COMMUTATORS MODULO THE CENTER IN A PROPERLY INFINITE VON NEUMANN ALGEBRA

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1. Introduction. An element \( C \) in a von Neumann algebra \( \mathcal{A} \) is said to be a commutator in \( \mathcal{A} \) if there are elements \( A \) and \( B \) in \( \mathcal{A} \) such that \( C = AB - BA \). For finite homogeneous discrete algebras and for properly infinite factor algebras the set of commutators has been completely described [1]-[5], [10]. In each of these special cases any element \( C \) is a commutator modulo a central element depending on \( C \). In this paper we show that given any element \( C \) in a properly infinite von Neumann algebra \( \mathcal{A} \) there is an element \( C_0 \) in the center of \( \mathcal{A} \) depending on \( C \) such that \( C - C_0 \) is a commutator in \( \mathcal{A} \). The element \( C_0 \) is an arbitrary element in the intersection \( \mathcal{H}_C \) of the center with the uniform closure of the convex hull of \( \{ U^*CU \mid U \text{ unitary in } \mathcal{A} \} \) [6, III, §5]. We then present a few facts about those elements \( C \) such that \( 0 \in \mathcal{H}_C \) or what is the same as far as determining commutators is concerned about those elements \( C \) such that \( 0 \in \mathcal{H}_{C_0}^{-1}cS \) for some invertible \( S \) in \( \mathcal{A} \).

2. Commutators. Let \( \mathcal{A} \) be a \( C^* \)-algebra with identity and let \( I \) be a closed two-sided ideal in \( \mathcal{A} \). The image of the element \( A \in \mathcal{A} \) in the factor algebra \( \mathcal{A}(I) = \mathcal{A}/I \) under the canonical homomorphism of \( \mathcal{A} \) onto \( \mathcal{A}/I \) will be denoted by \( A(I) \). If \( \zeta \) is a maximal ideal of the center of \( \mathcal{A} \), the smallest closed two-sided ideal in \( \mathcal{A} \) containing \( \zeta \) is denoted by \( [\zeta] \). For simplicity we write \( A([\zeta]) \) as \( A(\zeta) \). The set of maximal (respectively, primitive) ideals of \( \mathcal{A} \) with the hull-kernel topology is called the strong structure space (respectively, structure space) of \( \mathcal{A} \). If \( \mathcal{A} \) is a von Neumann algebra, then the strong structure space \( M(\mathcal{A}) \) of \( \mathcal{A} \) is homeomorphic with the spectrum of the center \( S \) of \( \mathcal{A} \) under the map \( M \rightarrow M \cap \mathcal{F} \) [13]. This means \( M(\mathcal{A}) \) is extremely disconnected.

**Proposition 1.** Let \( \mathcal{A} \) be a properly infinite von Neumann algebra and let \( A \) be a fixed element of \( \mathcal{A} \). The function \( M \rightarrow \|A(M)\| \) of the strong structure space \( M(\mathcal{A}) \) of \( \mathcal{A} \) into the real numbers is continuous.

**Proof.** For every \( \alpha \geq 0 \) we know that the set \( X = \{ M \in M(\mathcal{A}) \mid \|A(M)\| \leq \alpha \} \) is closed. If \( I = \cap X \), then \( \|A(I)\| \leq \alpha \) [8, Lemma 1.9] and so \( \|A(M)\| \leq \alpha \) for every \( M \in M(\mathcal{A}) \) containing \( I \). Thus \( X = \{ M \in M(\mathcal{A}) \mid I \subseteq M \} \).

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Conversely, let $\alpha > 0$; we show that

$$Y = \{ M \in M(\mathcal{A}) \mid \| A(M) \| \geq \alpha \}$$

is closed in $M(\mathcal{A})$. Let $J$ be the strong radical of $\mathcal{A}$ and let $\mathcal{P}$ be the structure space of $\mathcal{A}(J)$. The set

$$Y' = \{ K \in \mathcal{P} \mid \| A(J)(K) \| \geq \alpha \}$$

is compact (but not necessarily closed) in $\mathcal{P}$ [16, 4.9.18]. If $\mathcal{P}'$ is the structure space of $\mathcal{A}$, then $M \to M(J)$ defines a homeomorphism of $\{ M \in \mathcal{P}' \mid M \supset J \} = h(J)$ onto $\mathcal{P}$ [16, 2.6.6]. But if $M \in \mathcal{P}'$, then the intersection of $M$ with the center of $\mathcal{A}$ is a maximal ideal. So $M \in h(J)$ implies $M$ is of the form $J + [\xi]$ for some maximal ideal $\xi$ of the center. It is then clear that $h(J)$ is the set of maximal ideals of $\mathcal{A}$ [10, Proposition 2.3]. Furthermore, the topology of $h(J)$ and $M(\mathcal{A})$ coincide. This proves that $Y$ is compact in $M(\mathcal{A})$ since it is the inverse image of $Y'$ under the homeomorphism $M \to M(J)$ of $M(\mathcal{A})$ onto $\mathcal{P}$. Because $M(\mathcal{A})$ is homeomorphic to the spectrum of the center which is Hausdorff, every compact set of $M(\mathcal{A})$ is closed. Thus $Y$ is a closed subset of $M(\mathcal{A})$. Q.E.D.

**Remark.** If $\mathcal{A}$ is not properly infinite, Proposition 1 is certainly not true.

Let $H$ be a Hilbert space and let $A$ be a bounded linear operator on $H$. Let $F$ be a projection on $H$. Define the numerical gauge $\eta_A(F)$ to be

$$\eta_A(F) = \operatorname{ lub } \{ \| Ax - (Ax, x)x \| \mid x \text{ is a unit vector in } F(H) \}.$$ 

Let $\mathcal{W}_A(F)$ be the closure of the convex set

$$\{(Ax, x) \mid x \text{ a unit vector in } F(H)\}.$$ 

For every $\alpha \in \mathcal{W}_A(F)$ we have that

(1) \quad $\| (A - \alpha)F \| \leq 65 \eta_A(F)$.

This can be obtained by a simple reworking of Lemma 2.3 [2].

Let $\mathcal{A}$ be a properly infinite von Neumann algebra with no $\sigma$-finite type III direct summands; then for each projection $F$ in $\mathcal{A}$ and each element $A$ in $\mathcal{A}$ define $\nu_A(F)$ to be

(2) \quad $\nu_A(F) = \operatorname{ lub } \{ \| AE - EAE \| E \in (J), E \leq F \}$

where $(J)$ is the set of projections in the strong radical $J$ of $\mathcal{A}$. For every irreducible representation $\phi$ of $\mathcal{A}$ on a Hilbert space such that $\phi(J) \neq (0)$ we have that

(3) \quad $\eta_{\phi}(\phi(F)) \leq \nu_A(F)$

[10, Proposition 3.1]. Define $\nu(A)$ to be

$$\nu(A) = \operatorname{ glb } \{ \nu_A(F) \mid 1 - F \in (J) \}.$$
Let $\mathcal{A}$ now be the product of $\sigma$-finite type III algebras; let

$$
\nu(A) = \text{lub} \{ \|AE - EAE\| \mid E \text{ a projection in } \mathcal{A} \}
$$

for each $A \in \mathcal{A}$. If $A$ is in the complement in $\mathcal{A}$ of the set of all elements of $\mathcal{A}$ equal to scalar (zero included) multiples of the identity modulo some maximal ideal of $\mathcal{A}$, then there is a $\nu > 0$ such that $\nu(AP) \geq \nu$ for every nonzero central projection $P$ since there is a projection $E$ in $\mathcal{A}$ with $E \sim 1 - E \sim 1$ such that $EA^*(1 - E)AE \geq \alpha E$ for some scalar $\alpha > 0$ [10, Theorem 3.7]. Also it is easy to see from Proposition 3.1 [10] that

$$
(4) \quad \eta_{\phi(A)}(1) \leq \nu(A)
$$

for every irreducible representation of $\mathcal{A}$.

**Lemma 2.** Let $\mathcal{A}$ be a properly infinite von Neumann algebra such that

(i) either $\mathcal{A}$ has no $\sigma$-finite type III direct summands or

(ii) $\mathcal{A}$ is a product of $\sigma$-finite type III algebras.

Let $\{P_n\}$ be a set of mutually orthogonal central projections of $\mathcal{A}$ of sum $P$. Then $
u(AP) = \text{lub}_n \nu(AP_n)$ for every $A \in \mathcal{A}$.

**Proof.** Suppose $\mathcal{A}$ satisfies condition (i). Let $(I)$ be the set of projections in the strong radical of $\mathcal{A}$. Then given $\epsilon > 0$ there is a projection $F$ with $1 - F \in (I)$ such that

$$
\nu(AP) + \epsilon \geq \text{lub} \{ \| (1 - E)APE \| \mid E \leq F, E \in (I) \}.
$$

But

$$
\| (1 - E)APE \| \leq \| (1 - E)AEP_n \|
$$

for every $P_n$. So $\nu(AP_n) \leq \nu(AP) + \epsilon$ for every $n$. Since $\epsilon > 0$ is arbitrary, we have that $\text{lub} \nu(AP_n) \leq \nu(AP)$. Conversely, given that $\epsilon > 0$ there is for each $P_n$ a projection $F_n$ with $1 - F_n \in (I)$ such that

$$
\text{lub} \{ \| (1 - E)AP_n E \| \mid E \leq F_n, E \in (I) \} \leq \nu(AP_n) + \epsilon.
$$

Setting $F = \sum F_n P_n$, we have that $P - F \in (J)$ [10, Corollary, Proposition 2.2] and that

$$
\| (1 - E)APE \| = \text{lub}_n \| (1 - E)AP_n E \| \leq \nu(AP_n) + \epsilon
$$

for every $E$ in $(J)$ with $E \leq F$. Thus $\nu(AP) \leq \text{lub} \nu(AP_n) + \epsilon$. Since $\epsilon > 0$ is arbitrary, we have that $\nu(AP) \leq \text{lub} \nu(AP_n)$. This completes the proof for case (i).

The proof for case (ii) is similar.

**Lemma 3.** Let $\mathcal{A}$ be a properly infinite von Neumann algebra with no $\sigma$-finite type III direct summands. Let $G$ be a projection in $\mathcal{A}$ such that the orthogonal complement $1 - G$ of $G$ is in the set of projections $(J)$ of the strong radical of $\mathcal{A}$. Let $(J')$ be the strong radical of the reduced algebra $G\mathcal{A}G$. For any element $A$ in $\mathcal{A}$ we have that

$$
\nu'(GAG) = \text{glb} \{ \| (1 - E)GAGE \| \mid E \leq F, E \in (J') \mid G - F \in (J') \}
$$

is equal to $\nu(A)$.
 Proof. First we show that \((J') = \{ E \in (J) \mid E \subseteq G \}\). Suppose \(E \notin (J')\) for some \(E \in (J)\) majorized by \(G\). There is a nonzero projection \(P\) in the center of \(G\) such that \(PE \sim P\) [10, §2]. But \(P = QG\) for some nonzero \(Q\) in the center of \(\mathcal{A}\) [6, I, §2, Corollary, Proposition 2]. Then \(QE \sim QG \sim Q\) since \(G \cong 1\) [cf. 10, §2]. So \(E \notin (J)\). This is a contradiction. Hence \(\{ E \in (J) \mid E \subseteq G \} = (J')\). Conversely, suppose \(E \in (J')\). If \(Q\) is a central projection in \(\mathcal{A}\) with \(EQ \sim Q\), then \(EQ \sim Q \sim QG\). This implies \(Q = 0\). So \((J') \subseteq \{ E \in (J) \mid E \subseteq G \}\).

Now let \(\varepsilon > 0\) be given. There is by relation (2) a projection \(F\) in \(\mathcal{A}\) with \(1 - F \in (J)\) such that \(\nu(A(F)) \leq \nu(A) + \varepsilon\). Let \(G'\) be the least upper bound of \(1 - G\) and \(1 - F\). Then \(G' \in (J)\) and \(1 - G' \leq F\) and \(1 - G' \leq G\). We see that
\[
G - (1 - G') = G' - (1 - G) \in (J')
\]
by the first paragraph. So
\[
\nu'(GAG) \leq \text{lub} \{ \| (1 - E) GAG \| \mid E \in (J'), E \leq 1 - G' \}
= \text{lub} \{ \| G(1 - E)AE \| \mid E \in (J), E \leq 1 - G' \}
\leq \text{lub} \{ \| (1 - E) AE \| \mid E \in (J), E \leq F \} \leq \nu(A) + \varepsilon.
\]
Since \(\varepsilon > 0\) is arbitrary, we have that \(\nu'(GAG) \leq \nu(A)\).

Conversely, let \(\varepsilon > 0\) be given; there is a projection \(F \in G\mathcal{A}\) with \(G - F \in (J')\) and
\[
\text{lub} \{ \| (G - E) AE \| \mid E \leq F, E \in (J') \} \leq \nu'(GAG) + \varepsilon.
\]
The domain support \(G'\) of \((1 - G)A\) is a projection in \((J)\) since \(G' \preceq 1 - G\) [6, III, §1, Proposition 2]; similarly, the domain support \(G''\) of \(G'F\) is a projection in \((J)\) majorized by \(F\). So \(1 - (F - G'') = (1 - G) + (G - F) + G''\) is a projection in \((J)\). Then
\[
\nu(A) \leq \text{lub} \{ \| (1 - E) AE \| \mid E \leq F - G'', E \in (J) \}.
\]
But
\[
GAGE = AE - (1 - G)AE = AE - (1 - G)AG'FE = AE - (1 - G)AG'FG'E = AE.
\]
So we see that
\[
\nu(A) \leq \text{lub} \{ \| (1 - E) GAGE \| \mid E \leq F - G'', E \in (J') \}
\leq \text{lub} \{ \| (1 - E) GAGE \| \mid E \leq F, E \in (J') \} \leq \nu'(GAG) + \varepsilon.
\]
Since \(\varepsilon > 0\) is arbitrary, we have that \(\nu(A) \leq \nu'(GAG)\). Q.E.D.

THEOREM 4. Let \(C\) be an element in a properly infinite von Neumann algebra \(\mathcal{A}\) and let \(C_0\) be an element in the intersection \(\mathcal{N}_G\) of the center of \(\mathcal{A}\) with the uniform closure of the convex hull of \(\{ U^*CU \mid U\ \text{unitary in} \ \mathcal{A} \}\). Then \(C - C_0\) is a commutator in \(\mathcal{A}\).

Proof. We first make a preliminary reduction. There is a maximal central projection \(P\) in \(\mathcal{A}\) such that \((C - C_0)P\) is in the strong radical \(J\) of \(\mathcal{A}\) [10, Corollary,
Proposition 2.2]. However, the structure of the strong radical of \( \mathcal{A} \) allows us to conclude that \( (C - C_0)P \) is in the strong radical of \( \mathcal{A}P \) [10, §2]. Since \( (C - C_0)P \) is a commutator in \( \mathcal{A}P \), it is necessary to prove that \( (C - C_0)(1 - P) \) is a commutator in \( \mathcal{A}(1 - P) \). It is easy to see that the uniform closure of the convex hull of \( \{ U^*(C - C_0)U \mid U \text{ unitary in } \mathcal{A}(1 - P) \} \) contains 0. So without loss of generality we may assume that \( C \) is an element in \( \mathcal{A} \) such that \( 0 \in \mathcal{N}_C \) and such that \( CP \in J \) for some central projection \( P \) in \( \mathcal{A} \) implies that \( P = 0 \).

Now there is a central projection \( Q \) in \( \mathcal{A} \) such that

(i) \( \mathcal{A}Q \) has no \( \sigma \)-finite type III direct summands and
(ii) \( \mathcal{A}(1 - Q) \) is the product of \( \sigma \)-finite type III algebras.

It is clearly necessary to prove only that \( CQ \) and \( C(1 - Q) \) are commutators in \( \mathcal{A}Q \) and \( \mathcal{A}(1 - Q) \) respectively. Here \( 0 \in \mathcal{N}_{CQ} \) and \( 0 \in \mathcal{N}_{C(1 - Q)} \) is also true when these sets are formed relative to \( \mathcal{A}Q \) and \( \mathcal{A}(1 - Q) \) respectively. In the ensuing paragraphs we shall assume that either \( \mathcal{A} \) satisfies condition (i) or condition (ii).

Let \( \| C \| = \alpha \). We construct by induction a sequence \( \{ P_n \} \) of mutually orthogonal central projections of sum 1 such that for each nonzero central projection \( P \) majorized by \( P_n \) the number \( \nu(CP) \) lies in the real interval \( [2^{-n}\alpha, 2^{-n+1}\alpha] \). The induction hypothesis may be stated as follows: let \( P_0 = 0 \); then \( P_n \) is the largest central projection majorized by \( 1 - \sum \{ P_k \mid 0 \leq k \leq n - 1 \} \) such that \( \nu(CP) \in I_n \) for every nonzero central projection \( P \) majorized by \( P_n \). Suppose we have constructed \( P_1, P_2, \ldots, P_n \). We find \( P_{n+1} \). We may assume \( R = 1 - \sum \{ P_k \mid 0 \leq k \leq n \} \) is nonzero. There is a maximal ideal \( M \) in the strong structure space \( M(\mathcal{A}) \) of \( \mathcal{A} \) such that \( CR(M) \) is not in the center of \( \mathcal{A}(M) \). Indeed, suppose \( CR(M) \) is in the center of \( \mathcal{A}(M) \). Then \( R \neq 0 \) implies \( CR(M') \) is not in the center of \( \mathcal{A}(M') \) for every \( M \in M(\mathcal{A}) \). Given \( \varepsilon > 0 \) there are unitary operators \( U_1, U_2, \ldots, U_m \) in \( \mathcal{A} \) and positive scalars \( \alpha_1, \alpha_2, \ldots, \alpha_m \) of sum 1 such that \( \| \sum \alpha_i U_i^*CU_i \| < \varepsilon \) since \( 0 \in \mathcal{N}_C \). Thus

\[
\| CR(M) \| = \| \sum \alpha_i U_i^*CU_i \| (M) \| < \varepsilon
\]

for every \( M \in M(\mathcal{A}) \). Because \( \varepsilon > 0 \) is arbitrary, we have that \( CR(M) = 0 \) for every \( M \in M(\mathcal{A}) \). This means that \( CR \) is in the strong radical of \( \mathcal{A} \). This is contrary to the choice of \( C \). Hence, we must conclude that \( CR(M) \) is not in the center of \( \mathcal{A}(M) \) for at least one \( M \) in \( M(\mathcal{A}) \). Then there is a projection \( E \) in \( \mathcal{A} \) such that \( \| (1 - E)CRE(M) \| \neq 0 \). By the continuity of \( M' \rightarrow \| (1 - E)CRE(M') \| \) on \( M(\mathcal{A}) \) (Proposition 1), there is an open and closed neighborhood \( X \) of \( M \) in \( M(\mathcal{A}) \) such that for every \( M' \in X \) the element \( CR(M') \) is not in the center of \( \mathcal{A}(M') \). Let \( Q \) be the nonzero central projection of \( \mathcal{A} \) which determines \( X \) by the relation \( X = \{ M' \in M(\mathcal{A}) \mid Q \neq M' \} \) [13]. Then \( Q \) is majorized by \( R \) and \( CQ(M') \) is not in the center of \( \mathcal{A}Q(M') \) for every \( M' \) in the strong structure space \( M(\mathcal{A}Q) \) of the algebra \( \mathcal{A}Q \). The latter is true because \( M' \rightarrow M'Q \) defines a homeomorphism of \( \{ M' \in M(\mathcal{A}) \mid Q \neq M' \} \) onto \( M(\mathcal{A}Q) \) [16, Theorem 2.6.6].
Then by 3.1 and 3.7 [10] there is a $\nu > 0$ such that $\nu(CQ') \geq \nu$ for every nonzero central projection $Q'$ majorized by $Q$. It is clearly immaterial whether $\nu(CQ')$ is evaluated in $\mathcal{A}Q$ or in $\mathcal{A}$. Let $m$ be the smallest integer for which there is a nonzero central projection $Q$ majorized by $R$ such that $\nu(CQ') \geq 2^{-\alpha_m}$ for every nonzero central projection $Q'$ majorized by $Q$. We then have that $\nu(CQ') \in I_m$ for every such $Q'$. In fact by the choice of $m$ the projection $Q$ is easily seen to be the least upper bound of a set $\{Q_i\}$ of nonzero mutually orthogonal central projections which satisfy $2^{-\alpha_m} \leq \nu(CQ_i) < 2^{-\alpha_m+1}\alpha$. By Lemma 2 we may conclude that $\nu(CQ') \leq 2^{-\alpha_m+1}\alpha$. So for every nonzero central projection $Q'$ majorized by $Q$ we have that $\nu(CQ') \leq \nu(CQ) \leq 2^{-\alpha_m+1}\alpha$. Now if $n+1 < m$ set $P_{n+1}$ equal to zero. If $m \leq n+1$, then $m=n+1$. Indeed suppose that $m < n+1$; the projection $P_n + Q \neq P_m$, and for any nonzero central projection $Q'$ majorized by $P_n + Q$ we have that $\nu(CQ') = \nu(CQ') = \nu(CQ')$ $\nu(CQ') = \nu(CQ') 
exists I_m$. This contradicts the definition of $P_m$. Therefore $m=n+1$. Now we argue as follows. Let $\{Q_i\}$ be a maximal set of nonzero mutually orthogonal central projections majorized by $R$ such that $\nu(CQ') \in I_{n+1}$ for every nonzero central projection $Q'$ majorized by $Q$ majorized by some $Q_i$. Let $P_{n+1} = \sum Q_i$. It is clear that $P_{n+1} \leq R$. Because $\nu(CQ) = \nu(CQ) \nu(CQ)$ for any nonzero central projection $Q$ majorized by $P_{n+1}$ (Lemma 2) and since at least one projection $Q_i$ is nonzero, we have that $\nu(CQ) \leq 2^{-(n+1)\alpha}$. On the other hand $Q \leq R$ and so by the induction hypothesis there is a set $\{R_i\}$ of nonzero mutually orthogonal central projections of sum $Q$ such that $\nu(CR_i) < 2^{-\alpha_m}$ for each $R_i$. Thus $\nu(CQ) \leq 2^{-\alpha_m} \nu(CQ)$ (Lemma 2). This proves that $\nu(CQ) \in I_{n+1}$. It is clear that $P_{n+1}$ is the largest central projection majorized by $R$ such that $\nu(CQ) \in I_{n+1}$ for every nonzero central projection majorized by $P_{n+1}$. Suppose that the sequence $\{P_n\}$ with the required properties has been constructed by induction. We show that $\sum P_n = 1$. If $R = 1 - \sum P_n$, then for each $n=1, 2, \ldots$ we may conclude that $\nu(CR) \leq 2^{-\alpha_m}$ by performing the construction of the previous paragraph. This means that $\nu(CR) = 0$. The results of the previous paragraph show that $R = 0$ by our choice of $C$. Hence $\sum P_n = 1$. Let $Z$ be the spectrum of the center of $\mathcal{A}$ and let $X_n = \{\xi \in Z \mid P_n \notin \xi\}$ for $n=1, 2, \ldots$. Now suppose that $\mathcal{A}$ has no $\sigma$-finite type III direct summands. For each $n=1, 2, \ldots$ there is a projection $F_n$ in $\mathcal{A}P_n$ with $P_n - F_n \in J$ such that $\nu(CP_n(F_n)) \leq 2\nu(CP_n)$ by definition. For each $\xi \in Z$ there is an irreducible representation $\psi_\xi = \psi_\xi$ of $\mathcal{A}$ on a Hilbert space whose kernel is the smallest closed two-sided ideal $[\xi]$ of $\mathcal{A}$ which contains $\xi$ [11, Theorem 4.7]. Then $\psi$ does not annihilate $J$. Indeed $J$ contains a projection $E$ of central support 1, cf. [10, §2]. The two-valued continuous function $\xi' \mapsto \|E(\xi')\|$ [9, Lemma 10] on $Z$ assumes the value 0 on an open and closed set given by $\{\xi' \in Z \mid R \notin \xi'\}$ where $R$ is a central projection. Then $\|E(1-R)-E(\xi')\| = 0$ for every $\xi' \in Z$. Since $\cap \{[\xi'] \mid \xi' \in Z\} = \{0\}$ cf. [9, §4, remarks preceding Lemma 9], we have that $E(1-R) = E$. But this means that $R=0$ since the central support of $E$ is 1. This proves that $E(\xi') \neq 0$ for every $\xi' \in Z$. 
and in particular $E(t) \neq 0$. So $\psi$ does not annihilate $J$. Thus relations (1) and (3) imply that

$$\|(\psi(CP_n) - \beta)\psi(F_n)\| \leq 65\gamma_{CP_n}(\psi(F_n)) \leq 65\nu_{CP_n}(F_n) \leq 130\nu(CP_n)$$

whenever $\beta \in W_{\psi(CP_n)}(\psi(F_n)) = W$ and $\zeta \in X_n$. We show that $0 \in W$. Indeed, given $\epsilon > 0$, there is a set $U_1, U_2, \ldots, U_m$ of unitary elements in $\mathfrak{A}$ and positive scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$ of sum 1 such that $\sum \alpha_i U_i^*CU_i \| x \| < \epsilon$. If $G_i$ is the range projection of $U_i^*(P_n - F_n)$ for $i = 1, 2, \ldots, m$, then $G_i$ is a projection in $J$ and $G = \text{lub} G_i$ is a projection in $J$. Thus the projection $P_n - G$ is equivalent to $P_n$. Let $x$ be a unit vector in the subspace determined by $\psi(P_n - G)$. For each $U_i$ we have that $\psi(U_i)x = y_i$ is in the orthogonal complement of the subspace determined by $\psi(F_n - F_n)$ and thus $\psi(F_n)y_i = y_i$. Therefore

$$\sum \alpha_i(\psi(CP_n)y_i, y_i) \in W$$

since $W$ is convex. But we have that

$$\left| \sum (\alpha_i\psi(CP_n)y_i, y_i) \right| \leq \|\psi\| \sum \alpha_i U_i^*CU_i \| x \|^2 < \epsilon.$$ 

Because $W$ is closed and because $\epsilon > 0$ is arbitrary, we see that $0 \in W$. The relation (5) now becomes

$$\|CP_nF_n(\zeta)\| = \|CF_n(\zeta)\| = \|\psi_2(CP_n)\psi_2(F_n)\| \leq 130\nu(CP_n),$$

for every $\zeta \in X_n$. The orthogonal complement of the projection $F = \sum F_n$ is in $J$ since $P_n(1 - F) \in J$ for every $n = 1, 2, \ldots$ [10, Proposition 2.2]. For every nonzero central projection $P$ majorized by $P_n$ we have that

$$\|CPF\| = \text{lub} \{\|CF(\zeta)\| \mid P \notin \zeta, \ z \in Z\} \leq 130\nu(CP_n) \leq 260\nu(CP).$$

So for any central projection $P$ we have that

$$\|CPF\| = \text{lub} \{\|CPF_Pn\| \mid n = 1, 2, \ldots\}$$

$$\leq \text{lub} \{260\nu(CPP_n) \mid n = 1, 2, \ldots\}$$

$$\leq 260\nu(CP)$$

by Lemma 2.

Now for an algebra which is the product of $\sigma$-finite type III algebras, we may show that $\|CP\| \leq 260\nu(CP)$ for any central projection $P$. The proof is entirely similar to that just given except that relation (4) replaces relations (1) and (3).

Now let us suppose that $\mathfrak{A}$ has no $\sigma$-finite type III direct summands. Let $D = FCF$ and let $\mathfrak{B}$ be the von Neumann algebra $F \mathfrak{A} F$. By setting $Q_n = P_nF$ we obtain a sequence $\{Q_n\}$ of mutually orthogonal central projections in $\mathfrak{B}$ of sum $F$ such that $\nu(DQ) \in I_n$ for any nonzero central projection $Q$ in $\mathfrak{B}$ majorized by $Q_n$. Here $\nu(DQ)$ is evaluated in $\mathfrak{B}$ and Lemma 3 is employed. By relation (6) we see that
\( \{\|2^nDQ_n\|\} \) is a bounded sequence and hence \( B = \sum 2^nDQ_n \) defines an element of \( \mathcal{A} \) such that \( \nu(BQ) \geq \alpha \) for every nonzero central projection \( Q \) in \( \mathcal{B} \). Indeed,

\[
\nu(BQ) = \mathrm{lub} \{\nu(BQQ_n) \mid n = 1, 2, \ldots\} = \mathrm{lub} \{2^n\nu(DQQ_n) \mid n = 1, 2, \ldots\} \geq \alpha
\]

since at least one projection \( QQ_n \) is nonzero. There is an invertible element \( S \) in \( \mathcal{B} \) and a projection \( G \) in \( \mathcal{B} \) with \( F \sim G \sim F - G \) such that \( U*S^{-1}BSU = 0 \) and \( V*S^{-1}BSV \) is a commutator in \( \mathcal{B} \). Here \( U \) and \( V \) are partial isometries in \( B \) such that \( U*U = V*V = F, \quad UU* = G \) and \( VV* = F - G \) [10, Theorem 3.6]. By multiplying both \( U*S^{-1}BSU \) and \( V*S^{-1}BSV \) by the central element \( \sum 2^{-n}Q_n \) we see that \( U*S^{-1}DSU = 0 \) and \( V*S^{-1}DSV \) is a commutator in \( \mathcal{B} \). Now let \( T = S + (1 - F) \) in \( \mathcal{A} \); the element \( T \) is invertible with inverse \( T^{-1} = S^{-1} + (1 - F) \) where \( S^{-1} \) still denotes the inverse of \( S \) in \( \mathcal{B} \). Let \( W \) be a partial isometry in \( \mathcal{A} \) with domain support \( 1 \) and range support \( F \). Then \( V_1 = VW \) is a partial isometric operator of domain support \( 1 \) and range support \( F - G \). Then it is easy to see that \( V_1^*T^{-1}CTV_1 = W*V*S^{-1}DSVW \) is a commutator in \( \mathcal{B} \). We have that \( 1 \sim F \sim G < G + (1 - F) \). Thus there is a partial isometry \( U_1 \) in \( \mathcal{A} \) with domain support \( 1 \) and range support \( G + (1 - F) \). Then

\[
U_1^*T^{-1}CTU_1 = U_1^*T^{-1}CT(1 - F)U_1 + U_1^*(1 - F)T^{-1}CTGU_1
\]

is an element of the strong radical of \( \mathcal{A} \) and therefore, is a commutator in \( \mathcal{A} \) [10, Theorem 2.5]. We have proved that there is an isomorphism of \( \mathcal{A} \) onto the algebra \( \mathcal{A}_2 \) of \( 2 \times 2 \) matrices over \( \mathcal{A} \) which carries \( T^{-1}CT \) into the matrix \( (B_{ij}) \) where \( B_{11} \) and \( B_{22} \) are commutators in \( \mathcal{A} \). But this matrix is a commutator in the algebra \( \mathcal{A}_2 \). Indeed, let \( B_{11} = S_{11}T_{11} - T_{11}S_{11} \) and \( B_{22} = S_{22}T_{22} - T_{22}S_{22} \) for \( S_{11}, S_{22}, T_{11}, T_{22} \) in \( \mathcal{A} \). We may assume that \( S_{11} \) and \( S_{22} \) have disjoint spectra since \( B_{11} = (S_{11} + \beta)T_{11} - T_{11}(S_{11} + \beta) \) for a scalar \( \beta \). There is an operator \( T_{12} \) and an operator \( T_{21} \) in \( \mathcal{A} \) such that \( S_{11}T_{12} - T_{12}S_{11} = B_{12} \) and \( S_{22}T_{21} - T_{21}S_{22} = B_{21} \) [12]. Setting \( S_{21} = S_{12} = 0 \), we find by direct calculation that \( (S_{ij})(T_{ij}) - (T_{ij})(S_{ij}) = (B_{ij}) \) in \( \mathcal{A}_2 \). This proves \( (B_{ij}) \) is a commutator in \( \mathcal{A}_2 \) and \( T^{-1}CT \) is a commutator in \( \mathcal{A} \).

Now if \( \mathcal{A} \) is the product of \( \sigma \)-finite type III algebras, the preceding paragraph allows us to conclude that there is an invertible \( S \) in \( \mathcal{A} \) such that \( S^{-1}CS \) may be identified with the \( 2 \times 2 \) matrix \( (B_{ij}) \) over \( \mathcal{A} \) with \( B_{11} = 0 \) and \( B_{22} \) a commutator in \( \mathcal{A} \). So \( (B_{ij}) \) is a commutator in the \( 2 \times 2 \) matrices over \( \mathcal{A} \) and \( S^{-1}CS \) is a commutator in \( \mathcal{A} \). Q.E.D.

3. Elements \( C \) with \( 0 \in \mathcal{K}_C \). The construction of Theorem 4 actually depended upon choosing a central element \( C_0 \) corresponding to a given element \( C \) in a properly infinite von Neumann algebra \( \mathcal{A} \) such that \( ES^{-1}(C-C_0)SE = 0 \) for some invertible \( S \) in \( \mathcal{A} \) and some projection \( E \) in \( \mathcal{A} \) equivalent to \( 1 \). The next proposition clarifies this choice.
Proposition 5. If $C$ is an element in a properly infinite von Neumann algebra $\mathcal{A}$ such that $ECE=0$ for some projection $E$ in $\mathcal{A}$ equivalent to 1, then 0 is an element of the intersection $\mathcal{Z}_C$ of the center of $\mathcal{A}$ with the uniform closure of the convex hull of the set $\{U^*CU \mid U \text{ unitary in } \mathcal{A}\}$.

Proof. There are projections $E'$ and $E''$ such that $E' \sim E'' \sim E$ and $E' + E'' = E$ [6, III, §8, Theorem 1, Corollary 2]. Then $1 \sim E' \leq E' + (1 - E) \leq 1$ implies that $E' + (1 - E) \sim 1$. So there is no loss of generality in supposing that $E \sim 1 - E \sim 1$.

The operator $U = E - (1 - E)$ is unitary in $\mathcal{A}$ and

$$2^{-1}(C + U^*CU) = (1 - E)C(1 - E).$$

Now let $E_1, E_2, \ldots, E_n$ be orthogonal projections of sum $E$ such that $E_1 \sim \cdots \sim E_n \sim E$. There are unitary operators $U_1, U_2, \ldots, U_n$ in $\mathcal{A}$ such that $(1 - E)U_j$ has domain support $E_j$ for $j = 1, 2, \ldots, n$ since $E_j \sim 1 - E$ and $1 - E_j \sim E$ for $j = 1, 2, \ldots, n$. So we have that

$$\left\| \sum \{n^{-1}U_j^*(1 - E)C(1 - E)U_jx \mid j = 1, 2, \ldots, n\} \right\|^2 \leq n^{-2}\|C\|^2 \sum \|E_jx\|^2 \leq n^{-2}\|C\|^2\|x\|^2$$

for every $x$ in the Hilbert space. Thus

$$\left\| \sum n^{-1}U_j^*(1 - E)C(1 - E)U_j \right\| \leq n^{-1}\|C\|.$$

This means that 0 is an element of the uniform closure of the convex hull of $\{V^*CV \mid V \text{ unitary in } \mathcal{A}\}$ because $n$ is arbitrary. Hence $0 \in \mathcal{Z}_C$. Q.E.D.

Let $\mathcal{A}$ be a properly infinite von Neumann algebra. If we could prove that $0 \in \bigcup \{\mathcal{Z}_{S^{-1}CS} \mid S \text{ invertible in } \mathcal{A}\}$ for every commutator $C$ in $\mathcal{A}$, then we would have a complete characterization of the set commutators. This characterization is certainly valid for factor algebras. Indeed $C$ is a commutator in the properly infinite factor algebra $\mathcal{A}$ if and only if $\mathcal{A}$ is not a nonzero scalar multiple of the identity modulo the unique maximal ideal $M$ of $\mathcal{A}$. If $C \in M$, then $0 \in \mathcal{Z}_C$ [10, Proposition 2.4]. If $C$ is not a scalar multiple of the identity modulo $M$, then the canonical form of Brown and Pearcy [2] in conjunction with the preceding proposition shows $0 \in \mathcal{Z}_{S^{-1}CS}$ for some invertible $S$ in $\mathcal{A}$. The characterization though is at odds with a conjecture that the set of commutators in $\mathcal{A}$ is the complement $(F')$ in $\mathcal{A}$ of the set of all elements equal to a nonzero scalar multiple of the identity modulo some maximal ideal of $\mathcal{A}$ [4]. In fact let $\{P_n\}$ be a sequence of nonzero mutually orthogonal central projections of sum 1. (This presupposes that $\mathcal{A}$ has a sufficiently large center.) Then let $E_n$ be a projection in $\mathcal{A}P_n$ such that $E_n \sim P_n \sim P_n - E_n \ (n = 1, 2, \ldots)$. Let $C = \sum (n^{-1}P_n + n^{-2}E_n)$. We have that

$$\bigcup_n \{M \in M(\mathcal{A}) \mid P_n \notin M\}$$

is dense in the strong structure space $M(\mathcal{A})$ of $\mathcal{A}$ by the remarks at the beginning of §2. Then $C(M) = n^{-1}1(M) + n^{-2}E_n(M)$ for every $M$ with $P_n \notin M$ and clearly
$C(M)$ is not a scalar (zero included) multiple of the identity. But $S^{-1}CS(M)$ is not a scalar multiple of the identity for every $M$ with $P_n \notin M$. If for example $0 \in \mathcal{K}_{S^{-1}CS}$, then by the proof of Theorem 4 we would be able to find an invertible $T$ and a projection $E$ equivalent to $1$ with $ET^{-1}CTE = 0$. Thus
\[\|n^{-1}EP_n\| = n^{-2}\|ET^{-1}E_nTE\| \leq n^{-2}\|T^{-1}\| \|T\|\]
for each $n=1, 2, \ldots$. This is obviously impossible.

It might be well to remark that there is no canonical matrix form with 0 on the diagonal in the sense of Brown and Pearcy [2] for operators of class $(F')$.

**Lemma 6.** Let $C$ be an element in a von Neumann algebra $\mathcal{A}$. Let $D_1$ and $D_2$ be elements in $\mathcal{K}_C$ and let $A$ be a central element of $\mathcal{A}$ with $0 \leq A \leq 1$. Then $AD_1 + (1-A)D_2 \in \mathcal{K}_C$.

**Proof.** First let $A$ be a projection in the center of $\mathcal{A}$. There are unitary operators $U_1, U_2, \ldots, U_n$ (respectively $V_1, V_2, \ldots, V_m$) and positive scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ (respectively $\beta_1, \beta_2, \ldots, \beta_m$) of sum 1 such that
\[\|\sum \alpha_i U_i^* U_i - D_1\| < \epsilon \quad \text{(respectively, } \|\sum \beta_i V_i^* V_i - D_2\| < \epsilon).\]
Here $\epsilon > 0$ is a preassigned constant. Then $U_i' = U_iA + (1-A)$ ($1 \leq i \leq n$) and $V_i' = V_i(1-A) + A$ ($1 \leq i \leq m$) are unitary in $\mathcal{A}$ with the property
\[\|\sum \alpha_i V_i'^* U_i'^* U_i U_i - (D_1A + D_2(1-A))\| < \epsilon.\]
Since $\mathcal{K}_C$ is closed, we have that $AD_1 + (1-A)D_2 \in \mathcal{K}_C$.

Suppose that the restriction that $A$ is a projection is removed. Let $Z$ be the spectrum of the center of $\mathcal{A}$ and let $D^\sim(\zeta)$ denote the Gelfand transform of the central element $D$ evaluated at $\zeta \in Z$. Since $\mathcal{K}_C$ is convex, the set $\mathcal{K}_C = \{D^\sim(\zeta) \mid D \in \mathcal{K}_C\}$ is convex and so $A^\sim(\zeta)D^\sim(\zeta) + (1-A)^\sim(\zeta)D^\sim(\zeta) \in \mathcal{K}_C$ for every $\zeta \in Z$. Thus there is for each $\epsilon > 0$ a finite set $P_1, P_2, \ldots, P_n$ of orthogonal central projections of sum 1 and corresponding elements $B_1, B_2, \ldots, B_n$ in $\mathcal{K}_C$ such that
\[\|(AD_1 + (1-A)D_2 - B_j)P_j\| < \epsilon\]
for $j = 1, 2, \ldots, n$. Since $\sum B_jP_j \in \mathcal{K}_C$ by the first paragraph and since $\epsilon > 0$ is arbitrary, we see that $AD_1 + (1-A)D_2$ is in the closed set $\mathcal{K}_C$. Q.E.D.

The next proposition corresponds to a result of C. R. Putnam [14]. We use many of his calculations cf. [15, 1.5.1].

**Proposition 7.** Let $A, B, C$ be elements in a properly infinite von Neumann algebra $\mathcal{A}$. If $A$ is seminormal (i.e., if either $\pm (AA^* - A^*A)$ is positive) and if $C = AB - BA$, then $0 \in \mathcal{K}_{S^{-1}CS}$ for some invertible $S$ in $\mathcal{A}$. 
Proof. There is a projection $P$ in the center of $\mathcal{A}$ such that $CP$ is in the strong radical of $\mathcal{A}P$ and

$$\{M \in M(\mathcal{A}) | C(1-P)(M) \neq 0\}$$

is dense in the subset $\{M \in M(\mathcal{A}) | 1-P \not\in M\}$ of the strong structure space $M(\mathcal{A})$ of $\mathcal{A}$. Since $X_{CP} = X_{CP} + X_{(1-P)} = X_{(1-P)}$ due to the fact that $X_{CP} = \{0\}$ [10, Proposition 2.4] and since $\{M \in M(\mathcal{A}) | 1-P \not\in M\}$ is identified with the strong structure space of $\mathcal{A}(1-P)$, we may assume that $\{M \in M(\mathcal{A}) | C(M) \neq 0\}$ is dense in $M(\mathcal{A})$.

As we have previously argued we may assume that either $\mathcal{A}$ has no $\sigma$-finite type III direct summands or that $\mathcal{A}$ is the product of $\sigma$-finite type III algebras.

Now for each nonzero real scalar $\alpha$ and each unitary element $U$ in the center $\mathcal{Z}$ of $\mathcal{A}$ let

$$F(\alpha, U) = |\alpha|^{1/2}UA + |\alpha|^{-1/2}B^*.$$

Then

$$\text{sgn}(\alpha)(FF^* - F^*F) = D(\alpha) + \text{sgn}(\alpha)S(U)$$

where

$$D(\alpha) = \alpha(AA^* - A^*A) - \alpha^{-1}(BB^* - B^*B)$$

and $S(U) = UC + U^*C^*$. The set $\{|D| = (D^*D)^{1/2} \mid D \in X_C\}$ is monotonely decreasing in $\mathcal{Z}$. Indeed, let $D_1$ and $D_2$ be elements of $X_C$; there is a central projection $P$ such that $|D_1|P \leq |D_2|P$ and $|D_2|(1-P) \leq |D_1|(1-P)$. But $D_1P + D_2(1-P) \in X_C$ and $|D_1P + D_2(1-P)| = |D_1P| + |D_2(1-P)|$ is majorized by both $|D_1|$ and $|D_2|$. This proves that $\{|D| \mid D \in X_C\}$ is monotonely decreasing. Let $D_0$ be the greatest lower bound of this set [6, Appendix II]. Suppose $D_x$ is a positive central element which majorizes $D(\alpha)$ for some $\alpha$. We show that $D_0 \leq D_1$. If not, there is an $\varepsilon > 0$ and a nonzero central projection $P$ such that $D_0P \geq (D_1 + \varepsilon)P$. By reducing to $\mathcal{A}P$ we may assume that $P = 1$. Let $U_1, U_2, \ldots, U_n$ be unitary elements in $\mathcal{Z}$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive scalars of sum 1; then for any central element $R$ in the sphere of radius $2^{-1}\varepsilon$ about 0 we have that

$$\left(\sum \alpha_i U_i\right)D_1 + R \leq \left(\sum \alpha_i D_1 + |R|\right)^2 = (D_1 + |R|)^2 \leq (D_1 + 2^{-1}\varepsilon)^2.$$ 

Hence $(D^*Dx, x) \leq ((D_1 + 2^{-1}\varepsilon)^2 x, x)$ for any $D$ in the strong closure $X''$ of the convex hull of the set

$$\{UD_1 + R \mid U, R \in \mathcal{Z}, U \text{ unitary}, \|R\| \leq 2^{-1}\varepsilon\}$$

and for any $x$ in the Hilbert space $H$ of $\mathcal{A}$. On the other hand we see that $(D^*Dx, x) \geq ((D_1 + \varepsilon)^2 x, x)$ for any $D$ in the strong closure $X'$ of $X_C$ and any $x$ in $H$ because $R^*R \geq (D_1 + \varepsilon)^2$ for every $R$ in $X_C$. By the standard separation theorem there is a nonzero strongly continuous functional $f$ on $\mathcal{Z}$ such that

$$\text{lub} \{\text{Re} f(R) \mid R \in X'\} \leq \text{glb} \{\text{Re} f(R) \mid R \in X''\}.$$
Here \( \text{Re} \beta \) denotes the real part of the complex number \( \beta \). Indeed the element \( 0 \) is not in the strong closure of \( \mathcal{H}' - \mathcal{H}'' \). Since \( f \) is also weakly continuous on \( \mathcal{Z} \) [6, I, §3, Theorem 1 (i)], there is a unitary \( U \) in \( \mathcal{Z} \) and a nonzero vector \( x \) in \( H \) such that \( f(R) = (RUx, x) \) for every \( R \in \mathcal{Z} \) [17] and [6, III, §1, Corollary, Theorem 4]. Now let \( F = F(\alpha, \text{sgn} (\alpha)U) \). We have that

\[
\text{sgn} (\alpha)(FF^* - F^*F) = D(\alpha) + S(U) \leq D_1 + S(U).
\]

Let \( D \) be an arbitrary element in \( \mathcal{H}_{\text{sgn} (\alpha)(FF^* - F^*F)} = \mathcal{H} \). There is an element \( S \) in \( \mathcal{H}_{S(U)} \) such that \( D \leq D_1 + S \) [6, III, §5, Problem 2a]. We may find a \( T \) in \( \mathcal{H}_{UC - UC} \) such that

\[
2^{-1}(S + T) \in \mathcal{H}_{UC} \quad \text{and} \quad 2^{-1}(S - T) \in \mathcal{H}_{UC}.
\]

[6, III, §5, Problem 2a]. We then have that \( 2^{-1}U^*(S + T) \) and \( 2^{-1}U^*(S - T) \) are elements of \( \mathcal{H}_C \). The latter is true because \( \mathcal{H}_C = \{R^* \mid R \in \mathcal{H}_C \} \). From relation (7) we obtain that both \( \text{Re} (2^{-1}(S + T)x, x) \) and \( \text{Re} (2^{-1}(S - T)x, x) \) are majorized by \( \text{glb} \{\text{Re} f(R) \mid R \in \mathcal{H}''\} \). Thus

\[
(Sx, x) = \text{Re} (Sx, x) \leq 2 \text{glb} \{\text{Re} f(R) \mid R \in \mathcal{H}''\}
\]

since \( S \) is clearly selfadjoint. But \( -U^*D_1 - 2^{-1}eU^* \) is an element of \( \mathcal{H}'' \). So

\[
2^{-1}(Sx, x) \leq \text{Re} (U(-U^*D_1 - 2^{-1}eU^*)x, x) = -((D_1 + 2^{-1}e)x, x).
\]

Therefore,

\[
(Dx, x) \leq (D_1x, x) - e(x, x) - 2(D_1x, x) \leq -e(x, x)
\]

by relation (8). Using reasoning similar to that which we used to prove that \( \{\mid R \mid \mid R \in \mathcal{H}_C \} \) is monotonely decreasing, we may prove that \( \mathcal{H} \) is monotonely increasing. Setting \( R_0 = \text{lub} \mathcal{H} \), we see that \( (R_0x, x) \leq -e(x, x) \). We show that this is impossible by showing \( (R_0x, x) \geq 0 \). Indeed, in proving this then we may certainly assume that \( F \) is invertible and that \( \alpha > 0 \). Because \( F \) is invertible, there is a unitary operator \( V \) in \( \mathcal{A} \) obtained from the polar decomposition of \( F \) [6, Appendix III] such that \( V^*FF^*V = F^*F \). If \( R \in \mathcal{H}_{FF} \), then there are unitary operators \( U_1, U_2, \ldots, U_n \) in \( \mathcal{A} \) and positive scalars \( \alpha_1, \alpha_2, \ldots, \alpha_n \) of sum 1 such that

\[
\left\| \sum \alpha_iU_i^*F*FU_i - R \right\| < \epsilon'
\]

for any preassigned constant \( \epsilon' > 0 \). But this means that

\[
\left\| \sum \alpha_i(VU_i)^*FF^*(VU_i) - R \right\| < \epsilon'.
\]

Because \( \epsilon' > 0 \) is arbitrary we have that \( R \in \mathcal{H}_{FF} \). By symmetry it is then clear that \( \mathcal{H}_{PP} = \mathcal{H}_{FP} \). Now for any \( \epsilon' > 0 \) there is an element \( R_1 \) in \( \mathcal{H}_{FP} \) such that

\[
(R_1x, x) \geq (R_0x, x) - \epsilon'(x, x)
\]

where \( R_0 \) is the least upper bound of the monotonely increasing set \( \mathcal{H}_{FP} \). But
there is an element $R \in \mathcal{K}_{P,F}$ such that $R_1 - R \in \mathcal{K}$ [6, III, §5, Problem 2a]. However we have that

$$(R_0 x, x) \geq ((R_1 - R)x, x) \geq (R_2 x, x) - (Rx, x) - \epsilon'(x, x) \geq -\epsilon'(x, x)$$

since $R_2 \geq R$. Because $\epsilon' > 0$ is arbitrary, we see that $(R_0 x, x) \geq 0$. This is a contradiction. We must conclude that $D_0 \leq D_1$.

We now show that $D_0 = 0$. We may assume that $AA^* - A^*A \leq 0$. Then for $\alpha > 0$ we have that

$$D(\alpha) \leq -\alpha^{-1}(BB^* - B^*B) \leq 2\alpha^{-1}\|B\|^2.$$ 

Thus we see that $D_0 \leq 2\alpha^{-1}\|B\|^2$ for every $\alpha > 0$. Therefore $D_0 = 0$. If $Q$ is a nonzero central projection in $\mathcal{A}$ and if $\epsilon > 0$ there is a net $\{Q_n\}$ of mutually orthogonal central projections of sum $Q$ such that each set $\mathcal{K}_{Q_n}$ contains an element $D_n$ of norm not exceeding $\epsilon$. Indeed, if $\{Q_n\}$ is a maximal set of mutually orthogonal nonzero central projections majorized by $Q$ with this property, then the assumption that $Q' = Q - \sum Q_n \neq 0$ gives a contradiction. Since $\text{glb} \{\|D\| \mid D \in \mathcal{K}_{Q_n}\} = 0$, there is a $D \in \mathcal{K}_{Q'}$ such that $\|D\| \geq 2^{-1}\epsilon Q'$ is not true. This means that there is a nonzero central projection $Q''$ majorized by $Q'$ such that $\|D\|Q'' \leq \epsilon Q''$. This contradicts the maximality of $\{Q_n\}$. Hence we have that $\sum Q_n = Q$.

Now suppose $\mathcal{A}$ has no $\sigma$-finite type III direct summands. In Theorem 4 we constructed a sequence $\{P_n\}$ of mutually orthogonal central projections of sum 1 and a projection $F$ whose orthogonal complement $1 - F$ was in the strong radical such that $\|(C - R)FP\| \leq 260v(CP)$ whenever $P$ is a central projection majorized by $P_n$ and whenever $R \in \mathcal{K}_{C,F}$. Also either $P_n = 0$ or $v(CP) \in [2^{-n}\|C\|, 2^{-n+1}\|C\|]$ for every nonzero central projection $P$ majorized by $P_n$. By the preceding paragraph there is a set $\{P_{nj}\}$ of mutually orthogonal central projections of sum $P_n$ such that each set $\mathcal{K}_{C,P_{nj}}$ contains an element $D_{nj}$ of norm not exceeding $v(CP_n)$. Then for each nonzero central projection $P$ majorized by $P_n$ we have that

$$\|CFP\| = \text{lub} \|CFPP_{nj}\| \leq \text{lub} \|(C - D_{nj})PP_{nj}\| + \text{lub} \|D_{nj}PP_{nj}\| \leq 260v(CP) + v(CP_n) \leq 262v(CP)$$

by relation (6). By the same reasoning as found in Theorem 4, we may find an invertible $W$ in $\mathcal{A}$ such that $EW^{-1}CWE = 0$ for some projection $E$ in $\mathcal{A}$ which is equivalent to 1. However, this means that $0 \in \mathcal{K}_{W^{-1}CW}$ by Proposition 5.

If $A$ is the product of $\sigma$-finite type III algebras a similar proof holds. Q.E.D.

**Corollary.** If $F$ is an element in a properly infinite von Neumann algebra $\mathcal{A}$, then there is an invertible $S$ in $\mathcal{A}$ such that $\mathcal{K}_{S^{-1}(FS-FS),S}$ contains 0.

**Proof.** If $A = 2^{-1}(F-F^*)$ and $B = 2^{-1}(F+F^*)$, then $2^{-1}(FF^* - FF^*) = AB - BA$. Now Proposition 7 applies.

**Added in proof** (April 25, 1970). I have improved Proposition 7 by showing that $0 \in \mathcal{K}_C$. 

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