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INFINITE PRODUCTS WHICH ARE HILBERT CUBES

BY
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Abstract. Let Q denote the Hilbert cube. It is shown that if P and P' are compact polyhedra of the same simple homotopy type then $P \times Q$ and $P' \times Q$ are homeomorphic. A consequence of this result is that the Cartesian product of a countable, locally finite simplicial complex with a separable, infinite-dimensional Fréchet space is a manifold modelled on the Fréchet space. It is also proved that a countably infinite product of nondegenerate spaces is a Hilbert cube provided that the product of each of the spaces with the Hilbert cube is a Hilbert cube. Together with the first result, this establishes that a countably infinite product of nondegenerate, compact, contractible polyhedra is a Hilbert cube. In addition, a proof is given of the (previously unpublished) theorem of R. D. Anderson that a countably infinite product of nondegenerate dendra is a Hilbert cube.

Introduction. In 1964 [1], R. D. Anderson proved that the Cartesian product of a triod with the Hilbert cube is itself homeomorphic to the Hilbert cube. He later generalized this result to the extent of showing that the product of countably infinitely many dendra is a Hilbert cube (a dendron is a nondegenerate, uniquely arcwise connected Peano continuum) and, together with R. H. Bing in [5], conjectured that the product of countably infinitely many nondegenerate, contractible, compact polyhedra is homeomorphic to the Hilbert cube.

In this paper it is proved (Theorem 5.2) that the products of two compact polyhedra with the Hilbert cube are homeomorphic if the polyhedra have the same simple homotopy type. Another theorem (Theorem 6.2) establishes that a countably infinite product of nondegenerate spaces is a Hilbert cube provided that the product of each with a Hilbert cube is. Taken together as Corollary 6.1, these two results give an affirmative answer to the above Anderson-Bing conjecture. A corollary (Corollary 5.3) to Theorem 5.2 establishes that the product of a separable, infinite-dimensional Hilbert space with a polyhedron triangulable by a countable, locally finite, simplicial complex is homeomorphic to an open subset of that Hilbert space. This is the converse of a theorem of D. W. Henderson [10], and consists essentially of settling Conjecture 2 of [9] (attributed to R. D. Anderson) and then applying another theorem of Henderson which says that all separable, paracompact manifolds modelled on infinite-dimensional Hilbert spaces are open subsets of the Hilbert spaces [11]. Because of work of N. H. Kuiper and D. Burghelea [12] and N.

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Moulis [13], all homotopy-equivalent open subsets of separable, infinite-dimensional Hilbert spaces are C^∞-diffeomorphic, hence homeomorphic. Corollary 5.4 combines Corollary 5.3 with this result to give that the products of two polyhedra with a separable, infinite-dimensional Hilbert space are homeomorphic if the polyhedra are of the same homotopy type and are triangulable by countable, locally finite, simplicial complexes.

The author wishes to express his thanks and acknowledge his debt to R. D. Anderson for several conversations on the subjects covered in this paper. These conversations date from February of 1966, when Anderson presented an exposition of his technique of proof that the product of a triod with a Hilbert cube is a Hilbert cube. At the same time, he mentioned the existence of Theorem 6.1 and recommended as a problem Corollary 5.1. Neither Theorem 6.1 nor Theorem 6.3 (the two announced in [1]) has appeared in print, and the author wishes to thank Professor Anderson for suggesting that they be included in this paper (with the author's proofs).

Concerning methods of proof, the technique used by Anderson to prove that the product of a finite tree with the Hilbert cube is a Hilbert cube was the following: He found two sequences $\{\mathscr{U}_i\}_{i=1}^\infty$ and $\{\mathscr{V}_i\}_{i=1}^\infty$ of finite closed covers of the product of the tree with the Hilbert cube and of the Hilbert cube, respectively. The covers were chosen so that $\mathscr{U}_{i+1}$ and $\mathscr{V}_{i+1}$ refined $\mathscr{U}_i$ and $\mathscr{V}_i$, respectively, that $\lim_{i\to\infty} \text{mesh}(\mathscr{U}_i) = 0 = \lim_{i\to\infty} \text{mesh}(\mathscr{V}_i)$, and that there were bijections ϕ_i of $\mathscr{U}_i$ onto $\mathscr{V}_i$, for each i, preserving incidence of cover elements and respecting the refinements (that is, if $U_{i+1}\subset U_i$, then $\phi_{i+1}(U_{i+1})\subset\phi_i(U_i)$). The sequence $\{\phi_i\}_{i=1}^\infty$ then in the limit determined a homeomorphism.

This was the method Anderson expected would be used in the more general case of the product of a compact contractible polyhedron with a Hilbert cube. The author, however, used instead the methods developed in §4.

The author wishes to note that he has been informed by R. D. Anderson that *he* has heard by letter that Andrej Szankowski, a student of Pełczyński, has recently also proved Corollary 6.1, if each polyhedron is a triod [14].

Notational conventions. If A is a subset of the positive integers N and if $\{X_i\}_{i\in A}$ is an indexed collection of spaces, then $\prod_{i\in A} X_i$ will denote the Cartesian product, and $p_i\colon \prod_{i\in A} X_i \to X_i$, the projection. Often, it will be convenient to denote $p_i(x)$ by x_i. If $\{X, Y, Z, \ldots\}$ is a nonindexed collection of spaces, than $X\times Y\times Z\times\cdots$ will denote the Cartesian product and $p_X, p_Y, p_Z, \ldots$, the projections. Also, $p_X(x)$ will often be written as x_X. The unit interval $[0, 1]$ will be denoted by I, and the Hilbert cube, by Q and by $\prod_{i>0} I_i$.

The symbol $d(\ ,\ )$ will be used for all metrics, with the following conventions understood to prevent ambiguity:

(1) If $\{X_i\}_{i\in A}$ is a collection of metric spaces indexed by a subset of N, then the metric for each X_i will be understood to be bounded so that the diameter of X_i,

dia (X_i), is less than or equal to one. Also, the metric d on $\prod_{i\in A} X_i$ will be the one given by the formula $d(x, y)=\sum_{i\in A} 2^{-i}d(x_i, y_i)$, for x and y points of $\prod_{i\in A} X_i$.

(2) Where a product of finitely many nonindexed spaces is considered, the metric on the product space will be taken to be given by the sum of the metrics on the individual factors.

(3) For functions from a space X into a bounded metric space Y, the metric used will be that of uniform convergence, i.e., $d(f, g)=\sup_{x\in X} d(f(x), g(x))$.

2. **Local homotopy negligibility.** This brief section contains an exposition, sufficient for the present purpose, of a condition which has been exploited recently to obtain many homogeneity results.

Following Anderson [4], we say that a closed subset A of a space X has *Property Z in X* if for each nonnull, homotopically trivial open set U of X, the set $U\backslash A$ is also nonnull and homotopically trivial. The principal result concerning Property Z that will be required is the following theorem of Anderson [4]:

THEOREM 2.1. *Any homeomorphism between two closed subsets of the Hilbert cube which have Property Z may be extended to a homeomorphism of the Hilbert cube onto itself.*

The next three lemmas provide easy criteria for determining that a set has Property Z.

LEMMA 2.1. *A finite union of sets which have Property Z has Property Z.*

Proof. Let $A_1, \ldots, A_n$ be subsets of X with Property Z. If U is any nonnull, homotopically trivial open set of X, then $U\backslash A_1$ is also nonnull, open, and homotopically trivial, and, by induction, $U\backslash\bigcup_{i=1}^n A_i=(((U\backslash A_1)\backslash A_2)\backslash\cdots\backslash A_n)$ is, too.

LEMMA 2.2. *A closed subset A of a metric space X has Property Z in X if for all* $\varepsilon>0$ *there is a homotopy* $F_\varepsilon\colon X\times I\to X$ *such that*

(1) $F_\varepsilon(x, 0)=x$ *for all x in X*,

(2) $F_\varepsilon(X\times\{1\})\cap A=\varnothing$, *and*

(3) dia $F_\varepsilon(\{x\}\times I)<\varepsilon$, *for all x in X.*

Proof. Let U be any nonnull, homotopically trivial, open set in X, and let f be any map of the unit n-sphere S^n (in R^{n+1}) into $U\backslash A$. Choose an extension $\bar f$ of f to a map of the $(n+1)$-ball B^{n+1} into U, which must exist by the homotopy triviality of U. Now let

$$\varepsilon = \tfrac{1}{2}\min\left[\inf\{d(f(a), x) \mid a\in S^n, x\in(X\backslash U)\cup A\},\ \inf\{d(\bar f(b), x)\mid b\in B^{n+1}, x\in X\backslash U\}\right].$$

Since B^{n+1} and S^n are compact, $\varepsilon>0$. Let $\delta\in(0, 1)$ be small enough that for a and b in B^{n+1}, $d(a, b)\leqq\delta$ implies that $d(\bar f(a), \bar f(b))\leqq\varepsilon$. Let F_ε be a homotopy which "uncovers" A as hypothesized, and define $\hat f\colon B^{n+1}\to U\backslash A$ by the formula $\hat f(a)$

$=F_\varepsilon(\bar{f}(a), \min\{1, (1-\|a\|)/\delta\})$, where $\|a\|$ is the distance from a to the origin of R^{n+1}. Now, indeed, $\hat{f}|S^n=f$. Also, for a in B^{n+1} with $\|a\| \geqq 1-\delta$,

$$d(\hat{f}(a), f(a/\|a\|)) < \varepsilon \leqq \inf\{d(f(a/\|a\|), x) \mid x \in (X\backslash U) \cup A\},$$

so $\hat{f}(a) \in U\backslash A$. Finally, for a in B^{n+1} with $\|a\| \leqq 1-\delta$, $\hat{f}(a)=F_\varepsilon(\bar{f}(a), 1) \notin A$. Thus $\hat{f}(B^{n+1}) \subset U\backslash A$ and $U\backslash A$ is homotopically trivial. Since A can contain no nontrivial open set, it must have Property Z.

LEMMA 2.3. *If X and Y are absolute neighborhood retracts and X has a basis of contractible open sets, then for each subset A of X with Property Z in X, the set $A \times Y$ has Property Z in $X \times Y$.*

Proof. Eells and Kuiper [8] proved that for any absolute neighborhood retract B and any closed subset C of B with the property that each point of C has a basis of neighborhoods U (in B) such that the inclusion $U\backslash A \to U$ is a homotopy equivalence, the global inclusion $B\backslash C \to B$ is also a homotopy equivalence. The statement that A has Property Z in X is merely the statement that A is nowhere dense in X and that for each homotopically trivial open set U of X, the inclusion $U\backslash A \to U$ is a weak homotopy equivalence (that is, induces isomorphisms on all homotopy groups). In an absolute neighborhood retract, however, all weak homotopy equivalences are homotopy equivalences (a theorem of J. H. C. Whitehead [16]). Since all open sets of absolute neighborhood retracts are absolute neighborhood retracts, the inclusion $U\backslash A \to U$ is a homotopy equivalence. Therefore, for any open set V of Y, the inclusion $(U \times V)\backslash(A \times Y) \to U \times V$ is a homotopy equivalence, so each open set W of $X \times Y$ has a basis of open sets $U \times V$ such that the inclusion $(U \times V)\backslash(A \times Y) \to U \times V$ is a homotopy equivalence. The theorem of Eells and Kuiper now gives the desired result, namely, that for any nonnull, homotopically trivial open set W of $X \times Y$, $W\backslash(A \times Y)$ is also homotopically trivial.

3. **Informalities.** In the next section, the approximation method used later in §5 is developed. Before launching into this, however, an informal, illustrated discussion of the problems involved may be of some use to the reader.

The simplest nontrivial example is the triod T, which may be represented as that subset of $J_0 \times I$ $(=[1, 1] \times [0, 1])$ consisting of all points whose first coordinate is zero or whose second coordinate is one. Let $Z \subset T \times I$ be $T \times \{0\} \cup (J_0 \times \{1\}) \times I$. Using Theorem 2.1, it is easily seen that $Z \times Q$ is homeomorphic to Q. (See the first paragraph of the proof of Theorem 5.1 for the method.) Therefore, one might try showing $T \times Q$ homeomorphic to $Z \times Q$, which is the approach adopted in this paper, is the origin of §4, and is achieved in Theorem 5.1.

A first idea might be to consider a sequence $\{Z_i\}_{i=1}^{\infty}$ of copies of $Z \times Q$ in $T \times Q$ where Z_i is restricted in the ith coordinate of Q instead of in the I-coordinate of $T \times I$. Then this sequence of homeomorphisms of $Z \times Q$ onto successive Z_i's might be considered: The first would consist of replacing the I_1-coordinate of Q by the

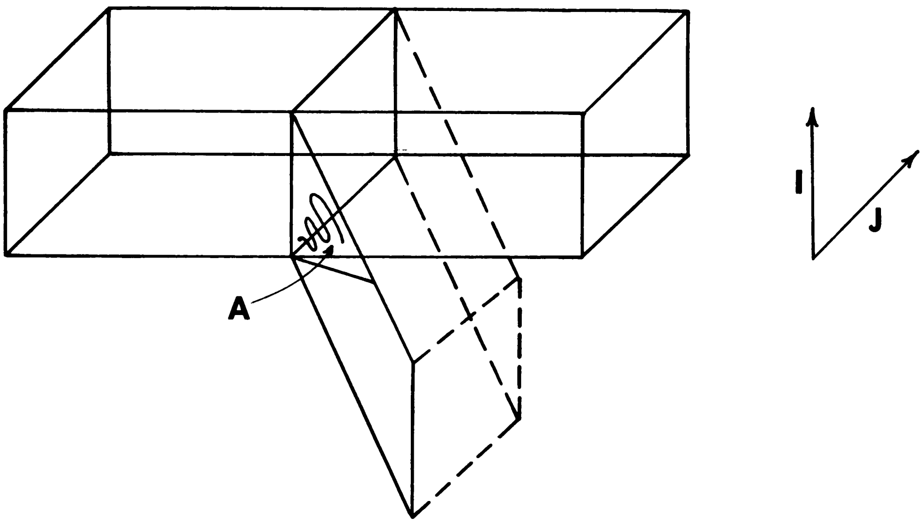

FIGURE 1. Z_2 (solid lines) in $T \times I \times J$.

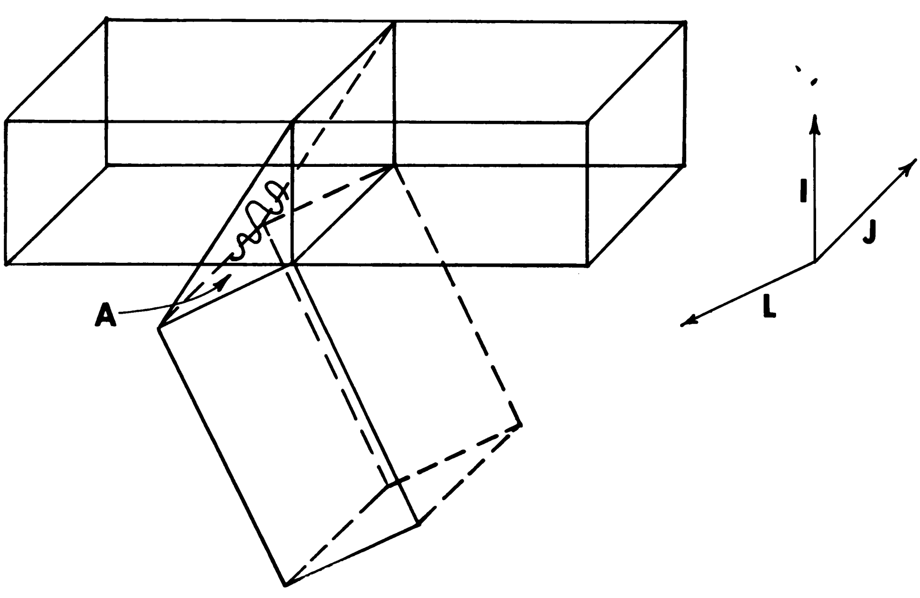

FIGURE 2. Image after "projecting" ($\bar{g}_\varepsilon$).

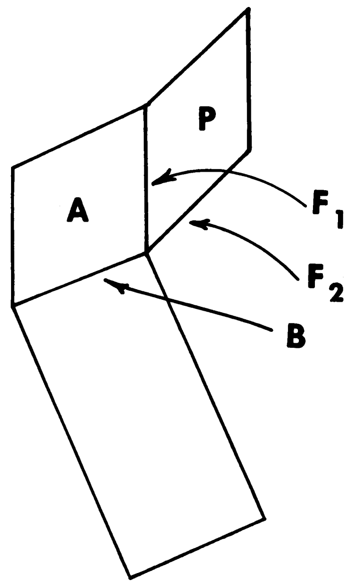

FIGURE 3. Image after straightening A (α_ε) and "folding flanges" into central plate P (ψ_ε). (Motion pointwise fixed on B, F_1, and F_2.)

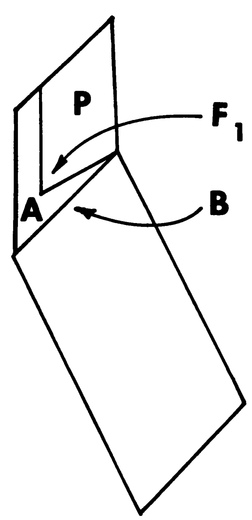

FIGURE 4. After absorbing A into P carrying B onto F_2 (β_ε).

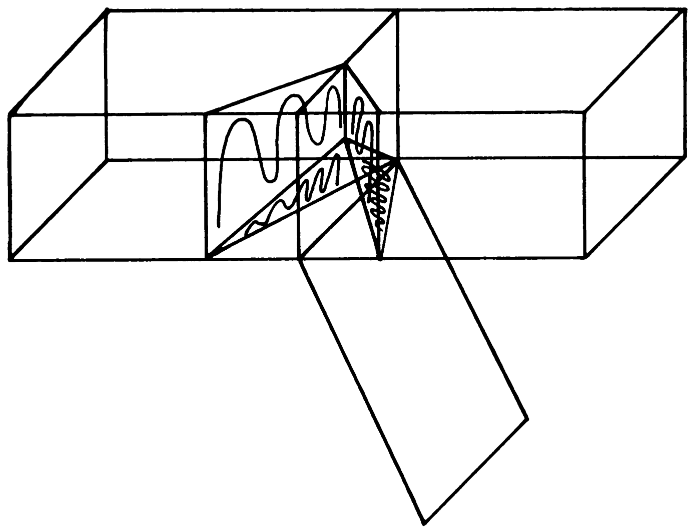

FIGURE 5. After unfolding flanges (ψ_ε^{-1}). Final image of Z_2 ($h_\varepsilon \circ \bar{g}_\varepsilon(Z_2)$), indicating images of A and F_1.

A = the closed "volume",
F_1 = its boundary in the "3-cell", indicated here by shading.

I-coordinate in $T\times I$ of Z, leaving the T-coordinate alone, and sending the I_i-coordinate (in Q) of $Z\times Q$ to the $(i+1)$th coordinate of Q in Z_1. The second homeomorphism might consist of following the first by the homeomorphism which merely exchanges the first and second coordinates of Q. The third would then consist of following the second by the homeomorphism which exchanges the second and third coordinates of Q, and so forth. The constructed sequence of homeomorphisms converges to a map of $Z\times Q$ onto $T\times Q$ which unfortunately is just the projection of $Z\times Q$ off the I-coordinate of $T\times I$ and so is not a homeomorphism.

Another insufficient try might be to let $X=\{0\}\times I\times I$, $A=\{0\}\times I\times\{0\}$, and $B=\{0\}\times\{1\}\times I$ and to observe that there is a homeomorphism of pairs

$$f\colon (X\times\{0\}, (A\cup B)\times\{0\})\to((A\cup B)\times I, (A\cup B)\times\{0\})$$

which is the identity on $(A\cup B)\times\{0\}$. This shows immediately that $X\times Q$ is homeomorphic to $(A\cup B)\times Q$, so perhaps there is one which is the identity on $B\times Q$. (If there were, it would extend to one of $T\times Q$ onto $Z\times Q$.) Of course, there is no such homeomorphism, so this, too, is not enough. A last observation, however, is that there is such a homeomorphism which is as close to the identity as may be desired on $(A\cup B)\times Q$ and may be had by following $f\times \mathrm{id}_Q\colon X\times Q\to(A\cup B)\times I\times Q$ by a homeomorphism which "inserts" the extra I-coordinate into Q as a high-indexed coordinate and shifts the replaced coordinate and each successive one to the next-indexed, leaving unchanged all other coordinates.

This suggests that a combination of the two tries might work when coupled with some method of "weaving" the unwanted "fibers" of $X\times Q$ lying "over" $(A\cup B)\times Q$ into the rest of $Z\times Q$, and that is what §4 and the proof of Theorem 5.1 are all about. (Theorem 4.1 establishes conditions under which a sequence of homeomorphic subsets, called an *interior approximation*, of a compact metric space is so nicely embedded that the space must be homeomorphic to the members of the sequence. Theorem 4.2 shows that under certain conditions a subset A of a product $X\times Y$ of compact metric spaces is so well situated that an analogue of the sequence of sets $\{Z_i\}_{i=1}^{\infty}$ in the first try above forms an interior approximation to $X\times\prod_{i>0} Y_i$, so that $A\times\prod_{i>0} Y_i$ and $X\times\prod_{i>0} Y_i$ are homeomorphic. The conditions required by Theorem 4.2 grew out of the second insufficient approach, and under them, A is referred to as a *Y-approximation to X*. The proof of this theorem involves a complicated variant of the sequence of homeomorphisms which did not work in the first try. Finally, Theorem 5.1 shows that $Z\times Q$ as a subset of $T\times I\times Q$ meets the conditions of being a Q-approximation to $T\times I$.)

Because the proofs of Theorems 4.2 and 5.1 are a little complicated and in order to illustrate the simple underlying geometric conception involved in the definition of Y-approximation, the following diagrams show the steps of the proof of Theorem 5.1 in the case of the triod. The terminology of these proofs is adopted for ease in reference.

4. **Approximations to compact metric spaces.** In this section the approximation method used later in §5 is developed. If X and Y are compact metric spaces, an *interior approximation to X by Y* is a sequence $\{Y_i\}_{i=1}^{\infty}$ of homeomorphic copies of Y in X such that $X=\liminf\{Y_i\}_{i=1}^{\infty}$ (that is, for every $\varepsilon>0$, there is an integer N such that if $i\geqq N$, then Y_i is ε-dense in X) for which there exists a collection $\{\alpha\}\cup\{\beta_i^\varepsilon\}_{i=1,\varepsilon>0}^{\infty}\cup\{\gamma_{i,j}^\varepsilon\}_{i=1,j=1,\varepsilon>0}^{\infty}$ of homeomorphisms as follows:

1. α is an embedding of X in the Hilbert cube,

2. for each positive integer i and positive number ε, β_i^ε is an embedding of X in the Hilbert cube with $d(\beta_i^\varepsilon,\alpha)\leqq 2^{-i}$, and

3. for each positive integer j, $\gamma_{i,j}^\varepsilon$ is a homeomorphism of $\alpha(Y_i)$ onto $\beta_i^\varepsilon(Y_{i+j})$ with $d(\gamma_{i,j}^\varepsilon,\mathrm{id})\leqq\varepsilon$.

Actually, the choice of the embedding α of X in the Hilbert cube may be arbitrary in the sense shown below by Lemma 4.1.

LEMMA 4.1. *If $\{Y_i\}_{i=1}^{\infty}$ is an interior approximation to X by Y and α' is any embedding of X in the Hilbert cube, then there is a subsequence $\{Y_{i_j}\}_{j=1}^{\infty}$ of $\{Y_i\}_{i=1}^{\infty}$ which is an interior approximation to X by Y in which α' may be taken instead of α in statement* (1) *above.*

Proof. Let $h\colon Q\to Q\times I$ be the embedding $x\to(x,0)$. Now, $Q\times I$ is a Hilbert cube, and by Lemma 2.2, both $h\alpha(X)$ and $h\alpha'(X)$ have Property Z in it. Hence, by Theorem 2.1, there is an extension g of $h\circ\alpha'\circ\alpha^{-1}\circ(h|\alpha(X))^{-1}$ to a homeomorphism of $Q\times I$ onto itself.

For each $\varepsilon>0$, let $\delta(\varepsilon)$ be a positive number small enough that if x and y are in $Q\times I$ and $d(x,y)\leqq\delta(\varepsilon)$, then $d(g(x),g(y))\leqq\varepsilon$. Also, for each $\varepsilon>0$, let η_ε be a homeomorphism of $Q\times I$ onto Q such that $d(\eta_\varepsilon,p_Q)\leqq\varepsilon$.

Now, let $\{i_j\}_{j=1}^\infty$ be a subsequence of the positive integers such that for each j, $2^{-i_j}\leqq\delta(2^{-j-1})$. The subsequence $\{Y_{i_j}\}_{j=1}^\infty$ of $\{Y_i\}_{i=1}^\infty$ is the one desired.

In order to verify this, consider the homeomorphisms

$$\{\beta_i^\varepsilon\}_{i=1,\varepsilon>0}^\infty \quad\text{and}\quad \{\gamma_{i,j}^\varepsilon\}_{i=1,j=1,\varepsilon>0}^{\infty\ \ \infty}$$

postulated for $\{Y_i\}_{i=1}^\infty$ in the definition of interior approximation. For each $j>0$, $k>0$, and $\varepsilon>0$, let $\beta_j^\varepsilon(\alpha')=\eta_{m(j,\varepsilon)}\circ g\circ h\circ\beta_{i_j}^{\delta(\varepsilon/2)}$ and

$$\gamma_{j,k}^\varepsilon(\alpha') = \eta_{m(j,\varepsilon)}\circ g\circ h\circ\gamma_{i_j,i_j+k-i_j}^{\delta(\varepsilon/2)}\circ\alpha\circ\alpha'^{-1},$$

where $m(j,\varepsilon)=\frac{1}{2}\min\{2^{-j},\varepsilon\}$. These functions, together with α', satisfy the requirements of statements (1), (2), and (3) in the definition of interior approximation. Statement (1) is satisfied because α' is an embedding of X in Q. Statement (2) is satisfied because each $\beta_j^\varepsilon(\alpha')$ is an embedding of X in Q and

$$\begin{aligned} d(\beta_j^\varepsilon(\alpha'),\alpha') &= d(\eta_{m(j,\varepsilon)}\circ g\circ h\circ\beta_{i_j}^{\delta(\varepsilon/2)},\alpha')\\ &= d(\eta_{m(j,\varepsilon)}\circ g\circ h\circ\beta_{i_j}^{\delta(\varepsilon/2)},p_Q\circ g\circ h\circ\alpha)\\ &\leqq d(\eta_{m(j,\varepsilon)},p_Q)+d(p_Q\circ g\circ h\circ\beta_{i_j}^{\delta(\varepsilon/2)},p_Q\circ g\circ h\circ\alpha)\\ &\leqq m(j,\varepsilon)+d(g\circ h\circ\beta_{i_j}^{\delta(\varepsilon/2)},g\circ h\circ\alpha).\end{aligned}$$

However, $d(\beta_{i_j}^{\delta(\varepsilon/2)},\alpha)\leqq 2^{-i_j}\leqq\delta(2^{-j-1})$, so by the choice of that number and the fact that h is an isometry, $d(g\circ h\circ\beta_{i_j}^{\delta(\varepsilon/2)},g\circ h\circ\alpha)\leqq 2^{-j-1}$. Therefore,

$$d(\beta_j^\varepsilon(\alpha'),\alpha') \leqq m(j,\varepsilon)+2^{-j-1} \leqq 2^{-j}.$$

Statement (3) is satisfied because

(a) $$\gamma_{j,k}^\varepsilon(\alpha')\circ\alpha'(Y_{i_j}) = \eta_{m(j,\varepsilon)}\circ g\circ h\circ\gamma_{i_j,i_j+k-i_j}^{\delta(\varepsilon/2)}\circ\alpha\circ\alpha'^{-1}(\alpha'(Y_{i_j})),$$

which is

$$\eta_{m(j,\varepsilon)}\circ g\circ h(\gamma_{i_j,i_j+k-i_j}^{\delta(\varepsilon/2)}\circ\alpha(Y_{i_j})) = \eta_{m(j,\varepsilon)}\circ g\circ h\circ\beta_{i_j}^{\delta(\varepsilon/2)}(Y_{i_{j+k}}) = \beta_j^\varepsilon(\alpha')(Y_{i_{j+k}}),$$

and

(b) $$\begin{aligned} &d(\gamma_{j,k}^\varepsilon(\alpha'),\mathrm{id})\\ &= d(\eta_{m(j,\varepsilon)}\circ g\circ h\circ\gamma_{i_j,i_j+k-i_j}^{\delta(\varepsilon/2)}\circ\alpha\circ\alpha'^{-1},p_Q\circ g\circ h\circ\alpha\circ\alpha'^{-1})\\ &\leqq d(\eta_{m(j,\varepsilon)},p_Q)+d(p_Q\circ g\circ h\circ\gamma_{i_j,i_j+k-i_j}^{\delta(\varepsilon/2)}\circ\alpha\circ\alpha'^{-1},p_Q\circ g\circ h\circ\alpha\circ\alpha'^{-1})\\ &\leqq \varepsilon/2+d(g\circ h\circ\gamma_{i_j,i_j+k-i_j}^{\delta(\varepsilon/2)},g\circ h)\\ &\leqq \varepsilon/2+\varepsilon/2=\varepsilon,\end{aligned}$$

since $d(\gamma_{i_j,i_j+k-i_j}^{\delta(\varepsilon/2)},\mathrm{id})\leqq\delta(\varepsilon/2)$ and $\delta(\varepsilon/2)$ was chosen so that $d(g(x),g(y))\leqq\varepsilon/2$ if $d(x,y)\leqq\varepsilon/2$. The only thing left to verify in order to have $\{Y_{i_j}\}_{j=1}^\infty$ an interior approximation to X by Y is that $X=\liminf\{Y_{i_j}\}_{j=1}^\infty$, but this is immediate.

From Lemma 1 it is possible to show that if X admits an interior approximation by Y, then X and Y are homeomorphic, which is Theorem 4.1.

THEOREM 4.1. *If X and Y are compact metric spaces and there is an interior approximation to X by Y, then X and Y are homeomorphic.*

Proof. Let $\{Y_i\}_{i=1}^{\infty}$ be an interior approximation to X by Y, with associated homeomorphisms α, $\{\beta_i^{\varepsilon}\}_{i=1,\varepsilon>0}^{\infty}$, and $\{\gamma_{i,j}^{\varepsilon}\}_{i=1,j=1,\varepsilon>0}^{\infty\ \infty}$. A sequence $\{f_j\}_{j=1}^{\infty}$ of embeddings of X in Q will be constructed inductively with Lemma 4.1 used to select subsequences $\{_jY_i\}_{i=1}^{\infty}$ of $\{Y_i\}_{i=1}^{\infty}$ such that f_j may be the embedding of X in Q corresponding to α. In fact, the subsequences $\{_jY_i\}_{i=1}^{\infty}$ will be selected so that $\{_jY_i\}_{i=1}^{\infty}$ is a subsequence of $\{_{j-1}Y_i\}_{i=1}^{\infty}$. The embeddings of X in Q associated to $\{_jY_i\}_{i=1}^{\infty}$ and analogous to $\{\beta_i^{\varepsilon}\}_{i=1}^{\infty}$ will be denoted by $\{_j\beta_i^{\varepsilon}\}_{i=1,\varepsilon>0}^{\infty}$, and those analogous to $\{\gamma_{i,j}^{\varepsilon}\}_{i=1,j=1,\varepsilon>0}^{\infty\ \infty}$ will be denoted by $\{_j\gamma_{i,k}^{\varepsilon}\}_{i=1,k=1,\varepsilon>0}^{\infty\ \infty}$. Another sequence

$$\{m_j = g_j \circ \cdots \circ g_1 \circ \alpha\}_{j=1}^{\infty}$$

of embeddings of Y_1 into Q will be defined at the same time as $\{f_j\}_{j=1}^{\infty}$. The sequences $\{f_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty}$ will converge to homeomorphisms f and g onto a common image, proving the theorem.

Let $\varepsilon_0 = 2^{-3}\min\{d(\alpha(x),\alpha(x')) \mid x, x' \in X, d(x,x') \geqq 1\}$, and let $f_1 = \beta_1^{\varepsilon_0}$. To construct g_1, let

$$\delta_1 = 2^{-3}\min\{d(f_1(x), f_1(x')) \mid x, x' \in X \text{ and } d(x,x') \geqq \tfrac{1}{2}\}.$$

Now use Lemma 4.1 to select a subsequence $\{_1Y_i\}_{i=1}^{\infty}$ of $\{Y_i\}_{i=1}^{\infty}$ so that f_1 may be taken in the place of α. Choose $i_1 > 1$ large enough that $2^{-i_1} \leqq \delta_1$, and choose k_1 so that $Y_{k_1+1} = {}_1Y_{i_1}$. Let $g_1 = \gamma_{1,k_1}^{\varepsilon_0}$. Now, $g_1\colon \alpha(Y_1) \to f_1({}_1Y_{i_1})$, and $d(g_1, \mathrm{id}) \leqq \varepsilon_0$.

To define f_2, let $\varepsilon_1 > 0$ be less than $\frac{1}{2}\varepsilon_0$ and

$$2^{-3}\min\{d(m_1(x), m_1(x')) \mid x, x' \in Y_1 \text{ and } d(x,x') \geqq \tfrac{1}{2}\}.$$

Let $f_2 = {}_1\beta_{i_1}^{\varepsilon_1}$. Then, $d(f_1, f_2) \leqq 2^{-i_1} \leqq \delta_1$. Now use Lemma 3.1 to obtain a subsequence $\{_2Y_i\}_{i=1}^{\infty}$ of $\{_1Y_i\}_{i=1}^{\infty}$ for which f_2 may be taken in the place of α. Letting $\delta_2 > 0$ be less than $\frac{1}{2}\delta_1$ and

$$2^{-3}\min\{d(f_2(x), f_2(x')) \mid x, x' \in X \text{ and } d(x,x') \geqq 2^{-2}\},$$

select $i_2 > i_1$ so large that $2^{-i_2} \leqq \delta_2$. If now k_2 is chosen so that ${}_1Y_{i_1+k_2} = {}_2Y_{i_2}$, define g_2 to be ${}_1\gamma_{i_1,k_2}^{\varepsilon_1}$. Let $\varepsilon_2 > 0$ be less than $\frac{1}{2}\varepsilon_1$ and

$$2^{-3}\min\{d(m_2(x), m_2(x')) \mid x, x' \in Y_1 \text{ and } d(x,x') \geqq 2^{-2}\}.$$

From the above, it is easy to see that one may proceed inductively to obtain sequences $\{f_j\}_{j=1}^{\infty}$, $\{g_j\}_{j=1}^{\infty}$, and $\{_jY_{i_j}\}_{j=1}^{\infty}$ such that

(1) f_j embeds X in Q,

(2) g_j is a homeomorphism of $m_{j-1}(Y_1)$ onto $f_j({}_jY_{i_j})$,

(3) $\{{}_jY_{i_j}\}_{j=1}^{\infty}$ is a subsequence of $\{Y_i\}_{i=1}^{\infty}$,

(4) for each $k>0$, $d(f_{j+k}, f_{j+k-1})$ is less than

$$2^{-k-2}\min\{d(f_j(x), f_j(x')) \mid x, x' \in X;\ d(x, x') \geqq 2^{-j}\},$$

and

(5) for each $k>0$, $d(g_{j+k}, \mathrm{id})$ is less than

$$2^{-k-2}\min\{d(m_j(x), m_j(x')) \mid x, x' \in Y_1;\ d(x, x') \geqq 2^{-j}\}.$$

From (4) and (5) above, the sequences $\{f_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty}$ are uniformly Cauchy sequences of mappings of X and Y_1, respectively, into Q and hence converge uniformly to maps f and g of X and Y_1, respectively, into Q. However, (4) and (5) also guarantee that f and g are one-to-one and thus embeddings, for if x and x' are distinct points of X, then there is a $j>0$ such that $d(x, x')\geqq 2^{-j}$, and (4) gives that

$$\begin{aligned} d(f_{j+k}, f_j) &\leqq \sum_{m=1}^{k} d(f_{j+m}, f_{j+m-1}) \leqq 2^{-2}d(f_j(x), f_j(x'))\sum_{m=1}^{k} 2^{-m} \\ &< 2^{-2}d(f_j(x), f_j(x')). \end{aligned}$$

Thus, $d(f(x), f_j(x))\leqq 2^{-2}d(f_j(x), f_j(x'))$ and $d(f(x'), f_j(x'))\leqq 2^{-2}d(f_j(x), f_j(x'))$, so $d(f(x), f(x'))\geqq \frac{1}{2}d(f_j(x), f_j(x'))>0$. The analogous proof demonstrates that g is one-to-one.

For any $\varepsilon>0$, there is a $j>0$ such that for $j'\geqq j$, $d(f_j, f_{j'})\leqq \varepsilon/5$ and $d(m_j, m_{j'})\leqq \varepsilon/5$. Because $X=\liminf\{Y_i\}_{i=1}^{\infty}$, there is a $j'\geqq j$ such that $f_j(Y_k)$ is $(\varepsilon/5)$-dense in $f_j(X)$ for each $k\geqq j'$. Also, there is a $k\geqq j'$ such that $Y_k={}_{j'}Y_{i_{j'}}$. Combining these statements, we have that for y in Y_1, there is an x in X such that $f_{j'}(x)=g_{j'}\circ\cdots\circ g_1\circ\alpha(y)$ and that

$$\begin{aligned} d(f(x), g(y)) \leqq\ & d(f(x), f_j(x))+d(f_j(x), f_{j'}(x)) \\ &+d(f_{j'}(x), m_{j'}(y))+d(m_{j'}(y), m_j(y))+d(m_j(y), g(y)) < \varepsilon, \end{aligned}$$

and thus, $g(Y_1)\subset f(X)$. On the other hand, for x in X, there is an x' in Y_k for which $d(f_j(x), f_j(x'))\leqq \varepsilon/5$. For some y in Y_1, $f_{j'}(x)=m_{j'}(y)$, so

$$\begin{aligned} d(f(x), g(y)) &\leqq d(f(x), f_j(x))+d(f_j(x), f_j(x'))+d(f_j(x'), f_{j'}(x')) \\ &\quad +d(f_{j'}(x'), g_{j'}\circ\cdots\circ g_1\circ\alpha(y))+d(g_{j'}\circ\cdots\circ g_1\circ\alpha(y), g(y)) \\ &\leqq d(f_{j'}(x'), m_{j'}(y))+d(m_{j'}(y), g(y)) \leqq \varepsilon. \end{aligned}$$

Hence $f(X)\subset g(Y_1)$. Thus, $g\circ f^{-1}$ is a homeomorphism of X onto Y_1. As Y_1 is homeomorphic to Y, X is homeomorphic to Y.

The final portion of this section develops an application of Theorem 4.1 to product spaces in the form used in §5. In some vague sense, Theorem 4.2 may be regarded as a "stabilization" theorem.

If $\prod_{i\in S} Y_i$ is a product, it is convenient to have a compact notation for the "switching" homeomorphisms induced by coordinate-permutations. Thus, if

$S=\{1, 2, 3, 4, 5\}$ and the permutation of S leaving 1 fixed and exchanging the members of the two pairs (2, 3) and (4, 5) is denoted by (2 3)(4 5), then

$$s_{(2\ 3)(4\ 5)}\colon \prod_{i=1}^{5} Y_i \to \prod_{i=1}^{5} Y_i$$

is defined by $s_{(2\ 3)(4\ 5)}(y_1, y_2, y_3, y_4, y_5)=(y_1, y_3, y_2, y_5, y_4)$, *etc.*

For X and Y compact metric spaces, a closed subset Z of $X\times Y$ will be termed a *Y-approximation to X* provided that for each positive number ε there is an embedding g_ε of $X\times Y$ in $Z\times Q$ with $d(p_X g_\varepsilon, p_X)\leqq \varepsilon$ which satisfies condition (*) below.

(*) Let $\bar{g}_\varepsilon = s_{(2\ 3)} \circ (g_\varepsilon \times \mathrm{id}_Y)\colon X\times Y\times Y=(X\times Y)\times Y \to (Z\times Q)\times Y = Z\times Q\times Y \to Z\times Y\times Q$, and, regarding $Z\times Y$ as a subset of $X\times Y\times Y$, let $\bar{Z}=s_{(2\ 3)}(Z\times Y)$. There is a homeomorphism h_ε of $\bar{g}_\varepsilon(\bar{Z})$ onto $Z\times Y$ with $d(p_Z\circ h_\varepsilon, p_Z)\leqq\varepsilon$.

The effect of this definition, of course, is to guarantee a two-step move from $\bar{Z}$ to $Z\times Y$ in which the motion is restricted in the X-coordinate in the first step and in the $(X\times Y)$-coordinate in the second, and where, moreover, the first step is a homeomorphism of the entire space $X\times Y\times Y$. The first step will appear in the construction of β's, and the second, in the construction of γ's for an interior approximation.

Simple examples of Y-approximations include the graphs of continuous functions from X to Y, for if f is such a map and Z is its graph then given any embedding μ of Y in Q, the embedding g of $X\times Y$ in $Z\times Q$ sending (x, y) to $((x, f(x)), \mu(y))$ will do for each g_ε and the homeomorphism $h\colon \bar{g}(\bar{Z})\to Z\times Y$ sending $((x, f(x)), f(x), \mu(y))$ to $(x, f(x), y)$ will serve for each h_ε.

If Y is contractible, it is easily seen that any Y-approximation to X must have the same homotopy type as X. However, if Z is a Y-approximation to X and Y' is any compact metric space, then $Z\times Y'$ is a $(Y\times Y')$-approximation to X, so this need not be true. In fact, from considerations already made, the torus is an S^1-approximation to S^1 (=the circle).

THEOREM 4.2. *If Z is a Y-approximation to X, then $Z\times\prod_{i>0} Y_i$ is homeomorphic to $X\times\prod_{i>0} Y_i$ (where $\{Y_i\}_{i>0}$ is a collection of homeomorphic copies of Y).*

Proof. Let $\bar{Z}_i=\{(x, y_1, y_2, \dots)\in X\times\prod_{j>0} Y_j \mid (x, y_{2i})\in Z\}$. It will be shown that $\{\bar{Z}_i\}_{i>0}$ is an interior approximation to $X\times\prod_{j>0} Y_j$ by $Z\times\prod_{j>0} Y_j$.

Let z be the point of $Q=\prod_{i>0} I_i$ with all coordinates zero. Identifying $X\times\prod_{i>0} Y_i$ with $X\times\prod_{i>0} Y_i\times\{z\}\subset X\times\prod_{i>0} Y_i\times Q$, it is sufficient to consider embeddings of $X\times\prod_{i>0} Y_i\times\{z\}$ in $X\times\prod_{i>0} Y_i\times Q$ in order to exhibit an interior approximation to $X\times\prod_{i>0} Y_i$ by $Z\times\prod_{i>0} Y_i$. (This is because if Q, Q_0, $Q_1, \dots$ are copies of the Hilbert cube, if $\mu_z\colon X\times\prod_{i>0} Y_i\to Q$ is the constant map to z, and if $\mu_0\colon X\to Q_0$ and $\mu_i\colon Y_i\to Q_i$, $i>0$, are embeddings, then

$$\alpha = \left(\left(\mu_0\times\prod_{i>0}\mu_i\right), \mu_z\right)\colon X\times\prod_{i>0} Y_i\to\left(Q_0\times\prod_{i>0} Q_i\right)\times Q$$

is an embedding into a Hilbert cube. This embedding may be used as the required

α of an interior approximation with the other homeomorphisms constructed from the image of α in the same way as is done below but with due care taken to adjust the sizes of the constants involved.)

For a positive integer i and a positive number ε, let $k=k(i,\varepsilon)$ be a positive integer so large that 2^{-k} is less than or equal to $2^{-1}\min\{\varepsilon, 2^{-i-2}\}$, and let $\phi_{i,\varepsilon}$ be the homeomorphism of $X\times\prod_{i>0} Y_i\times Q$ onto itself defined by

$$\phi_{i,\varepsilon}(x,(y_1,y_2,\dots),q) = (x,(y'_1,y'_2,\dots),q),$$

with

1. $y'_j=y_j$, if $j\leqq 2i$,
2. $y'_{2k+2j}=y_{2i+2j}$, if $j>0$,
3. $y'_{2i+2j}=y_{2i+2j-1}$, if $1\leqq j\leqq k-1$,
4. $y'_{2i+2j-1}=y_{2k+2j-1}$, if $j>0$.

Let R denote the real numbers, and for $t\in R$, let

$$\psi_t\colon X\times\prod_{i>0} Y_i\times\prod_{i>0} R_i\to X\times\prod_{i>0} Y_i\times\prod_{i>0} R_i$$

be the map which changes no point's X- or $(\prod_{i>0} Y_i)$-coordinate and multiplies each R_i coordinate by t.

If i is a positive integer, let $r_i(x,y_1,q,y_2,y_3,\dots)=(x,y_i,y_2,\dots,y_{i-1},y_1,y_{i+1},\dots,q)$, and define $g_i^\varepsilon=r_{2i}\circ(g_{\varepsilon/4}\times\mathrm{id})\circ(\mathrm{id}\times s_{(1\ 2i)})\colon X\times\prod_{j>0} Y_j\times\{z\}\ (=X\times\prod_{j>0} Y_j)\to X\times\prod_{j>0} Y_j\times Q$. (The slight abuse of notation is adopted for clarity.)

In a similar manner, for $i<j$ let

$$u_{i,j}(x,y_1,y_2,\dots,q) = (x,y_i,y_j,q,y_1,y_2,\dots,y_{i-1},y_{i+1},\dots,y_{j-1},y_{j+1},\dots)$$

and

$$v_{i,j}(x,y_1,y_2,\dots) = (x,y_3,y_4,\dots,y_{i+1},y_1,y_{i+2},\dots,y_j,y_2,y_{j+1},\dots,z).$$

Then set $h_{i,j}^\varepsilon=v_{i,j}\circ(h_{\varepsilon/4}\times\mathrm{id})\circ u_{i,j}\colon g_i^\varepsilon(\overline{Z}_j\times\{z\})\to\overline{Z}_i\times\{z\}$.

From these definitions, it is possible to define β_i^ε and $\gamma_{i,j}^\varepsilon$ as follows: Let $\beta_i^\varepsilon\colon X\times\prod_{i>0} Y_i\times\{z\}\to\overline{Z}_{i+1}\times Q$ be $\psi_{\varepsilon/4}\circ g_{i+1}^\varepsilon\circ\phi_{i+1,\varepsilon}$, and let $\gamma_{i,j}^\varepsilon\colon\overline{Z}_{i+1}\times\{z\}\to\beta_i^\varepsilon(\overline{Z}_{i+j+1}\times\{z\})$ be $\psi_{\varepsilon/4}\circ(h_{i+1,j+k(i+1,\varepsilon)})^{-1}$. (Of course, this is for $\varepsilon\leqq 4$.)

These functions $\{\beta_i^\varepsilon\}_{i=1,\varepsilon>0}^\infty$ and $\{\gamma_{i,j}^\varepsilon\}_{i=1,j=1,\varepsilon>0}^{\infty\ \ \infty}$, for $\varepsilon\leqq 2^{-i}$, suffice to demonstrate that $\{\overline{Z}_{i+1}\}_{i=1}^\infty$ is an interior approximation to $X\times\prod_{i>0} Y_i$ by $Z\times\prod_{i>0} Y_i$. The remainder of the proof is a verification of this.

In the first place, one need only consider, for each i, ε small with respect to i, that is, $\varepsilon\leqq 2^{-i}$. Because $\phi_{i+1,\varepsilon}$, g_{i+1}^ε, and $\psi_{\varepsilon/4}$ are all homeomorphisms, β_i^ε is an embedding of $X\times\prod_{i>0} Y_i\times\{z\}$ in $\overline{Z}_{i+1}\times Q$. Also,

$$\begin{aligned} d(\beta_i^\varepsilon,\mathrm{id}) &= d(\psi_{\varepsilon/4}\circ g_{i+1}^\varepsilon\circ\phi_{i+1,\varepsilon},\mathrm{id})\\ &\leqq d(p_Q\circ\psi_{\varepsilon/4}\circ g_{i+1}^\varepsilon,p_Q)+d(\phi_{i+1,\varepsilon},\mathrm{id})\\ &\quad +2^{-2i-2}\operatorname{dia}(Y_{2i+2})+d(p_X\circ g_{i+1}^\varepsilon,p_X)\\ &\leqq \varepsilon/4+\sum_{j=1}^\infty 2^{-2i-2-j}+2^{-2i-2}+\varepsilon/4\\ &= \varepsilon/2+2^{-2i-1} < 2^{-i-1}+2^{-i-1} = 2^{-i}. \end{aligned}$$

Concerning $\gamma^{\varepsilon}_{i,j}=\psi_{\varepsilon/4}\circ(h^{\varepsilon}_{i+1,j+k(i+1,\varepsilon)})^{-1}$, it carries $\bar{Z}_{i+1}\times\{z\}$ onto $\beta^{\varepsilon}_{i}(\bar{Z}_{i+j+1}\times\{z\})$ because

$$\begin{aligned}\beta^{\varepsilon}_{i}(\bar{Z}_{i+j+1}\times\{z\}) &= \psi_{\varepsilon/4}\circ g^{\varepsilon}_{i+1}\circ\phi_{i+1,\varepsilon}(\bar{Z}_{i+j+1}\times\{z\})\\ &= \psi_{\varepsilon/4}\circ g^{\varepsilon}_{i+1}(\bar{Z}_{k(i+1,\varepsilon)+j}\times\{z\})\\ &= \psi_{\varepsilon/4}(g^{\varepsilon}_{i+1}(\bar{Z}_{k(i+1,\varepsilon)+j}\times\{z\}))\\ &= \psi_{\varepsilon/4}((h^{\varepsilon}_{i+1,k(i+1,\varepsilon)+j})^{-1}(\bar{Z}_{i+1}\times\{z\}))\\ &= \gamma^{\varepsilon}_{i,j}(\bar{Z}_{i+1}\times\{z\}).\end{aligned}$$

Finally,

$$\begin{aligned}d(\gamma^{\varepsilon}_{i,j},\mathrm{id}) &\leqq d(p_{X\times Y_{2i}}\circ\gamma^{\varepsilon}_{i,j},p_{X\times Y_{2i}})\\ &\quad +2^{-2k(i+1,\varepsilon)-2j}\operatorname{dia}(Y_{2k(i+1,\varepsilon)+2j})+d(p_Q\circ\gamma^{\varepsilon}_{i,j},p_Q)\\ &\leqq \varepsilon/4+2^{-2k(i+1,\varepsilon)-2j}+\varepsilon/4\\ &\leqq \varepsilon/4+\varepsilon/2+\varepsilon/4 = \varepsilon.\end{aligned}$$

Thus, $\{\bar{Z}_{i+1}\}_{i=1}^{\infty}$ is an interior approximation to $X\times\prod_{i>0}Y_i$ by $Z\times\prod_{i>0}Y_i$, and by Theorem 4.1, $X\times\prod_{i>0}Y_i$ is homeomorphic to $Z\times\prod_{i>0}Y_i$.

5. **Products with the Hilbert cube.** Here the apparatus of the previous two sections is used to study products of compact metric spaces, primarily of compact polyhedra, with the Hilbert cube.

THEOREM 5.1. *If $X=Q_1\cup Q_2$, where Q_1, Q_2 and $Q_1\cap Q_2$ are all Hilbert cubes and $Q_1\cap Q_2$ has Property Z in Q_1, then $X\times Q$ is a Hilbert cube.*

Proof. Let $Z=(Q_1\times\{0\})\cup(Q_2\times I)\subset X\times I$. It will be shown that Z is an I-approximation to X. Theorem 4.2 then will yield that $Z\times Q$ and $X\times Q$ are homeomorphic. However, Z is easily seen to be a Hilbert cube itself, for it is the union of two Hilbert cubes which intersect in a third $(Q_1\cap Q_2)\times\{0\}$ with Property Z in each. (By hypothesis on the one hand and by Lemma 2.2 on the other.) Thus, there are homeomorphisms θ, η, and ζ of $Q_1\times\{0\}$, $Q_2\times I$, and $Q_3\times\{0\}$ (where $Q_3=Q_1\cap Q_2$) onto $[0,\frac{1}{2}]\times Q$, $[\frac{1}{2},1]\times Q$, and $\{\frac{1}{2}\}\times Q$, respectively. Because Property Z is a topological invariant, $\theta(Q_3\times\{0\})$ and $\eta(Q_3\times\{0\})$ have Property Z in $[0,\frac{1}{2}]\times Q$ and $[\frac{1}{2},1]\times Q$, respectively. Lemma 2.2 shows that $\{\frac{1}{2}\}\times Q$ has Property Z in each, so by Theorem 2.1, there exist homeomorphisms λ and μ of $[0,\frac{1}{2}]\times Q$ and $[\frac{1}{2},1]\times Q$ onto themselves, respectively, such that $\lambda|\theta(Q_3\times\{0\})=\zeta\circ\theta^{-1}$ and $\mu|\eta(Q_3\times\{0\})=\zeta\circ\eta^{-1}$. Then the function $\nu\colon Z\to I\times Q$ defined by

$$\begin{aligned}\nu(x) &= \lambda\circ\theta(x),\quad \text{if } x\in Q_1\times\{0\},\\ &= \mu\circ\eta(x),\quad \text{if } x\in Q_2\times I\end{aligned}$$

is a homeomorphism onto a Hilbert cube. Since Z is a Hilbert cube, so is $Z\times Q$ and, hence, $X\times Q$.

Let, now, J and L, as well as I, denote $[0,1]$. In the proof that Z is an I-approximation to X, it will not be necessary to use more than an interval of the "room" provided by Q in the definition of Y-approximation above, so L will be substituted for Q here for the sake of simplicity. For each $\varepsilon>0$, then, there must be an embedding g_ε of $X\times I\times J\times\{0\}$ in $Z\times J\times L$ with $d(p_X\circ g_\varepsilon, p_X)\leqq\varepsilon$ such that $p_J\circ g_\varepsilon=p_J$ and a homeomorphism h_ε of $g_\varepsilon(\bar{Z}\times\{0\})$ onto $Z\times J\times\{0\}$ with $d(p_{X\times I}\circ h_\varepsilon, p_{X\times I})\leqq\varepsilon$, where $\bar{Z}=\{(x, s, t)\in X\times I\times J \mid (x, t)\in Z\}$.

Fix $\varepsilon>0$, and construct a retraction f_ε of $X\times I$ onto Z as follows: Since Q_3 has Property Z in Q_1, there is a homeomorphism of compact pairs $(Q_3, Q_1) \xrightarrow{\phi} (Q\times\{0\}, Q\times I)$. Now consider the retractions r_1 and r_2 with

$$r_1: I\times I\to(I\times\{0\}, \{0\}\times I)$$

defined as the projection from the point $(1, \delta^{-1})$ for some small $\delta>0$, and

$$r_2 = \mathrm{id}\times r_1: Q\times I\times I\to Q\times I\times\{0\}\cup Q\times\{0\}\times I.$$

For small enough δ, $d(p_{Q\times I}\circ r_2, p_{Q\times I})$ is as small as may be required. Finally, carry r_2 over to $Q_1\times I$ by the homeomorphism $\phi\times\mathrm{id}$, obtaining $(\phi^{-1}\times\mathrm{id})\circ r_2\circ(\phi\times\mathrm{id})$, and extend over $Q_2\times I$ by the identity to obtain f. With $\delta=\delta(\varepsilon)$ sufficiently small, $d(p_X\circ f, p_X)\leqq\varepsilon$, so f_ε may be set equal to f for an appropriate choice of δ.

Define $\bar{g}_\varepsilon$: $X\times I\times J\times\{0\}\to Z\times J\times L$ by

$$\bar{g}_\varepsilon(x, s, t, 0) = (f_\varepsilon(x, s)_X, f_\varepsilon(x, s)_I, t, s-f_\varepsilon'(x, s)_I).$$

By construction, $d(p_X\circ\bar{g}_\varepsilon, p_X)\leqq\varepsilon$, since $d(s, p_{\delta(\varepsilon)}(s, t))\leqq\delta(\varepsilon)$ for all (s, t) in $I_1\times I$.

To define h_ε, first let α_ε be a homeomorphism of

$$\{(s, t)\in I\times L \mid 0\leqq t\leqq 1-s\}$$

onto $I\times L$ with $d(p_I\circ\alpha_\varepsilon, p_I)\leqq\varepsilon/2$ which is the identity on $(\{0\}\times L)\cup(I\times\{0\})$. Let $\bar{\alpha}_\varepsilon$: $\bar{g}_\varepsilon(\bar{Z}\times\{0\})\to Z\times J\times L$ be the extension of α_ε by the identity in the other coordinates, that is, $p_X\circ\bar{\alpha}_\varepsilon=p_X$ and $p_J\circ\bar{\alpha}_\varepsilon=p_J$.

Now, $\bar{\alpha}_\varepsilon\circ\bar{g}_\varepsilon(\bar{Z}\times\{0\})=(Q_2\times I\times J\times\{0\})\cup(Q_1\times\{0\}\times\{0\}\times L)\cup(Q_3\times I\times\{0\}\times L)$. Here, the map which replaces the L-coordinate by zero and adds it to the J-coordinate carries $\bar{\alpha}_\varepsilon\circ\bar{g}_\varepsilon(\bar{Z}\times\{0\})$ onto $Z\times J\times\{0\}$ and commutes with the projection onto $Z\times I$, so its satisfies all the requirements for h_ε but that of being a homeomorphism. Finding a homeomorphism whose projection onto $Z\times I$ is close to the direct projection is made possible by Theorem 2.1. It is described in Diagrams 3, 4, and 5 in the case that X is the product of a triod with the Hilbert cube ($\bar{\alpha}_\varepsilon$ is also included in Diagram 3).

Since $Q_3\times I\times\{0\}$ and $Q_3\times\{0\}\times J$ have Property Z in $Q_3\times I\times I$ and in $Q_2\times I\times J$ so does their union, and by the same method used in showing $\bar{Z}$ to be homeomorphic to the Hilbert cube, there is a homeomorphism ψ of $Q_2\times I\times J$ onto $Q_3\times I\times J$ which is the identity on $(Q_3\times I\times\{0\})\cup(Q_3\times\{0\}\times J)$. Let $\bar{\psi}$ be the homeomorphism of $(Q_3\times I\times\{0\}\times L)\cup(Q_2\times I\times J\times\{0\})$ onto $(Q_3\times I\times\{0\}\times L)\cup(Q_3\times I\times J\times\{0\})$ obtained by extending ψ by the identity in the L-coordinate.

Let $\lambda(\varepsilon)\in(0,\varepsilon/16)$ be small enough that for x and x' in $(Q_3\times I\times\{0\}\times L)\cup(Q_3\times I\times J\times\{0\})$ with $d(x,x')\leqq 2\lambda(\varepsilon)$, $d(\bar{\psi}^{-1}(x),\bar{\psi}^{-1}(x'))\leqq\varepsilon/8$, and let

$$\beta_\varepsilon\colon (I\times J\times\{0\})\cup(I\times\{0\}\times L)\to I\times J\times\{0\}$$

be a homeomorphism satisfying the following:

(1) β_ε is the identity on $[\lambda(\varepsilon),1]\times[\lambda(\varepsilon),1]\times\{0\}$,

(2) $\beta_\varepsilon((\{s\}\times J\times\{0\})\cup(\{s\}\times\{0\}\times L))=\{s\}\times J\times\{0\}$, for each $s\geqq\lambda(\varepsilon)$, and

(3) $\beta_\varepsilon(\{0\}\times\{0\}\times L)=\{0\}\times J\times\{0\}$.

Let $\bar{\beta}_\varepsilon\colon (Q_3\times I\times J\times\{0\})\cup(Q_3\times I\times\{0\}\times L)\to Q_3\times I\times J\times\{0\}$ be the extension of β_ε by the identity in the X-coordinate. Now consider $\bar{\psi}^{-1}\circ\bar{\beta}_\varepsilon\circ\bar{\psi}$, a homeomorphism of $(Q_2\times I\times J\times\{0\})\cup(Q_3\times I\times\{0\}\times L)$ onto $Q_2\times I\times J\times\{0\}$. No point of $Q_3\times I\times J\times\{0\}$ is moved by $\bar{\beta}_\varepsilon$ unless it is within $2\lambda(\varepsilon)$ of some point of

$$(Q_3\times I\times\{0\}\times\{0\})\cup(Q_3\times\{0\}\times J\times\{0\}),$$

which lies in the fixed-point set of $\bar{\psi}$. Hence, for any point x of $Q_3\times I\times J\times\{0\}$, $d(\bar{\beta}_\varepsilon(x),\bar{\psi}^{-1}\circ\bar{\beta}_\varepsilon(x))\leqq 2\varepsilon/8=\varepsilon/4$. Also, for any point x in $Q_3\times I\times\{0\}\times L$, $d(p_{X\times I}\circ\bar{\beta}_\varepsilon(x),p_{X\times I}(x))\leqq\lambda(\varepsilon)$ and $\bar{\beta}_\varepsilon(x)$ is within $2\lambda(\varepsilon)$ of the fixed-point set of $\bar{\psi}$. Therefore, $\bar{\beta}_\varepsilon(x)$ is moved at most a distance of $\varepsilon/4$ by $\bar{\psi}^{-1}$. Adding these statements yields $d(p_{X\times I}\circ\bar{\psi}^{-1}\circ\bar{\beta}_\varepsilon\circ\bar{\psi},p_{X\times I})\leqq\varepsilon/4+\lambda(\varepsilon)\leqq\varepsilon/2$. Extend $\bar{\psi}^{-1}\circ\bar{\beta}_\varepsilon\circ\bar{\psi}$ to $Q_1\times\{0\}\times\{0\}\times L$ by sending $(x,0,0,t)$ to $(x,0,\beta_\varepsilon(0,t)_J,0)$, (this is a continuous map because the function $\bar{\psi}^{-1}\circ\bar{\beta}_\varepsilon\circ\bar{\psi}$ reduces to this formula on $Q_3\times\{0\}\times\{0\}\times L$), and call it γ_ε. The image of γ_ε is $(Q_1\times\{0\}\times J\times\{0\})\cup(Q_2\times I\times J\times\{0\})$, which is $Z\times J\times\{0\}$, and $d(p_{X\times I}\circ\gamma_\varepsilon,p_{X\times I})\leqq\varepsilon/2$. Let $h_\varepsilon=\gamma_\varepsilon\circ\bar{\alpha}_\varepsilon$. Then $h_\varepsilon\colon\bar{g}_\varepsilon(\bar{Z}\times\{0\})\to Z\times J\times\{0\}$ and $d(p_{X\times I}\circ h_\varepsilon,p_{X\times I})\leqq d(p_{X\times I}\circ\gamma_\varepsilon,p_{X\times I})+d(p_{X\times I}\circ\bar{\alpha}_\varepsilon,p_{X\times I})\leqq\varepsilon/2+\varepsilon/2\leqq\varepsilon$. Therefore, Z is an I-approximation to X, and $X\times Q$ is homeomorphic to Q, as noted earlier.

A polyhedron is a space homeomorphic to a geometric simplicial complex. Compact polyhedra are homeomorphic to finite geometric simplicial complexes. A homeomorphism from a geometric simplicial complex K onto a polyhedron X is said to define a triangulation of X by K. If K is a finite geometric simplicial complex and L is a subcomplex of K such that $K\backslash L=\Delta^\circ\cup\Delta'^\circ$ where Δ is a simplex of K not contained in any other simplex of K and Δ' is a proper face of Δ which lies in no other simplex of K, then K is said to *collapse to L by an elementary simplicial collapse*. If Y is a subpolyhedron of the compact polyhedron X, and if there is a finite geometric simplicial pair (K,L) such that K collapses to L by an elementary simplicial collapse and a triangulation $f\colon (K,L)\to(X,Y)$, then X is said to *collapse to* Y by an *elementary collapse*. A finite sequence of elementary (simplicial) collapses is called a (simplicial) collapse, and the initial polyhedron (geometric simplicial complex) is said to collapse to the terminal one. If the terminal one is a point, the initial one is said to be *collapsible*. Two compact polyhedra X and Y are said to be of the same *simple homotopy type* if there is a finite sequence $X_1, X_2,\dots, X_n$ of polyhedra with X homeomorphic to X_1 and Y homeomorphic to X_n such that for each $i=1,\ldots,n-1$, either X_i collapses to X_{i+1} by an elementary collapse or X_{i+1} collapses to X_i by an elementary collapse.

COROLLARY 5.1. *The product of a collapsible polyhedron with the Hilbert cube is a Hilbert cube.*

Proof. By induction on the number of elementary collapses necessary to reach a single point, it is sufficient to show that if X and Y are compact polyhedra and X collapses to Y by an elementary collapse, then $X\times Q$ is a Hilbert cube if $Y\times Q$ is. However, $\operatorname{cl}(X\setminus Y)$ is an n-cell, for some $n>0$, and $\operatorname{cl}(X\setminus Y)\cap Y$ is an $(n-1)$-cell in the boundary of $\operatorname{cl}(X\setminus Y)$. Therefore, by Lemma 2.2, $\operatorname{cl}(X\setminus Y)\cap Y$ has Property Z in $\operatorname{cl}(X\setminus Y)$, and by Lemma 2.3, $(\operatorname{cl}(X\setminus Y)\cap Y)\times Q$ has Property Z in $\operatorname{cl}(X\setminus Y)\times Q$. Finally, all three of $\operatorname{cl}(X\setminus Y)\times Q$, $Y\times Q$, and $(\operatorname{cl}(X\setminus Y)\cap Y)\times Q$ are Hilbert cubes, so by Theorem 5.1, $(X\times Q)\times Q$ is a Hilbert cube. Since $(X\times Q)\times Q$ is homeomorphic to $X\times Q$, $X\times Q$ is a Hilbert cube.

COROLLARY 5.2. *If X is a polyhedron which can be triangulated by a locally finite simplicial complex, then the product of X with the Hilbert cube is locally homeomorphic to the Hilbert cube.*

Proof. Every point of X has a compact neighborhood which is a collapsible polyhedron, so by Corollary 5.1, each point of $X\times Q$ has a neighborhood which is homeomorphic to the Hilbert cube.

The next (Corollary 5.3) appeared as Conjecture 2 in [9] and as such is attributed to R. D. Anderson. It is the converse of a theorem of D. W. Henderson, who proved in [10] that each open subset of a separable, infinite-dimensional Hilbert space is homeomorphic to the product of that space with a countable, locally finite, simplicial complex.

COROLLARY 5.3. *If X is a polyhedron which can be triangulated by a countable, locally finite, simplicial complex and if H is a separable, infinite-dimensional Hilbert space, then $X\times H$ is homeomorphic to an open subset of H.*

Proof. R. D. Anderson showed in [2] that H is homeomorphic to s, the countably infinite product of lines, and in [3] that $s\times Q$ is homeomorphic to s. (Bessaga and Klee in [6] have also shown this last.) Thus, $H\times Q$ is homomorphic to H and $X\times H$ is homeomorphic to $(X\times Q)\times H$, which is, by Corollary 5.2, locally homeomorphic to $Q\times H$, or H. However, $X\times H$ is now a separable, paracompact H-manifold and, by a theorem of Henderson [11], thus homeomorphic to an open subset of H.

Since work by N. H. Kuiper and D. Burghelea [12] together with work of N. Moulis [13] has shown all homotopy-equivalent open subsets of separable, infinite-dimensional Hilbert spaces to be C^∞-diffeomorphic, Corollary 5.3 shows

COROLLARY 5.4. *If X and Y are polyhedra which can be triangulated by separable, locally finite, simplicial complexes, and if H is a separable, infinite-dimensional*

Hilbert space, then $X\times H$ is homeomorphic to $Y\times H$ if X and Y are of the same homotopy type.

REMARK. In the above, one may substitute "separable, infinite-dimensional Fréchet space" for "separable, infinite-dimensional Hilbert space" because all such spaces have been shown to be homeomorphic. (A bibliography and brief discussion of the results may be found in the introduction of [5].)

THEOREM 5.2. *If X and Y are compact polyhedra of the same simple homotopy type, then $X\times Q$ is homeomorphic to $Y\times Q$.*

Proof. It is sufficient to show that if X collapses to Y by an elementary collapse, then $X\times Q$ is homeomorphic to $Y\times Q$, so suppose $f\colon (K, L)\to(X, Y)$ is a triangulation of (X, Y) such that K collapses to L by an elementary simplicial collapse. Let (Δ, Δ') be the pair of simplices of K determining the collapse. It is easy to see that there is a polyhedral neighborhood C of $\operatorname{cl}(X\setminus Y)$ in X which collapses to $\operatorname{cl}(X\setminus Y)$ and whose (topological) boundary, $\operatorname{cl}(X\setminus C)\cap C$, in X has Property Z in C. One way to see this is to take the piecewise-linear function $g\colon K\to I$ which sends each vertex of Δ to zero and each vertex of $K\setminus\Delta$ to one. The neighborhood C may be set equal to $f\circ g^{-1}([0,\frac{1}{2}])$, for K may be subdivided in such a way as to yield $g^{-1}([0,\frac{1}{2}])$ as a subcomplex, $g^{-1}([0,\frac{1}{2}])$ collapses to Δ, and the topological boundary $g^{-1}(\frac{1}{2})$ of $g^{-1}([0,\frac{1}{2}])$ in K has Property Z in $g^{-1}([0,\frac{1}{2}])$, (by Lemma 2.2, as it is even collared in $g^{-1}([0,\frac{1}{2}])$, that is, there is a homeomorphism of a neighborhood of $g^{-1}(\frac{1}{2})$ in $g^{-1}([0,\frac{1}{2}])$ onto $g^{-1}(\frac{1}{2})\times[0, 1]$ carrying each point x of $g^{-1}(\frac{1}{2})$ to $(x, 0)$).

Since C collapses to $C\cap Y$ which collapses to $\operatorname{cl}(X\setminus Y)\cap Y$ which collapses to a point, Corollary 5.1 gives a homeomorphism θ of $(C\cap Y)\times Q$ onto Q and a homeomorphism ζ of $C\times Q$ onto Q. By Lemma 2.3, $(\operatorname{cl}(X\setminus C)\cap C)\times Q$ has Property Z in $C\times Q$ and in $(C\cap Y)\times Q$, so

$$\theta((\operatorname{cl}(X\setminus C)\cap C)\times Q) \quad\text{and}\quad \zeta((\operatorname{cl}(X\setminus C)\cap C)\times Q)$$

have Property Z in Q. Therefore, by Theorem 2.1 there is a homeomorphism η of Q onto itself extending $\theta\circ\zeta^{-1}|\zeta((\operatorname{cl}(X\setminus C)\cap C)\times Q)$. Now, $\theta^{-1}\circ\eta\circ\zeta$ is a homeomorphism of $C\times Q$ onto $(C\cap Y)\times Q$ which is the identity on $(\operatorname{cl}(X\setminus C)\cap C)\times Q$ and so may be extended to a homeomorphism of $X\times Q$ onto $Y\times Q$.

COROLLARY 5.5. *If X and Y are simply connected compact polyhedra of the same homotopy type, then $X\times Q$ is homeomorphic to $Y\times Q$.*

Proof. For simply connected compact polyhedra, the concepts of homotopy type and simple homotopy type coincide [15].

COROLLARY 5.6. *The product of a compact contractible polyhedron with the Hilbert cube is a Hilbert cube.*

REMARK. It is possible to generalize Theorem 5.2 in the following way: Let K be a locally finite simplicial complex and let $\{(\Delta_\alpha, \Delta'_\alpha)\}_\alpha$ be an indexed collection of pairs of simplices of K, each of which determines an elementary simplicial collapse of K. Let $L=K\setminus\bigcup_\alpha (\mathring{\Delta}_\alpha \cup \mathring{\Delta}'_\alpha)$, and say that L is obtained from K by a *formal contraction*. In analogy to the definition of simple homotopy type for compact polyhedra, say that two polyhedra X and Y which may be triangulated by locally finite simplicial complexes are *formally equivalent* if there exists a sequence $X=X_1, \ldots, X_n=Y$ of polyhedra and triangulations $f_i: K_i \to X_i$, $i=1,\ldots,n$, by locally finite simplicial complexes such that for each $i=1,\ldots,n-1$ there is either a formal contraction of K_i onto a subcomplex L_i with $f_i(L_i)=f_{i+1}(K_{i+1})$ or a formal contraction of K_{i+1} onto a subcomplex L_{i+1} with $f_i(K_i)=f_{i+1}(L_{i+1})$. (This definition follows J. H. C. Whitehead in [16].) The generalization of Theorem 5.2 is that for two polyhedra X_1 and X_2 which are formally equivalent and may be triangulated by locally finite simplicial complexes, $X\times Q$ and $Y\times Q$ are homeomorphic. It is not difficult to prove this. The necessary modifications of the proof of Theorem 5.2 are (1) to observe that the hypothesis of local finiteness for a (geometric) simplicial complex forces the components to be separable and locally compact. As the homeomorphism may be constructed component by component, the problem is immediately reduced to this case. (2) The neighborhood C in the proof of Theorem 5.2 may be chosen inside any other given neighborhood of $\operatorname{cl}(X\setminus Y)$ in X. (3) Thus, it is possible to break up $\{(\Delta_\alpha, \Delta'_\alpha)\}_\alpha$ into a countable collection of finite sets of pairs $\{A_i\}_{i=1}^\infty$ such that for each i there is a neighborhood U_i of

$$\bigcup \{\Delta_\alpha \cup \Delta'_\alpha \mid (\Delta_\alpha, \Delta'_\alpha) \in A_i\}$$

in X such that $\bar{U}_i$ is a compact polyhedron and $\bar{U}_i \cap \bar{U}_j = \varnothing$ if $|i-j|>1$. Now, for each i, it is possible to make a homeomorphism of $\bar{U}_{2i}\times Q$ onto

$$(\bar{U}_{2i}\setminus\bigcup \{\mathring{\Delta}_\alpha \cup \mathring{\Delta}'_\alpha \mid (\Delta_\alpha, \Delta'_\alpha) \in A_i\})\times Q$$

which is the identity on $(\operatorname{cl}(X\setminus U_{2i}) \cap \bar{U}_{2i})\times Q$. As these homeomorphisms are defined and supported on disjoint domains, they may be taken simultaneously to define a homeomorphism of $X\times Q$ onto

$$\left(X\setminus\bigcup\left\{\mathring{\Delta}_\alpha \cup \mathring{\Delta}'_\alpha \mid (\Delta_\alpha, \Delta'_\alpha) \in \bigcup_{i=1}^{\infty} A_{2i}\right\}\right)\times Q.$$

Analogously, one may then define a homeomorphism of this space onto

$$\left(X\setminus\bigcup\left\{\mathring{\Delta}_\alpha \cup \mathring{\Delta}'_\alpha \mid (\Delta_\alpha, \Delta'_\alpha) \in \bigcup_{i=1}^{\infty} A_{i}\right\}\right)\times Q.$$

A simple induction then finishes the proof.

Infinite products. Let X be a metric space and $\{X_i\}_{i=1}^\infty$ an indexed collection of copies of X. If Y is another space, the product $Y\times\prod_{i>0} X_i$ will be said to be *strongly homeomorphic to* $\prod_{i>0} X_i$ provided that for each $\varepsilon>0$ there is a homeomorphism g_ε of $Y\times\prod_{i>0} X_i$ onto $\prod_{i>0} X_i$ so that for each y in Y, the diameter of $p_1 \circ g_\varepsilon \circ p_Y^{-1} \circ (y)$ is less than ε.

The next theorem is an unpublished result of R. D. Anderson [1]; the proof given here was constructed by the author as a class exercise at Louisiana State University in February of 1966.

THEOREM 6.1. *Let X be a compact metric space and $\{X_i\}_{i=1}^{\infty}$, a countably infinite collection of copies of X. If $\{Y_i\}_{i=1}^{\infty}$ is a countably infinite collection of compact metric spaces with for each j, $Y_j \times \prod_{i>0} X_i$ strongly homeomorphic to $\prod_{i>0} X_i$, then $\prod_{i>0} Y_i$ is homeomorphic to $\prod_{i>0} X_i$.*

Proof. According to the conventions concerning metrics, each space X_i or Y_j, considered separately, is equipped with a normalized metric, but for x and x' in $Y_j \times \prod_{i>j} X_i$, $d(x, y) = 2^{-j} d(p_j(x), p_j(x')) + \sum_{i>j} 2^{-i} d(p_i(x), p_i(x'))$. This should be kept in mind when considering the subsequent choices of constants.

Let $g_1\colon Y_1 \times \prod_{i>1} X_i \to \prod_{i>0} X_i$ be a homeomorphism such that for all y in Y_1, dia $(p_1 \circ g_1 \circ p_1^{-1}(y)) < \frac{1}{2}$. Then for any pair x, x' of points of $\prod_{i>0} X_i$ with $d(p_1(x), p_1(x')) \geqq \frac{1}{2}$, $p_1 \circ g_1^{-1}(x) \neq p_1 \circ g_1^{-1}(x')$. Thus, for x and x' in $\prod_{i>0} X_i$ with

$$d(x, x') \geqq \tfrac{3}{4} = \tfrac{1}{4} + \tfrac{1}{2} \geqq \tfrac{1}{4} + \text{dia}\left(\prod_{i>1} X_i\right), \quad d(p_1(x), p_1(x')) \geqq \tfrac{1}{2},$$

and $p_1 \circ g_1^{-1}(x) \neq p_1 \circ g_1^{-1}(x')$.

Select an integer $n_2 > 1$ large enough that

$$2^{-n_2} \leqq \min \{d(g_1^{-1}(x), g_1^{-1}(x')) \mid d(x, x') \geqq 2^{-2}\}.$$

Now, because dia $(\prod_{i>n_2+1} X_i) \leqq 2^{-n_2-1}$, it must be true that for $x, x' \in \prod_{i>0} X_i$ with

$$\begin{aligned} d(x, x') \geqq \tfrac{1}{4}, \ \tfrac{1}{2} d(p_1 \circ g_1^{-1}(x), p_1 \circ g_1^{-1}(x')) + \sum_{i=2}^{n_2+1} 2^{-i} d(p_i \circ g_1^{-1}(x), p_i \circ g_1^{-1}(x')) \\ \geqq 2^{-n_2-1}. \end{aligned}$$

Let $\{\bar{g}_i\}_{i=2}^{n_2+1}$ be homeomorphisms of $Y_i \times \prod_{j>n_2+1} X_j$ onto $X_i \times \prod_{j>n_2+1} X_j$, respectively, such that for each y in Y_i, dia $(p_i \circ \bar{g}_i \circ p_i^{-1}(y)) < 2^{1-n_2}/n_2$. Let g_i be the extension of $\bar{g}_i$ to a homeomorphism of $\prod_{j=1}^{i} Y_j \times \prod_{j>i} X_j$ onto $\prod_{j=1}^{i-1} Y_j \times \prod_{j \geqq i} X_j$ by the identity in the other coordinates. If, now, x and x' are in $\prod_{i>0} X_i$ with $d(x, x') \geqq \frac{1}{4}$, then either $p_1 \circ g_1^{-1}(x) \neq p_1 \circ g_1^{-1}(x')$, or

$$\sum_{i=2}^{n_2+1} 2^{-i} d(p_i \circ g_1^{-1}(x), p_i \circ g_1^{-1}(x')) \geqq 2^{-n_2-1},$$

in which case for some i, $2^{-i} d(p_i \circ g_1^{-1}(x), p_i \circ g_1^{-1}(x')) \geqq 2^{-n_2-1}/n_2$, and so

$$d(p_i \circ g_1^{-1}(x), p_i \circ g_1^{-1}(x')) \geqq 2^{1-n_2}/n_2.$$

Since $p_i \circ g_{i-1}^{-1} \circ \cdots \circ g_1^{-1} = p_i \circ g_1^{-1}$,

$$d(p_i \circ g_{i-1}^{-1} \circ \cdots \circ g_1^{-1}(x), p_i \circ g_{i-1}^{-1} \circ \cdots \circ g_1^{-1}(x')) \geqq 2^{1-n_2}/n_2,$$

and $p_i \circ g_i^{-1} \circ \cdots \circ g_1^{-1}(x) \neq p_i \circ g_i^{-1} \circ \cdots \circ g_1^{-1}(x')$. Because $p_i \circ g_j^{-1} = p_i$ for each $j > i$, this shows that $g_{n_2+1}^{-1} \circ \cdots \circ g_1^{-1}(x)$ and $g_{n_2+1}^{-1} \circ \cdots \circ g_1^{-1}(x')$ differ in some Y_i-coordinate, $1 \leqq i \leqq n_2+1$.

Now, there exists an integer $n_3 > n_2+1$ such that for x and x' in $\prod_{i>0} X_i$ with $d(x, x') \geqq 2^{-3}$, then

$$d(g_{n_2+1}^{-1} \circ \cdots \circ g_1^{-1}(x), g_{n_2+1}^{-1} \circ \cdots \circ g_1^{-1}(x')) \geqq 2^{-n_3}.$$

The stage is now set for an inductive construction of a collection $\{g_i\}_{i=1}^{\infty}$ of homeomorphisms with each g_i being a homeomorphism of $\prod_{j=1}^{i} Y_j \times \prod_{j>i} X_j$ onto $\prod_{j=1}^{i-1} Y_j \times \prod_{j \geqq i} X_j$ for which $p_j \circ g_i = p_j$, if $j < i$, such that for any $\varepsilon > 0$ there is an integer n_ε with the property that for x and x' in $\prod_{i>0} X_i$ with $d(x, x') \geqq \varepsilon$, $g_{n_\varepsilon+1}^{-1} \circ \cdots \circ g_1^{-1}(x)$ differs from $g_{n_\varepsilon+1}^{-1} \circ \cdots \circ g_1^{-1}(x')$ in some Y_j-coordinate with $j \leqq n_\varepsilon + 1$. For any such collection $\{g_i\}_{i=1}^{\infty}$, the function $g: \prod_{i>0} X_i \to \prod_{i>0} Y_i$ defined by $p_i \circ g = p_i \circ g_i^{-1} \circ \cdots \circ g_1^{-1}$ is easily seen to be a homeomorphism.

The next theorem is a sharper version of Theorem 6.1 for the setting of the Hilbert cube (that is, if $X = I$). It is made possible by the homogeneity theorem of Anderson (Theorem 2.1).

THEOREM 6.2. *A countably infinite product of nondegenerate spaces is a Hilbert cube if the product of each space with the Hilbert cube is.*

Proof. Theorem 6.1 reduces the problem immediately to that of showing that if $\{X_i\}_{i=1}^{\infty}$ is a collection of nondegenerate spaces such that for each $i \geqq 1$, $X_i \times Q$ is a Hilbert cube, then $\prod_{i>0} X_i \times Q$ is strongly homeomorphic to $I \times Q$. This is because then given any such collection $\{X_i\}_{i=1}^{\infty}$, it may be written as the union of infinitely many, pairwise disjoint, infinite subcollections $A_j = \{X_{i_{j,k}}\}_{k=1}^{\infty}$. Then $\prod_{i>0} X_i$ is homeomorphic to $\prod_{j>0} (\prod_{k>0} X_{i_{j,k}})$, which by Theorem 5.1, would be homeomorphic to $Q = \prod_{i>0} I_i$.

In order to see that $\prod_{i>0} X_i \times Q$ is strongly homeomorphic to $I \times Q$, it is first necessary to observe that it is *homeomorphic* to $I \times Q$. This is a simple matter, for a "refactorization" as in the preceding paragraph gives Q homeomorphic to $\prod_{i>0} Q_i$, with each Q_i a Hilbert cube. Then $\prod_{i>0} X_i \times Q$ is homeomorphic to $\prod_{i>0} X_i \times \prod_{i>0} Q_i$, which is homeomorphic to $\prod_{i>0} (X_i \times Q_i)$, which, by hypothesis, is homeomorphic to $\prod_{i>0} Q_i$, hence to Q and to $I \times Q$.

Now, each point x of $\prod_{i>0} X_i$ has Property Z in $\prod_{i>0} X_i$. This is because for each i, X_i must be contractible. The contractibility of each X_i yields for any $\varepsilon > 0$, a homotopy F_ε from the identity map of $\prod_{i>0} X_i$ to a map of $\prod_{i>0} X_i$ into $\prod_{i>0} X_i \backslash \{x\}$ such that for each point y of $\prod_{i>0} X_i$, $\operatorname{dia}(F_\varepsilon(\{y\} \times I)) < \varepsilon$. (To construct F_ε, merely select an i large enough that $2^{-i} < \varepsilon$ and a contraction $G: X_i \times I \to X_i$ with $G(z, 0) = z$ for all z in X_i and $G(X_i \times \{1\}) = z_0 \neq p_i(x)$. The natural extension of G to a homotopy on $\prod_{i>0} X_i$ by the identity in the other coordinates will do for F_ε.) Lemma 2.2 now gives that x has Property Z in $\prod_{i>0} X_i$.

By Lemma 2.3, for any point x of $\prod_{i>0} X_i$, the Hilbert cube $\{x\}\times Q$ has Property Z in $\prod_{i>0} X_i \times Q$. Fix, then, x in $\prod_{i>0} X_i$ and let $\varepsilon>0$. By Theorem 2.1, there is a homeomorphism h of $\prod_{i>0} X_i\times Q$ onto $I\times Q$ carrying $\{x\}\times Q$ onto $\{0\}\times Q$. Let n be an integer large enough that $1/n<\varepsilon/2$, and select a sequence $1=\delta_0>\delta_1>\cdots>\delta_{n-1}>\delta_n=0$ such that the projections of $\{h^{-1}(\{\delta_i\}\times Q)\}_{i=0}^n$ into $\prod_{i>0} X_i$ are pairwise disjoint. Define $\bar{g}_\varepsilon$ to be a homeomorphism of I onto itself carrying each δ_i to $(n-i)/n$, and let g_ε be the homeomorphism of $I\times Q$ onto itself which is the extension of $\bar{g}_\varepsilon$ by the identity in the Q-coordinate. Let $f_\varepsilon=g_\varepsilon\circ h\colon \prod_{i>0} X_i\times Q \to I\times Q$. If z is any element of $\prod_{i>0} X_i$, then z is in at most one of the sets $p\ X_i\circ h^{-1}(\{\delta_i\}\times Q)$, with p the projection to $\prod_{i>0} X_i$, so $\{z\}\times Q$ meets at most two of the sets $h^{-1}([\delta_{i+1},\delta_i]\times Q)$, $i=0,\cdots,n-1$, and those must intersect. Therefore,

$$p_I\circ f_\varepsilon(\{z\}\times Q) = \bar{g}_\varepsilon\circ p_I\circ h(\{z\}\times Q) \subset \bar{g}_\varepsilon((\delta_{i+2},\delta_i)) = \left(\frac{n-i-2}{n},\frac{n-i}{n}\right),$$

for some $i=0,\ldots,n-2$. But $(n-i)/n-(n-i-2)/n=2/n<\varepsilon$, so dia $(p_I\circ f_\varepsilon(\{z\}\times Q))<\varepsilon$ for each z in $\prod_{i>0} X_i$. Therefore, $\prod_{i>0} X_i\times Q$ is strongly homeomorphic to $I\times Q$, and the theorem follows.

COROLLARY 6.1. *The product of a countably infinite collection of* (*nondegenerate*) *compact, contractible polyhedra is a Hilbert cube.*

The next theorem was also announced by Anderson in 1964 [1].

THEOREM 6.3. *The product of a countably infinite family of dendra is a Hilbert cube.*

Proof. By Theorem 6.2, it will suffice to show that the product of a dendron with the Hilbert cube is a Hilbert cube, so let X be a dendron.

According to Whyburn [17], there is a countable collection (which may be assumed infinite) $\{\alpha_i\}_{i=1}^\infty$ of arcs whose union is dense in X and for which, $\alpha_i\cap(\bigcup_{j=1}^{i-1}\alpha_j)=\{a_i\}$, an endpoint of α_i, $i>1$. Furthermore, $\lim_{i\to\infty}\text{dia}(\alpha_i)=0$. Also, the metric on X may be taken to be convex in the sense that if x, y, and z are in X and y separates x from z, then $d(x,y)+d(y,z)=d(x,z)$.

(See Bing [6] for the construction of convex metrics for finite-dimensional Peano spaces. In the simple case of a dendron X, one may be constructed by embedding X in $\prod_{i>0} I_i$ in such a way that α_1 goes to $I_1\times\prod_{i>1}\{0\}_i$, that α_2 goes to $\{p_1(\text{im}(a_2))\}\times I_2\times\prod_{i>2}\{0\}_i$, etc....)

Let f_1 be a homeomorphism of $\prod_{j>0} I_j$ onto $\alpha_1\times\prod_{j>0} I_j$. Now set $\delta_1>0$ small enough that for x and x' in $\prod_{j>0} I_j$ with $d(x,x')\geqq 2^{-1}$, $d(f_1(x),f_1(x'))\geqq\delta_1$. Let n_1 be a positive integer large enough that $2^{-n_1}\leqq\delta_1/2$. Now for x and x' in $\prod_{j>0} I_j$ with $d(x,x')\geqq 2^{-1}$, either $f_1(x)$ differs from $f_1(x')$ in an I_j-coordinate with $j\leqq n_1$ or $d(p_X\circ f_1(x),p_X\circ f_1(x'))\geqq\delta_1/2$. A sequence $\{f_i\}_{i=1}^\infty$ of homeomorphisms will be constructed with the intent that $\lim_{i\to\infty} f_i\circ\cdots\circ f_1$ will define a homeomorphism of $\prod_{j>0} I_j$ onto $X\times\prod_{j>0} I_j$. For each $j>1$, $p_i\circ f_j$ will be p_i for $i\leqq n_1$, so in order to ensure that in the limit, x and x' are carried to different points, it will suffice to guarantee that two points of $\alpha_1 \times \prod_{j>0} I_j$ differing in the X-coordinate by as much as $\delta_1/2$ cannot in the limit be carried onto points with identical X-coordinates.

Let f_2 be a homeomorphism of $\alpha_1 \times \prod_{j>0} I_j$ onto $(\alpha_1 \cup \alpha_2) \times \prod_{j>0} I_j$ which changes the I_j-coordinate of no point if $j \leqq n_1$, and is the identity off the open $(\delta_1/8)$-neighborhood of $\{a_2\} \times \prod_{j>0} I_j$. (This may be done by the same method used in the proof of Theorem 4.2.) If, now, x and x' lie in $\alpha_1 \times \prod_{j>0} I_j$ with $d(p_X(x), p_X(x')) \geqq \delta_1/2$, then at least one, say x, is left fixed by f_2. If a_2 separates $p_X(x)$ from $p_X \circ f_2(x')$, then

$$\begin{aligned} d(p_X(x), p_X \circ f_2(x)) &= d(p_X(x), a_2) + d(a_2, p_X \circ f_2(x')) \geqq d(p_X(x), a_2) \\ &\geqq (\delta_1/2) - \delta_1/8 = 3\delta_1/8. \end{aligned}$$

If, on the other hand, a_2 does not separate them, then $d(p_X(x), p_X \circ f_2(x')) \geqq (\delta_1/2) - \delta_1/4 = \delta_1/4$.

Let $\delta_2 > 0$ be small enough that for x and x' in $\prod_{j>0} I_j$ with $d(x, x') \geqq 2^{-2}$, $d(f_2 \circ f_1(x), f_2 \circ f_1(x')) \geqq \delta_2$. Assume, too, that $\delta_2 \leqq 2^{-2}\delta_1$ and that if $a_3 \neq a_2$, then $\delta_2 < d(a_2, a_3)$. Now let $n_2 \geqq n_1$ be an integer large enough that $2^{-n_2} \leqq \delta_2/2$. Let f_3 be a homeomorphism of $(\alpha_1 \cup \alpha_2) \times \prod_{j>0} I_j$ onto $(\bigcup_{i=1}^3 \alpha_i) \times \prod_{j>0} I_j$ which changes the I_j-coordinate of no point if $j \leqq n_2$ and which is the identity off the open $(\delta_2/8)$-neighborhood of $\{a_3\} \times \prod_{j>0} I_j$. If x and x' lie in $(\alpha_1 \cup \alpha_2) \times \prod_{j>0} I_j$ with $d(x, x') \geqq \delta_2/2$, then at least one, say x, is left fixed under f_3, and $d(p_X(x), p_X \circ f_3(x')) \geqq d(p_X(x), p_X(x')) - \delta_2/4$ for the following reasons: if $p_X(x')$ is in that component of the $(\delta_2/8)$-neighborhood of a_3 in $\alpha_1 \cup \alpha_2$ which contains a_3, then $p_X \circ f_3(x')$ must lie in the union of that set with α_3. This is because f_3 is the identity off the product of the open $(\delta_2/8)$-neighborhood of a_3 with $\prod_{j>0} I_j$. Hence,

$$\begin{aligned} d(p_X(x), p_X \circ f_3(x')) &\geqq d(p_X(x), a_3) + d(p_X \circ f_3(x'), a_3), \quad \text{if } a_3 \text{ separates them,} \\ &\geqq d(p_X(x), a_3) - d(p_X \circ f_3(x'), a_3), \\ &\qquad\qquad \text{if } a_3 \text{ does not separate them.} \end{aligned}$$

In either case,

$$\begin{aligned} d(p_X(x), p_X \circ f_3(x')) &\geqq d(p_X(x), a_3) - \delta_2/8 \\ &\geqq d(p_X(x), p_X(x')) - d(p_X(x'), a_3) - \delta_2/8 \\ &\geqq d(p_X(x), p_X(x')) - \delta_2/4. \end{aligned}$$

As $\delta_2 \leqq \frac{1}{4}\delta_1$, $\frac{1}{4}\delta_2 \leqq \frac{1}{16}\delta_1$, if x and x' are in $\prod_{j>0} I_j$ with $d(x, x') \geqq 2^{-1}$, then supposing that $p_j \circ f_1(x) = p_j \circ f_1(x')$ for all $1 \leqq j \leqq n_1$, we have

$$d(p_X \circ f_3 \circ f_2 \circ f_1(x), p_X \circ f_3 \circ f_2 \circ f_1(x')) \geqq \delta_1/2 - \delta_1/4 - \delta_1/16.$$

By induction, one may construct $\{f_i\}_{i=1}^{\infty}$ so that

(1) there is a sequence $\{\delta_i\}_{i=1}^{\infty}$ of positive numbers, with

$$\delta_{i+1} \leqq \min\left(\{\tfrac{1}{4}\delta_i\} \cup \{d(a_i, a_j) \mid a_j \neq a_i\}\right),$$

such that for x and x' in $\prod_{j>0} I_j$ with $d(x, x') \geqq 2^{-i}$,

$$d(f_i \circ \cdots \circ f_1(x), f_i \circ \cdots \circ f_1(x')) \geqq \delta_i,$$

(2) there is a monotonic sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers such that $2^{-n_i} \leqq \delta_i/2$, and

(3) f_i is a homeomorphism of $(\bigcup_{j=1}^{i-1} \alpha_j) \times \prod_{j>0} I_j$ onto $(\bigcup_{j=1}^{i} \alpha_j) \times \prod_{j>0} I_j$ which is the identity off the product with $\prod_{j>0} I_j$ of the open $(\delta_{i-1}/8)$-neighborhood in $\bigcup_{j=1}^{i-1} \alpha_j$ of a_i and which has the property that $p_j \circ f_i = p_j$ if $j \leqq n_{i-1}$.

Because of the fact that $\lim_{i\to\infty} \text{dia}(\alpha_i) = 0$ and the fact that condition (2) forces $\{n_i\}_{i=1}^{\infty}$ to be unbounded, condition (3) and the definition of X give that $\{f_i \circ \cdots \circ f_1\}_{i=1}^{\infty}$ is uniformly Cauchy. The fact that $\bigcup_{i=1}^{\infty} \alpha_i$ is dense in X together with the fact that $f_i \circ \cdots \circ f_1(\prod_{j>0} I_j) = (\bigcup_{k=1}^{\infty} \alpha_k) \times \prod_{j>0} I_j$ and the uniform Cauchiness of $\{f_i \circ \cdots \circ f_1\}_{i=1}^{\infty}$ yields that $f = \lim_{i\to\infty} f_i \circ \cdots \circ f_1$ is a map of $\prod_{j>0} I_j$ onto $X \times \prod_{j>0} I_j$. Therefore, in order to establish that f is a homeomorphism, there only remains to show that it is one-to-one. However, if x and x' are in $\prod_{j>0} I_j$ and if i is a large-enough integer that $2^{-i} \leqq d(x, x')$ then

$$d(f_i \circ \cdots \circ f_1(x), f_i \circ \cdots \circ f_1(x')) \geqq \delta_i.$$

Since $2^{-n_i} \leqq \frac{1}{2}\delta_i$ and since $\text{dia}(\prod_{j>n_i} I_j) = 2^{-n_i}$, either there is a $j \leqq n_i$ for which $p_j \circ f_i \circ \cdots \circ f_1(x) \neq p_j \circ f_i \circ \cdots \circ f_1(x')$ or

$$d(p_X \circ f_i \circ \cdots \circ f_1(x), p_X \circ f_i \circ \cdots \circ f_1(x')) \geqq \delta_i - 2^{-n_i} \geqq \tfrac{1}{2}\delta_i.$$

In the former case, condition (3) guarantees that $p_j \circ f(x) \neq p_j \circ f(x')$ and hence $f(x) \neq f(x')$. In the latter case, observe that for all $k > 1$, for y and z in $(\bigcup_{m=1}^{k-1} \alpha_m) \times \prod_{j>0} I_j$ with $d(p_X(y), p_X(z)) \geqq \delta_{k-1}/2$, the distance $d(p_X \circ f_k(y), p_X \circ f_k(z))$ from the X-coordinate of $f_k(y)$ to that of $f_k(z)$ is greater than or equal to $d(p_X(y), p_X(z)) - (\delta_{k-1}/4)$. (The verification of this is exactly like that of the special cases $k = 2, 3$ considered before: At most one, say z, of y and z is moved by f_k, as both cannot be within $\delta_{i-1}/8$ of $\{a_k\} \times \prod_{j>0} I_j$ simultaneously. If a_k separates $p_X(y)$ from $p_X \circ f_k(z)$, then

$$\begin{aligned} d(p_X(y), p_X \circ f_k(z)) &= d(p_X(y), a_k) + d(a_k, p_X \circ f_k(z)) \\ &\geqq d(p_X(y), a_k) \\ &\geqq d(p_X(y), p_X(z)) - d(p_X(z), a_k) \\ &\geqq d(p_X(y), p_X(z)) - \delta_{k-1}/8. \end{aligned}$$

If a_k does not separate $p_X(y)$ from $p_X \circ f_k(z)$, then $p_X \circ f_k(z)$ is in the open $(\delta_{k-1}/8)$-neighborhood of a_k, so $d(p_X \circ f_k(z), p_X(z)) \leqq \delta_{k-1}/4$, and hence

$$\begin{aligned} d(p_X \circ f_k(y), p_X \circ f_k(z)) &\geqq d(p_X(z), p_X \circ f_k(y)) - d(p_X(z), p_X \circ f_k(z)) \\ &= d(p_X(z), p_X(y)) - d(p_X(z), p_X \circ f_k(z)) \\ &\geqq d(p_X(z), p_X(y)) - \delta_{k-1}/4.) \end{aligned}$$

Therefore,

$$\begin{aligned} d(p_X \circ f(x), p_X \circ f(x')) &\geqq \delta_i/2 - \sum_{j=i}^{\infty} \delta_j/4 \\ &\geqq \delta_i/2 - \sum_{j=i}^{\infty} 2^{-2} \cdot 2^{-2(j-i)}\delta_i \\ &\geqq (\delta_i/2)\left(1 - \tfrac{1}{2} \sum_{j=0}^{\infty} 2^{-2j}\right) = \delta_i/6 > 0. \end{aligned}$$

Therefore, f is a homeomorphism and the theorem is proved.

References

1. R. D. Anderson, *The Hilbert cube as a product of dendrons*, Notices Amer. Math. Soc. **11** (1964). Abstract #614–649.

2. ———, *Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **72** (1966), 515–519. MR **32** #8298.

3. ———, *Topological properties of the Hilbert cube and the infinite product of open intervals*, Trans. Amer. Math. Soc. **126** (1967), 200–216. MR **34** #5045.

4. ———, *On topological infinite deficiency*, Michigan Math. J. **14** (1967), 365–383. MR **35** #4893.

5. R. D. Anderson and R. H. Bing, *A complete elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **74** (1968), 771–792. MR **37** #5847.

6. C. Bessaga and V. L. Klee, *Every non-normable Fréchet space is homeomorphic with all of its closed convex bodies*, Math. Ann. **163** (1966), 161–166. MR **34** #1826.

7. R. H. Bing, *A convex metric for a locally connected continuum*, Bull. Amer. Math. Soc. **55** (1949), 812–819. MR **11**, 194.

8. J. Eells and N. H. Kuiper, *Homotopy negligible subsets of infinite-dimensional manifolds*, Compositio Math. **21** (1969), 155–161.

9. D. W. Henderson, *Infinite-dimensional manifolds*, Proc. Internat. Sympos. Topology and its Applications, Herceg Novi, Jugoslavia, 1968.

10. ———, *Open subsets of Hilbert space*, Compositio Math. (to appear).

11. ———, *Infinite-dimensional manifolds are open subsets of Hilbert space*, Topology **9** (1970), 155–161.

12. N. H. Kuiper and D. Burghelea, *Hilbert manifolds*, Ann. of Math. (2) **90** (1969), 379–417.

13. N. Moulis, *Sur les variétés Hilbertiennes et les fonctions non-dégénérées*, Nederl. Akad. Wentensch. Proc. Ser. A **71** (1968), 497–512.

14. A. Szankowski, *On factors of the Hilbert cube*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **17** (1969), 703–709.

15. J. H. C. Whitehead, *Simplicial spaces, nuclei, and m-groups*, Proc. Lond. Math. Soc. (2) **45** (1939), 243–327.

16. ———, *Simple homotopy types*, Amer. J. Math. **72** (1950), 1–57. MR **11**, 735.

17. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R. I., 1942. MR **4**, 86.

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