple, then this is true for arbitrary $k$ and $l$, and the factor $(2n-k-l)^{-1}$ can be omitted.

2. $\int_G M(S \cap g \# T) d\Psi g \leq \gamma(n, k, l) M(S) M(T)$.

If $S^{-}$ and $T^{-}$ are simple, then equality holds. If for $x$ in a set of positive $\|S\|$ measure, $S^{-}(x)$ is not simple and the smallest linear subspace of $T_x(X)$ containing $\ast S^{-}(x)$ has dimension not greater than $2n-k-l+1$, or if an analogous condition holds for $T^{-}$, then the inequality is strict.

These statements follow from use of 5.7 in conjunction with [1, 5.5].

5.14. Remark. We will now examine the relationship between the intersections defined in 4.1 and those defined in [1, §4].

First note that the slicing theory in [4, §4.3] can be used to extend the intersection theory in [1, §4] to flat chains having finite mass.

Fix $T \in F_k(\mathbb{R}^n)$, $M(T) < \infty$, let $U$ be a compact neighborhood of spt $T$, and use [4, 4.1.23] to obtain polyhedral chains $T_1, T_2, \ldots$ lying in $U$ such that $M(T_i) \leq 2M(T)$ and $T_i \to T$ as $i \to \infty$. Fix $u_1 \in D^0(\mathbb{R}^n - U)$, $M(u_1) \leq 1$, $u_2 \in D^0[SO(n)]$, $M(u_2) \leq 1$, and $\psi \in D^{k+1-n}(\mathbb{R}^n)$. Denoting intersection in the sense of [1] by $\land_1$, we use [1, 4.4(4) and 4.9] to conclude that whenever $\rho > \text{diam} (U \cup \text{spt} u_1)$,

$$
T_i \land_1 (z+g \# B) = T_i \land_1 [z+g \# (B \cap \{x : |x| < R\})]
$$

for $H^n \times H^n$ almost all $(z, g) \in (\text{spt} u_1) \times SO(n)$, hence by [1, 4.4(3)]

$$
T \land_1 (z+g \# B) = T \land_1 [z+g \# (B \cap \{x : |x| < R\})]
$$

for $H^n \times H^n$ almost all $(z, g) \in (\text{spt} u_1) \times SO(n)$. Therefore, by [1, 4.4(2)]

$$
\mathcal{F} = \int_{\mathbb{R}^n \times SO(n)} T \land_1 (z+g \# B)(\psi)u_1(z)u_2(g) \, dH^n \times H^m (z, g) < \infty,
$$

and Fubini’s theorem applies to give

$$
\mathcal{F} = \int_{\mathbb{R}^n} \int_{SO(n)} T \land_1 (z+g \# B)(\psi)u_1(z)u_2(g) \, dH^m g \, dH^n z.
$$
By [1, 4.4(4)] and 4.2(3) we have for $H^n$ almost all $z \in \mathbb{R}^n \sim U$,

$$(-1)^cT_i \cap (z + g\# B) = T_i \cap (z + g\# B)$$

for $H^m$ almost all $g \in SO(n)$, where $c = n(k + l) + kl$. We use [1, 4.4(3), 4.4(2)] and Fubini’s theorem to infer that

$$(-iy)^n \lim_{t \to \infty} \int_{\mathbb{R}^n} \int_{SO(n)} T_i \cap (z + g\# B)(\psi)u_i(z)v_2(g) \, dH^m g \, dH^n z.$$ 

Now by 5.9(1) each of the inner integrals is not greater in absolute value than

$$
la(l)2M(T)M(\psi) \text{ dist} (spt u_1, U)^{1-n}|u_1(z)|H^n[SO(n)],
$$

hence by 4.2(2) we can apply Lebesgue’s dominated convergence theorem to obtain

$$(-1)^cT_i = \lim_{t \to \infty} \int_{\mathbb{R}^n} \int_{SO(n)} T_i \cap (z + g\# B)(\psi)u_i(z)v_2(g) \, dH^m g \, dH^n z.$$ 

We therefore conclude that for $H^n$ almost all $z \notin spt T$,

$$(-1)^cT \cap (z + g\# B) = T \cap (z + g\# B)$$

for $H^m$ almost all $g \in SO(n)$.

Next suppose $T$ lies in the image of $E_k(S^{n-1})$ under the map induced by the inclusion of $S^{n-1}$ in $\mathbb{R}^n$. Orient $S^{n-1}$ so that for $x \in S^{n-1}$, $x \wedge (S^{n-1})^\ast(x)$ is positively oriented in $\mathbb{R}^n$. Denote

$$B' = B \cap S^{n-1};$$

orient $B'$ so that for $x \in B'$, $x \wedge B'^\ast$ is positively oriented in $B$. Then for $H^m$ almost all $g \in SO(n)$,

$$(-1)^cT \cap g\# B = -T \cap g\# B',$$

as defined in [1, §4] with $X = S^{n-1}$, $G = SO(n)$.

In order to show this, we infer from [4, 4.1.23], [1, 4.4(3) and (4)], and 4.2(2) and (3) that it is only necessary to show that for $x \in B'$ and $\nu \in \wedge_k [T_x(S^{n-1})]$,

$$(-1)^c\nu \cap B'^\ast = -\nu \cap_0 B'^\ast,$$

where $\cap_0$ refers to use of the star operator in $\wedge_k [T_x(S^{n-1})]$. One verifies this directly from the definitions of the star operators.


6.1. Lemma. Suppose $S \in F_{k+1}(\mathbb{R}^n)$, $M(S) < \infty$, and $T$, $U \in E_k(\mathbb{R}^n)$ with

$$\partial S = T + U, \quad M(T) < \infty,$$

and for some $z \in \mathbb{R}^n$,

$$(spt U) \cap (z + gB) = \emptyset$$

and for some $z \in \mathbb{R}^n$,
for $H^m$ almost all $g \in \text{SO}(n)$. Then

$$
\gamma^2(n-1, k, l-1)\partial S(\varphi) \leq M(T)M(\varphi) \quad \text{for} \ \varphi \in D^k(\omega),
$$

$$
\gamma^2(n-1, k-1, l-1)\partial S(\varphi) \leq M(T)M(\varphi) \quad \text{for} \ \varphi \in D^{k-1}(\omega) \land \partial S(\varphi) \land dr_n, \ k+l > n.
$$

**Proof.** Our proof was suggested by the proof of [3, 2.2].

We can assume $z = 0 \notin (\text{spt} \ S) \cup (\text{spt} \ T)$.

Using local coordinates for the bundle $\mathcal{A}$ in conjunction with a suitable partition of unity for $\mathbb{R}^n \sim \{0\}$, one applies [1, 3.3] to verify that

$$
\text{spt} \big[ \partial S(\varphi) - L_{\mathcal{A}}(T) \big] \subset C = p^{-1}(\text{spt} \ U) \cap \Phi.
$$

Choose $h_1, h_2, \ldots \in E^0(\text{SO}(n))$ such that for $j = 1, 2, \ldots$,

$$
\text{spt} \ h_j \subset \text{SO}(n) \sim q(C), \quad 0 \leq h_j \leq 1,
$$

and $\lim_{j \to \infty} h_j(g) = 1$ for $g \in \text{SO}(n) \sim q(C)$. Then for each $j = 1, 2, \ldots$,

$$
\partial [L_{\mathcal{A}}(S) \land q^h(h, \omega)] = \partial L_{\mathcal{A}}(S) \land q^h(h, \omega) = L_{\mathcal{A}}(T) \land q^h(h, \omega),
$$

because $d(h, \omega) = 0$ and

$$
\text{spt} \ q^h(h, \omega) \land \text{spt} \big[ \partial L_{\mathcal{A}}(S) - L_{\mathcal{A}}(T) \big] = \emptyset.
$$

Now $C' = q^{-1}[q(C)]$ is closed and $H^m[q(C')]=0$, hence for $H^m$ almost all $g \in \text{SO}(n)$, $\|L_{\mathcal{A}}(S), q|\Phi, g\|(C')=0$, and therefore by [4, 4.3.2(2)],

$$
\|L_{\mathcal{A}}(S) \land q^h(\omega)\|(C') = 0.
$$

We apply Lebesgue’s dominated convergence theorem and [1, 3.5(2)] to obtain for $\psi \in D^{k+1-n}(\Phi)$,

$$
L_{\mathcal{A}}(S)[(q^\omega) \land d\psi] = \lim_{j \to \infty} L_{\mathcal{A}}(S)[(q^h h_j, \omega) \land d\psi]
$$

$$
= \lim_{j \to \infty} L_{\mathcal{A}}(T)[(q^h h_j, \omega) \land \psi]
$$

$$
= L_{\mathcal{A}}(T) \cap (\Phi \sim C')[(q^\omega) \land \psi].
$$

Consider $\varphi \in D^k(0)$ and define $\Xi \in D^{k+1-n}(\Phi)$ as in the proof of 5.3. Then from [4, 4.3.2(1)], 3.6, 4.2(5) and 5.3 we have

$$
(-1)^t \partial [L_{\mathcal{A}}(S) \land q^\omega](r^{n-1} \circ p \Xi) = (-1)^t \int_{\text{SO}(n)} \partial (L_{\mathcal{A}}(S), q|\Phi, g)(r^{n-1} \circ p \Xi) \ dH^m g
$$

$$
= \int_{\text{SO}(n)} \partial S \cap g_B(r^{n-1} \varphi \cap g^{-1} \varphi_B) \ dH^m g
$$

$$
= io(l)H^m(I)\gamma^2(n-1, k, l-1)\partial S(\varphi),
$$

where $t = (k+l-n)(m+l+1)+m$. Finally, we conclude using 3.4(ii), 3.4(i), 5.2, 3.6 and 5.1 that this is equal to

$$
(-1)^t(L_{\mathcal{A}}(T) \land q^\omega) \cap (\Phi \sim C')(r^{n-1} \circ p \Xi) \leq io(l)H^m(I)M(T)M(\varphi).
$$
Verification of the inequality for $\varphi \in D^{k-1}(z) \wedge dr_z$, $k + l > n$, is analogous.

6.2. Theorem. Assume $S \in F_{k+1}(\mathbb{R}^n)$, $M(S) < \infty$, and $z \in \mathbb{R}^n$.

1. $\delta(n, k, l)M(\partial S) \leq 4 \binom{n}{k} V^S(z) \sup \{|x - z| : x \in \text{spt}\ \partial S\}^{n-l}$.

2. Under the hypothesis of 6.1,

$$\delta(n, k, l)M(\partial S) \leq 4 \binom{n}{k} l \alpha(l)M(T).$$

3. Suppose $\|S\|[(r^{n-k})] < \infty$, $h \in E^0(\mathbb{R}^n)$, $\varphi \in E^\alpha(S^{n-1})$, $M(h) \leq 1$ and $M(\varphi) \leq 1$. Then

$$\gamma^2(n-1, k, n-k-1)S[d(h \wedge \rho^k\varphi)]$$

$$\leq 4V^S_n(z) + \gamma^2(n-1, k+1, n-k-2)^{-1}h(z)M(d\varphi)V^S_{n-k-1}(z).$$

($V_i$ indicates use of $l$ planes; the second term is 0 if $k+1 = n$.)

Proof. We can assume that $z = 0$.

Consider $h \in E^0(\mathbb{R}^n)$ such that $h(0) = 0$. Corresponding to each $j = 1, 2, \ldots$ we choose $e_j > 0$, $\delta_j > 0$ and $x_j \in \mathbb{R}^n$ such that

$$\lim_{j \to \infty} e_j = \lim_{j \to \infty} \delta_j = 0, \quad |h(x_j)| = \sup \{|h(x)| : |x| \leq \delta_j\} = e_j,$$

and $|x_j| = \delta_j$. Using the mean value theorem, we see that we can assume that

$$\lim_{j \to \infty} \frac{e_j}{\delta_j} = |h(0)| \quad \text{and} \quad 2e_j \geq |h(0)| \delta_j.$$ 

There exists $\eta_j \in E^0(\mathbb{R})$ such that

$$\eta_j(r) = 0 \quad \text{for} \quad r \leq \delta_j/j,$$

$$\eta_j(r) = (1 - j^{-1})e_j^{-1} \delta_j |h(0)| \quad \text{for} \quad r \geq \delta_j,$$

$$0 \leq \eta_j(r) \leq 2|h(0)|/e_j \quad \text{for} \quad r \in \mathbb{R}.$$ 

define $h_j = \eta_j \circ r$. Then for $0 \neq x \in \mathbb{R}^n$,

$$0 \leq h_j(x) \leq 2, \quad |dh_j(x)| |h(x)| \leq 2|h(0)|,$$

$$\lim_{j \to \infty} h_j(x) = 1, \quad \lim_{j \to \infty} dh_j(x) = 0.$$

Recalling from [4, 4.1.21] that $\|S\|[0] = 0$, we write $\varphi_0 = h dx^1 \wedge \cdots \wedge dx^k$,

$$\partial S(h_j \varphi_0) = S(dh_j \wedge \varphi_0) + S(h_j d\varphi_0),$$

and use Lebesgue's dominated convergence theorem to conclude that

$$\lim_{j \to \infty} \partial S(h_j \varphi_0) = \partial S(\varphi_0).$$

It follows that $\partial S(\varphi_0) \leq 2M(\varphi_0)M[(\partial S)_z]$, hence for $\varphi \in E^\alpha(\mathbb{R}^n)$, $\varphi(0) = 0$, $M(\varphi) \leq 2$,

$$\partial S(\varphi) \leq 4 \binom{n}{k} M[(\partial S)_z].$$
We obtain this relation for arbitrary $\varphi$ with $M(\varphi) \leq 1$ by observing that $\partial S(\varphi) = \partial S(\varphi - \varphi_0)$. (1) now follows from 5.9(1).

Since $\delta(n, k, n-k)=0$, we assume $k+1>n$ for the proof of (2). Consider $\varphi \in D^k(\mathbb{R}^n \sim \{0\})$, $M(\varphi) \leq 1$. Using 5.4 to write $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in D^k(0)$ and $\varphi_2 \in D^k-1(0) \wedge d\varphi$, we apply 6.1 to $\varphi_1$ and to $\varphi_2$ to obtain $\delta(n, k, l) \partial S(\varphi) \leq la(l)M(T)$. Consequently, $\delta(n, k, l)M[(\partial S)_a] \leq la(l)M(T)$, whence follows (2).

Turning to (3), we choose $h_j$ for $h_0 = h - h(0)$ and infer from 5.3 that since $\|\rho^\varphi\| \leq r^{-k},$

$$(n-k)\alpha(n-k)\gamma(\varphi_{-k}, k, n-k-1)S[d(h, h_0 \wedge \rho^\varphi)] \leq 4V_{n-k}(0)$$

for $j=1, 2, \ldots$. As we found previously,

$S[d(h, h_0 \wedge \rho^\varphi)] \rightarrow S(dh_0 \wedge \rho^\varphi) = S(dh \wedge \rho^\varphi)$ as $j \rightarrow \infty$.

If $k+1=n$, then $d\varphi = 0$ and the proof is complete. On the other hand, if $k+1<n$, then we use 5.7, 5.6 and [4, 4.1.21] and proceed as in the derivation of equation (**) in the proof of 5.9 to infer that

$$\int_{\mathbb{R}^n} \|S^\varphi \| = M(d\varphi) \int_{\mathbb{R}^n} \|\rho^\varphi(x)S^\varphi(x)\| \|dS\|$$

$$\leq [(n-k-1)\alpha(n-k-1)\gamma^2(n-1, k+1, n-k-2)]^{-1}M(d\varphi)V_{n-k-1}(0).$$

Consequently, Lebesgue's dominated convergence theorem implies that if $V_{n-k-1}(0) < \infty$, then

$$S(h, h_0 \wedge d\rho^\varphi) \rightarrow S(h_0 \wedge d\rho^\varphi) \quad \text{as} \quad j \rightarrow \infty,$$

whence

$$(n-k)\alpha(n-k)\gamma^2(n-1, k, n-k-1)S[d(h \wedge \rho^\varphi)] \leq 4V_{n-k}(0) + \|h(0)S(d\rho^\varphi)\|,$$

and our assertion is clear.

6.3. Theorem. Suppose $S \in F_{k+1}(\mathbb{R}^n)$, $M(S) < \infty$, and $z_1, \ldots, z_{k+2} \in \mathbb{R}^n$ do not lie on a $k$-plane. If either $V_{n-k}(z_i) < \infty$ or the hypothesis of 6.1 holds for $z_i$ for each $i = 1, \ldots, k+2$, then

$$S \in N_{k+1}(\mathbb{R}^n).$$

Proof. The proof was suggested by the proof of [6, 2.10]. Let $\varphi = dx_1^i \wedge \cdots \wedge dx^k$. It suffices to show that

$$M(\partial S \wedge \varphi) < \infty \quad \text{for} \quad 1 \leq i_1 < \cdots < i_k \leq n.$$

Let $\Pi_j$ denote the $k$-plane containing $\{z_k : k \neq j\}$. Observing that

$$\{\mathbb{R}^n \sim \Pi_j : j = 1, \ldots, k+2\}$$

covers $\mathbb{R}^n$, we choose a subordinate partition of unity $\alpha_1, \ldots, \alpha_{k+2}$ with $\Pi_j \cap \text{spt } \alpha_j = \emptyset$. Thus it is sufficient to show that

$$M(\partial S \wedge \alpha_j\varphi) < \infty \quad \text{for} \quad j = 1, \ldots, k+2.$$
Consider, for instance, \( j = k + 2 \). By 5.4 there exist \( \beta_1, \ldots, \beta_{k+1} \in E^0(\mathbb{R}^n) \), \( \text{spt} \beta_i \subset \text{spt} \alpha_{k+2} \), and \( \varphi_i \in \rho^2_{E^0}([E^k(S^{n-1})]) \) such that

\[
\alpha_{k+2} = \sum_{i=1}^{k+1} \beta_i \varphi_i.
\]

Thus it is sufficient to show that

\[
M(\partial S \wedge \beta_i \varphi_i) < \infty \quad \text{for} \quad i = 1, \ldots, k+1.
\]

Fix \( h \in E^0(\mathbb{R}^n) \), \( M(h) \leq 1 \). Then from 5.3 and 4.2 we infer that

\[
|a(l)\gamma^{2(n-1)}(n-k, l-1)\partial S(h \beta_i \varphi_i) \leq V^{\partial S}(z) M(\beta_i \varphi_i) \sup \{|x-z_i| : x \in \text{spt} \partial S\}^{n-i}.
\]

Moreover, if the hypothesis of 6.1 is satisfied, then

\[
\gamma^{2(n-1}, k, l-1)\partial S(h \beta_i \varphi_i) \leq M(T)M(\beta_i \varphi_i).
\]

Therefore, \( M(\partial S \wedge \beta_i \varphi_i) < \infty \).

6.4. Remark. Let \( S \) be a rectifiable \( k+1 \) current. If one of the following conditions is satisfied, then \( M(\partial S) < \infty \), hence \( S \) is an integral current by [4, 4.2.16]:

(1) The hypothesis of 6.3.

(2) \( k+l > n \) and for some \( z \in \mathbb{R}^n \) either \( V^{\partial S}(z) < \infty \) or the hypothesis of 6.1 is satisfied.

Conversely, if \( M(\partial S) < \infty \), then from 5.9(1) it follows that \( V^{\partial S}(z) < \infty \) for \( z \notin \text{spt} \partial S \).

7. \( V^T \) for \( T \) a manifold. Let \( T \in F_{k}(\mathbb{R}^n) \), \( M(T) < \infty \), be obtained by integration over an oriented \( k \) dimensional proper submanifold of class 1, which is also denoted by \( T \). Assume \( z \in T \).

7.1. Theorem. \( \|T\|(r^{-k+1}_{k}) = \int_T r^{-k+1}_{k} dH^k < \infty \). Consequently, if \( k+l > n \), then by 5.9(1), \( V^T(z) < \infty \).

Proof. This is easily verified by introducing a bi-Lipschitzian coordinate system for \( T \) in a neighborhood of \( z \).

For the remainder of this section we assume \( k+l = n \).

7.2. Lemma.

\[
V^T(0) = (n-k)\alpha(n-k)\gamma(n-1, k, n-k-1) \int |T^{-}(x) \wedge x| |x|^{-k-1} dH^k x.
\]

Proof. Recalling equation (***) in the proof of 5.9(3), we observe that for \( x \neq 0 \),

\[
|\rho_p(x[T^-(x)]| = |T^-(x) \wedge x| |x|^{-k-1}.
\]

7.3. Example. Let \( n = 2 \) and \( k = 1 \). Whenever \( |t| < \frac{1}{2} \) define

\[
g(t) = \int_0^{|t|} \int_0^u -[ln|v|]^{-1} dH^1 v \ dH^1 u.
\]
Let $T$ be the graph of the function $F$ defined by

$$F(t) = g(t) \sin \left( \frac{1}{t} \right),$$

oriented so that

$$T^{-1}(t, F(t)) = \left[ 1 + F'(t)^2 \right]^{-1/2} [e_1 + F'(t)e_2].$$

Then $T$ is of class 1, but not of class 2, and by 7.2

$$V^T(0) \geq \int_{-\epsilon}^{\epsilon} \left| \frac{F'(t)}{t} - \frac{F(t)}{t^2} \right| \, dH^1 = \infty$$

for sufficiently small positive numbers $\epsilon$.

7.4. **Theorem.** Suppose there exist an open subset $U \subseteq \mathbb{R}^k$, a diffeomorphism $F : U \to T$ with $z \in F(U)$, and constants $A > 0$ and $0 < \alpha \leq 1$ for which

$$|F_\theta(x)(v) - F_\theta(y)(v)| < A|x - y|^\alpha$$

whenever $x, y \in U$ and $|v| \leq 1$. Then $V^T(z) < \infty$.

**Proof.** We can assume that $z = 0$, that $T_0(T)$ is spanned by $e_1, \ldots, e_k$ and, by 5.9(1), that $T = F(U)$. Let $P : \mathbb{R}^k \to T_0(T)$ be the orthogonal projection. It is easy to see that the condition on $F_\theta$ holds (with a different $A$) for $(F \circ \varphi)_\theta$ where $\varphi$ is a Lipschitzian diffeomorphism of an open subset $V$ of $\mathbb{R}^k$ onto $U$ and

$$\sup \{|\varphi_\theta(x)(v)| : x \in V, |v| \leq 1\} < \infty.$$ 

Thus we can assume that $U = \mathbb{R}^k \cap \{x : |x| < \epsilon\}$,

$$F^{-1} = P_0 = P|T$$

and the Jacobian $J_\theta P_0 > \frac{1}{2}$. Observe that

$$\sigma = \sup \{|F_\theta(x)(v)| : x \in U \text{ and } |v| \leq 1\} < \infty.$$ 

Then by 7.2 and the area formula [4, 3.2.3],

$$[(n-k)\alpha(n-k)-1]^{-1} V^T(0)$$

$$\leq 2 \int_0^{\epsilon} |T^{-1}[F(x)] \wedge F(x)| |F(x)|^{-2}|x|^{-k+1} \, dH^k x$$

$$= 2 \int_0^{\epsilon} \int_0^{\theta} |T^{-1}[F(t\theta)] \wedge F(t\theta)| |F(t\theta)|^{-2} \, dH^1 t \, dH^{k-1} \theta$$

$$\leq 2 \int_0^{\epsilon} \int_0^{\theta} |F_\theta(t) \wedge F_\theta(t)| |F_\theta(t)|^{-2} \, dH^1 t \, dH^{k-1} \theta,$$

where $F_\theta(t) = F(t\theta)$. Thus fix $\theta$ and observe that in evaluating the inner integral we can assume that

$$F_\theta(t) = te_1 + \varphi(t),$$

where $\varphi(t)$ is a smooth function.
where \( \varphi(t) \) involves only \( e_{k+1}, \ldots, e_n \) and \( \varphi'(0) = 0 \). It is a routine matter to verify with the help of the mean value theorem that for \( 0 < t < \epsilon \),

\[
\frac{|F_\varphi'(t) \wedge F_\varphi(t)| |F_\varphi(t)|^{-2}}{\left(1 + \sigma t^{-2}\varphi(t) + At^{-1}\right)} \leq A + (1 + \sigma)(n - k)A \leq \left[ A + (1 + \sigma)(n - k)A \right] t^{-1},
\]

whence \( V^\sigma(0) < \infty \).

**References**


**Indiana University,**

**Bloomington, Indiana 47401**