COEFFICIENT ESTIMATES FOR DIRICHLET SERIES

BY

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1. Introduction. The primary purpose of this paper is to study coefficient estimates of Dirichlet series

\[ f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x), \quad \lambda_n \to \infty, \quad 0 < \lambda_1 < \lambda_2 < \ldots, \]

for which the sequence \( \{\lambda_n\} \) satisfies certain lacunarity conditions. D. Gaier [2] has used such estimates in one of his proofs of the high-indices theorem for Borel summability, in which results on the order of an entire function are inferred from its rate of growth along the real line. A by-product of his work is the estimate

\[ |a_n| \leq 2\lambda_n \|p_n\|_1 \] when \( \sum \lambda_n^{-1} < \infty, f \in L^1(0, \infty) \), and

\[ p_n = \prod_{k=n}^{\infty} \frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n}. \]

Some subsequent papers in much the same vein are due to G. Halász [3] and to J. M. Anderson and K. G. Binmore [1]. Halász has obtained estimates for \( a_n, \sum a_n, \) and \( \sum |a_n| \) when \( \lambda_n+1/\lambda_n \geq q > 1 \) and \( f \) satisfies appropriate conditions. The paper of Anderson and Binmore is concerned with the application of coefficient estimates to the study of entire functions. Their work includes the estimate

\[ |a_n| \leq (2\lambda_n)^{1/2} p_n \|f\|_2 \] where \( p_n \) is as above, \( \sum \lambda_n^{-1} < \infty, \) and \( f \in L^2(0, \infty) \).

L. Schwartz [8] had earlier obtained results which essentially include both the estimates of Gaier and of Anderson and Binmore. Indeed, as the referee has pointed out, S. Mandelbrojt [5] had also established inequalities of the same kind and these appear in larger form in [6]. Later, Mandelbrojt’s results were extended by F. Sunyer Belaguer [9], [10].

There is some question left to the sharpness of Schwartz’s estimates and in §2 it is proved that if \( 1 \leq p \leq 2 \) and \( f \in L^p(0, \infty) \) then

\[ |a_n| \leq (2\lambda_n)^{1/p} p_n \|f\|_p. \]

I do not know if the constant is sharp when \( p < 2 \), or if the order of the estimate still holds when \( p > 2 \) but in §2 there are estimates when \( p > 2 \) for functions which are subject to more stringent conditions. Also in §2 there are estimates for the sequence of partial sums.

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These results are applied in §3 to obtain some gap theorems. W. Rudin [7] showed that if $\lim \inf \lambda_{n+1}/\lambda_n = 1$ then there is a Dirichlet series so that $f' \in L^1(0, \infty)$ yet $\sum a_n$ diverges. In §3 it is shown that there are sequences $\{\lambda_n\}$ so that $\lim \lambda_{n+1}/\lambda_n = 1$, but so that if $f' \in L^p(0, \infty)$ when $p > 1$, or if $f'' \in L^1(0, \infty)$, then $\sum |a_n| < \infty$.

2. Coefficient estimates.

Theorem 2.1. If $f(x) = \sum_{n=1}^{\infty} a_n \exp (-\lambda_n x)$ converges for each $x > 0$, $\sum \lambda_n^{-1} < \infty$, and $f \in L^p(0, \infty)$, $1 \leq p \leq 2$, then $|a_n| \leq (2\lambda_n)^{1/p} p_n ||f||_p$ where

$$p_n = \prod_{k \neq n} \frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n}.$$

Proof. Let

$$f_N(x) = \sum_{k=1}^{N} a_k \exp (-\lambda_k x)$$

and

$$F(z) = \sum_{k=1}^{N} \frac{a_k \exp (-\lambda_k z)}{\lambda_k - z}.$$

Then if $\operatorname{Re} z < 0$

$$F(z) = \int_{0}^{\infty} e^{t} f_N(t + e) \, dt.$$

Let

$$B(z) = \prod_{k=1}^{N} \frac{\lambda_k - z}{\lambda_k + z}$$

so that

$$G(z) = F(z)B(z) = \sum_{k=1}^{N} \frac{a_k \exp (-\lambda_k z)}{z + \lambda_k} B_k(z)$$

where

$$B_k(z) = \left( \prod_{j \neq k} \frac{\lambda_j - z}{\lambda_j + z} \right).$$

Now $B_k(z)/(z + \lambda_k) \in H^2$, a Hardy class of functions in the right half-plane, and so the same is true of $G$. Consequently, $G$ may be represented by the Poisson integral of its values on the imaginary axis, and

$$\|G(x + iy)\|_q \leq \|G(iy)\|_q$$

if $x > 0$ and $q \geq 1$ [4, p. 124, and p. 128].

Next, let $\Gamma$ be the boundary of a rectangle in the right half-plane with sides parallel to the real and imaginary axes, and enclosing $\lambda_n$.

Let $\zeta^{2/p} = \exp (2/p \log \zeta)$ for $|\arg \zeta| < \pi/2$, with $\log 1 = 0$. Then by the residue theorem

$$\frac{a_n \exp (-\lambda_n e) B_n(\lambda_n)}{(2\lambda_n)^{2/p}} = -\frac{1}{2\pi i} \int_{\Gamma} \frac{F(z)B_n(z)}{(z + \lambda_n)^{2/p}} \, dz.$$
But since $G(z)$ goes to zero uniformly in any fixed half-plane $\text{Re } z = \delta > 0$ [4, p. 125] it follows that

$$a_n \exp\left(-\lambda_n e\right)B_n(\lambda_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(x'_n + iy)B_n(x'_n + iy)}{(x'_n + iy + \lambda_n)^{2/p}} \, dy$$

$$- \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(x''_n + iy)B_n(x''_n + iy)}{(x''_n + iy + \lambda_n)^{2/p}} \, dy$$

$$= I_1 + I_2$$

where $x'_n < \lambda_n < x''_n$.

Then

$$|I_1| \leq \frac{1}{2\pi} \|G(x''_n + iy)\|_q \left[\int_{-\infty}^{\infty} \left|\frac{x'_n + iy + \lambda_n}{x'_n + iy - \lambda_n}\right|^p \, dy \right]^{1/p}$$

where $1/p + 1/q = 1$. A similar estimate holds for $I_2$. Since $\|G(x''_n + iy)\|_q \leq \|G(y)\|_q$ we may let $x''_n \to \infty$ and $x'_n \to 0$ to conclude that

$$\left|\frac{B_n(\lambda_n)a_n \exp\left(-\lambda_n e\right)}{(2\lambda_n)^{2/p}}\right| \leq \frac{1}{2\pi} \|G(y)\|_q \left(\int_{-\infty}^{\infty} \frac{dy}{y^2 + \lambda_n^2}\right)^{1/p}$$

$$= \frac{1}{2\pi} \left(\frac{\pi}{\lambda_n}\right)^{1/p} \|G(y)\|_q.$$

Moreover, since $|B(y)| = 1$, then $\|G(y)\|_q = \|F(y)\|_q$, but $F(y)$ is the Fourier transform of $f_N(t + \epsilon)$, so that by the Hausdorff-Young inequality [11, p. 96], $\|G(y)\|_q \leq (2\pi)^{1/q} ||f_N(t + \epsilon)||_p$.

Combining these inequalities gives

$$|a_n \exp\left(-\lambda_n e\right)| \leq \frac{(2\lambda_n)^{1/p}}{|B_n(\lambda_n)|} ||f_N(t + \epsilon)||_p.$$
Theorem 2.2. If
(a) \( f(z) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n z) \) converges when \( \Re z > 0 \); 
(b) \( \|f(re^{\theta})\|_p \leq M \) whenever \( |\theta| \leq \eta = \delta \pi/2 < \pi/2 \); 
(c) \( \sum \lambda_n^{-1/\beta} < \infty \), where \( \beta + \delta = \gamma > 1 \), and \( \beta < 2 \); then

\[
\left| a_n \right| \leq \frac{2M\beta\lambda_n^{1/p}}{|q \cos(\gamma \pi/2)|^{1/q}} \prod_{k \neq n} \left| \frac{\lambda_k^{1/\beta} + \lambda_n^{1/\beta}}{\lambda_k^{1/\beta} - \lambda_n^{1/\beta}} \right|
\]

where \( 1/p + 1/q = 1 \).

Proof. Let

\[
\phi_N(z) = \sum_{k=1}^{N} a_k \exp(-\lambda_k z)
\]

and

\[
F(z) = \sum_{k=1}^{N} \frac{a_k \exp(-\lambda_k \varepsilon)}{\lambda_k - z}, \quad \varepsilon > 0.
\]

Then if \( \Re ze^{i\alpha} < 0 \), and \( |\alpha| < \pi/2 \)

\[
F(z) = \int_{0}^{\infty} \exp(zte^{i\alpha})\phi_N(te^{i\alpha} + \varepsilon)e^{i\alpha} \, dt.
\]

But if \( z = re^{i\theta} \) and \( \Re ze^{i\alpha} < 0 \), then

\[
|F(z)| \leq \left[ \int_{0}^{\infty} \exp(rtq \cos(\theta + \alpha)) \, dt \right]^{1/q} \left\| \phi_N(te^{i\alpha} + \varepsilon) \right\|_p,
\]

\[
= \left\| \phi_N(te^{i\alpha} + \varepsilon) \right\|_p \frac{1}{|qr \cos(\theta + \alpha)|^{1/q}}
\]

Now let \( (\zeta)^{1/\beta} \) be defined for \( |\arg \zeta| < \pi \), \( 1^{1/\beta} = 1 \), and set

\[
B(z) = \prod_{k=1}^{N} \frac{1 - (z/\lambda_k)^{1/\beta}}{1 + (z/\lambda_k)^{1/\beta}}.
\]

Similarly define \( (\zeta)^{1/q} \), so that letting

\[
\alpha = \pm \eta, \quad \phi_N(te^{i\alpha} + \varepsilon) \left| z^{1/q} F(z) B(z) \right| \leq \left\| \phi_N(te^{i\alpha} + \varepsilon) \right\|_p = A.
\]

Now \( z^{1/q} F(z) B(z) \) is of relatively slow growth for \( |\arg z| < \beta \pi/2 \), so that by the Phragmen-Lindelöf theorems [12, p. 180]

\[
|z^{1/q} F(z) B(z)| \leq A, \quad |\arg z| \leq \beta \pi/2.
\]

In particular, if \( z = \lambda_n \), then

\[
\frac{\lambda_n^{1/q} |a_n| \exp(-\lambda_n \varepsilon)}{2\beta \lambda_n} \prod_{k \neq n} \left| \frac{\lambda_k^{1/\beta} + \lambda_n^{1/\beta}}{\lambda_k^{1/\beta} - \lambda_n^{1/\beta}} \right| \leq A.
\]

As in the proof of Theorem 2.1, let \( N \to \infty \) and then \( \varepsilon \to 0 \) to obtain the desired inequality.
The next theorem deals with sequences of partial sums.

**Theorem 2.3.** If \( f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x) \) converges for each \( x > 0 \), \( \sum \lambda_n^{-1} < \infty \) and \( f' \in L^p(0, \infty) \), \( 1 < p \leq 2 \), then

\[
\left| \sum_{k=r}^{s} a_k \right| \leq A_p \| f' \|_p [C_{r-1} + C_s]
\]

where \( A_p \) is a constant depending on \( p \) alone, and

\[
C_n = \min_{\lambda_n < x < \lambda_{n+1}} \frac{1}{x^{1-1/p}} \prod_{k=1}^{n} \left| \frac{x + \lambda_k}{x - \lambda_k} \right|.
\]

**Proof.** The proof is very much the same as that of Theorem 2.1 and hence many of the details are omitted.

Let

\[
\phi_N(x) = \sum_{k=1}^{N} \lambda_k a_k \exp(-\lambda_k x),
\]

\[
F(z) = \sum_{k=1}^{N} \frac{\lambda_k a_k \exp(-\lambda_k(z + \epsilon))}{\lambda_k - z}, \quad \epsilon > 0,
\]

and

\[
B(z) = \prod_{k=1}^{N} \frac{\lambda_k - z}{\lambda_k + z}.
\]

If \( \Gamma \) is a rectangle in the right half-plane enclosing \( \lambda_r, \ldots, \lambda_s \) and no others then

\[
\sum_{k=r}^{s} a_k \exp(-\lambda_k \epsilon) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{F(z)}{z} \, dz
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(x_r + iy)}{x_r + iy} \, dy - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(x_s + iy)}{x_s + iy} \, dy
\]

\[
= I_1 + I_2,
\]

where \( \lambda_{r-1} < x_r < \lambda_r \leq \lambda_s < x_s < \lambda_{s+1} \).

If \( G(z) = F(z)B(z) \) then

\[
|I_1| \leq \frac{1}{2\pi} \| G(x_r + iy) \|_q D_r \left( \int_{-\infty}^{\infty} \frac{dy}{|x_r + iy|^p} \right)^{1/p}
\]

where \( 1/p + 1/q = 1 \) and

\[
D_r = \max_{y} \frac{1}{|B(x_r + iy)|}
\]

A similar estimate holds for \( I_2 \) and since

\[
\| G(x_r + iy) \|_q \leq \| G(iy) \|_q = \| F(iy) \|_q \leq (2\pi)^{1/q} \| \phi_N(t + \epsilon) \|_p
\]

it follows that

\[
\left| \sum_{k=r}^{s} a_k \exp(-\lambda_k \epsilon) \right| \leq A_p \| \phi_N(t + \epsilon) \|_p \left( \frac{D_r}{x_r^q} + \frac{D_s}{x_s^q} \right).
\]

Letting \( N \to \infty \) and then \( \epsilon \to 0 \) completes the proof.
Theorem 2.4. If \( f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x) \) converges for each \( x > 0 \), \( \sum \lambda_n x < \infty \), and \( f^* \in L^1(0, \infty) \) then

\[
\left| \sum_{k=r}^{s} a_k \right| \leq \frac{\| f^* \|}{2} \left[ \min_{\lambda_n < \lambda < \lambda_{n+1}} \frac{1}{x} - \frac{x \lambda_k}{x - \lambda_k} + \min_{\lambda_{n+1} < \lambda < \lambda_{n+2}} \frac{1}{x} - \frac{x \lambda_k}{x - \lambda_k} \right].
\]

Proof. The proof is very similar to the proof of Theorem 2.3 and so will be omitted.

3. Some applications. Before applying the estimates of §2 it is necessary to establish a result about the special sequence \( \lambda_n = \exp(n^\alpha) \).

Theorem 3.1. If \( \lambda_n = \exp(n^\alpha) \) and \( 1 > \alpha > 0 \) then

\[
P_n = \prod_{k=n}^{\infty} \left| \frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n} \right| \leq \exp(C n^{2(1-\alpha)})
\]

where \( C \) is a constant, dependent only on \( \alpha \).

Proof. Since \( \lambda_k/\lambda_{k-1} = \lambda_{k+1}/\lambda_k \), it follows that

\[
\sum_{k=n+1}^{\infty} \frac{1}{\lambda_k} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} - \sum_{k=1}^{n+1} \frac{1}{\lambda_k} \leq \sum_{j=1}^{\infty} \left( \frac{\lambda_{n+1}}{\lambda_{n+1}} \right)^j + \int_{n+1}^{\infty} \exp(-jx^\alpha) \, dx
\]

so that

\[
\sum_{j=1}^{\infty} \frac{1}{\lambda_{n+1}^j} \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^j} \leq \sum_{j=1}^{\infty} \left( \frac{\lambda_{n+1}}{\lambda_{n+1}} \right)^j + \int_{n+1}^{\infty} \frac{\lambda_n x^\alpha e^{-u}}{1 - \lambda_n x^\alpha e^{-u}} \, du
\]

But

\[
\int_{(n+1)^{1/\alpha}}^{\infty} u \lambda_{n+1}^{1-\alpha} e^{-u} \, du = (n+1)^{1-\alpha} \lambda_{n+1}^{-1} + \frac{1-\alpha}{\alpha} \int_{(n+1)^{1/\alpha}}^{\infty} \frac{e^{-u} u^{1/\alpha-1}}{u} \, du
\]

so that

\[
\sum_{j=1}^{\infty} \frac{1}{\lambda_{n+1}^j} \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^j} \leq \sum_{j=1}^{\infty} \left( \frac{\lambda_{n+1}}{\lambda_{n+1}} \right)^j + \int_{n+1}^{\infty} \frac{\lambda_n e^{-u}}{1 - \lambda_n e^{-u}} \, du
\]

But

\[
\int_{(n+1)^{1/\alpha}}^{\infty} u^{1/\alpha-1} e^{-u} \, du = (n+1)^{1-\alpha} \lambda_{n+1}^{-1} + \frac{1-\alpha}{\alpha} \int_{(n+1)^{1/\alpha}}^{\infty} \frac{e^{-u} u^{1/\alpha-1}}{u} \, du
\]
so for large values of $n$

$$\sum_{k=n+1}^{\infty} \frac{1}{\lambda_k} \leq \frac{1}{1 - \lambda_n/\lambda_{n+1}} + \frac{1}{\alpha} \frac{(n+1)^{1-\alpha}}{1 - \lambda_n/\lambda_{n+1}} \left(1 - \frac{1 - \alpha}{\alpha(n+1)^{\alpha-1}}\right)^{-1}$$

$$\sim \frac{1}{an^{\alpha-1}} + \frac{1}{c^2} \frac{1}{(n+1)^{2(\alpha-1)}} \leq \frac{2}{\alpha^2} \frac{1}{(n+1)^{2(\alpha-1)}}.$$ 

So

$$p_n = \prod_{k=1}^{n-1} \frac{\lambda_k + \lambda_2}{\lambda_k - \lambda_1} \prod_{k=n+1}^{\infty} \frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n}$$

$$\leq \exp \left(2 \sum_{k=1}^{n-1} \frac{\lambda_k}{\lambda_n - \lambda_k}\right) \exp \left(2 \sum_{k=n+1}^{\infty} \frac{\lambda_n}{\lambda_k - \lambda_n}\right)$$

$$\leq \exp Cn^{2(1-\alpha)}$$

where $C$ is some constant.

With this estimate the following theorems may be proved.

**Theorem 3.2.** If $p > 1$ then there is a sequence $\{\lambda_n\}$ such that if

$$f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$$

converges for each $x > 0$ and $f' \in L^p(0, \infty)$ then $\sum |a_n|^e < \infty$ for each $e > 0$.

**Proof.** Assume without loss of generality that $1 < p \leq 2$, for if $p > 2$ and $f' \in L^p(0, \infty)$ then $f' \in L^r(0, \infty)$ for each $r < p$. This is true since the integrability of $f'$ depends only on its integrability over $(0, 1)$.

Now by Theorem 2.1 and Theorem 3.1, if $\lambda_n = \exp(n^a)$ and $2/3 < a < 1$ then

$$|a_n|^e = O(\exp(eCn^{2(1-a)} - eq^{-1}n^a)),$$

and this estimate shows that $\sum |a_n|^e$ converges.

**Theorem 3.3.** There exists a sequence $\{\lambda_n\}$ such that if $f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$ converges for each $x > 0$ and $f'' \in L^1(0, \infty)$ then $\sum |a_n|^e$ converges.

**Proof.** This follows from Theorem 2.4. The proof is similar to that of Theorem 3.2.

**References**


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