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SPECTRAL MAPPING THEOREMS AND PERTURBATION THEOREMS FOR BROWDER'S ESSENTIAL SPECTRUM

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Abstract. If T is a closed, densely defined linear operator in a Banach space, F. E. Browder has defined the essential spectrum of T, ess (T) [1]. We derive below spectral mapping theorems and perturbation theorems for Browder's essential spectrum. If T is a bounded linear operator and f is a function analytic on a neighborhood of the spectrum of T, we prove that $f(\text{ess}(T))=\text{ess}(f(T))$. If T is a closed, densely defined linear operator with nonempty resolvent set and f is a polynomial, the same theorem holds. For a closed, densely defined linear operator T and a bounded linear operator B which commutes with T, we prove that $\text{ess}(T+B)\subseteq\text{ess}(T)+\text{ess}(B)=\{\mu+\nu : \mu\in\text{ess}(T), \nu\in\text{ess}(B)\}$. By making additional assumptions, we obtain an analogous theorem for B unbounded.

Introduction. Let T be a closed, densely defined linear operator on a Banach space X. F. E. Browder [1] defined the essential spectrum of T, ess (T), to be the set of $\lambda\in\sigma(T)$, the spectrum of T, such that at least one of the following conditions holds: (1) $R(\lambda-T)$, the range of $\lambda-T$, is not closed; (2) λ is a limit point of $\sigma(T)$; (3) $\bigcup_{r>0} N[(\lambda-T)^r]$ is infinite dimensional, where $N(A)$ denotes the null space of a linear operator A. Browder proved that $\lambda_0\notin\text{ess}(T)$ if and only if $(\lambda-T)^{-1}$ is defined for $0<|\lambda-\lambda_0|<\delta$ and the Laurent expansion of $(\lambda-T)^{-1}$ around λ_0 has only a finite number of nonzero coefficients with negative indices.

Recall that a closed, densely defined operator T is called a Fredholm operator if and only if $\dim N(T)<\infty$, $R(T)$ is closed, and $\operatorname{codim} R(T)<\infty$. Let us write $G=\{\lambda\in \boldsymbol{C} \mid \lambda-T \text{ is a Fredholm operator}\}$. Since G is known to be an open set let us denote by G_i, $i\in I$, the open connected components of G. Gohberg and Kreĭn [2] proved (though they never formally defined the essential spectrum) that $\lambda_0\notin\text{ess}(T)$ if and only if λ_0 belongs to an open connected component G_i which has a nonempty intersection with the resolvent set of T.

There are several other characterizations of ess (T). D. Lay [5] has shown that if C is a compact, bounded linear operator such that C: $D(T)\to D(T)$ and $(CT)(x)=(TC)(x)$ for $x\in D(T^2)$, then $\text{ess}(T+C)=\text{ess}(T)$ and that conversely ess (T) is the largest subset of $\sigma(T)$ which remains invariant under such perturbations. Other authors have described the essential spectrum of T in terms of the ascent, descent, nullity and defect of T, but we shall not need these results.

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In this paper we shall find Lay's result especially useful. In the first section we shall establish spectral mapping theorems. If B is a bounded linear operator and f is a function analytic on a neighborhood of $\sigma(B)$, we shall prove that ess $(f(B)) = f(\text{ess}\,(B))$. If T is a closed, densely defined linear operator with nonempty resolvent and p is a polynomial, we shall show that ess $(p(T)) = p(\text{ess}\,(T))$.

In the second section we consider perturbation theorems for the essential spectrum. Let T be a closed, densely defined linear operator and B a bounded linear operator such that $B\colon D(T) \to D(T)$ and $(BT)(x) = (TB)x$ for $x \in D(T)$. Then we shall prove that $\text{ess}\,(T+B) \subset \text{ess}\,(T) + \text{ess}\,(B) = \{\nu + \mu \mid \nu \in \text{ess}\,(T),\ \mu \in \text{ess}\,(B)\}$. If $0 \notin \sigma(B)$ or if $0 \notin \text{ess}\,(B)$ and $\sigma(T) \neq \boldsymbol{C}$ (either hypothesis implies that BT is closed), then we shall show that $\text{ess}\,(BT) \subset \text{ess}\,(B)\ \text{ess}\,(T) = \{\mu\nu \mid \mu \in \text{ess}\,(B),\ \nu \in \text{ess}\,(T)\}$. Finally, we shall establish a perturbation theorem which makes no boundedness assumption on B. Let T and B be closed, densely defined linear operators with $D(T) \subset D(B)$. Let $D = \{x \in D(T) \mid Tx \in D(B)\}$ and assume that $B\colon D \to D(T)$ and $(TB)x = (BT)x$ for $x \in D$. Assume that $T+B$ is closed, $\rho(T+B)$ is nonempty, and ess (T) is compact. Then there is a compact subset S of ess (B) such that $\text{ess}\,(T+B) \subset \text{ess}\,(T) + S = \{\nu + s \mid \nu \in \text{ess}\,(T),\ s \in S\}$.

I would like to thank Richard Beals for a number of helpful discussions about the last mentioned theorem. In particular he suggested a simplified proof of an early version of the theorem, and it is that proof which will be given here.

1. **Spectral mapping theorems for the essential spectrum.** We begin with some trivialities. It is clear from the definition of ess (B) that if X is a finite dimensional vector space and $B\colon X \to X$ is a linear operator, then ess (B) is empty. Thus we shall always assume that X is an infinite dimensional Banach space, since our theorems are trivial otherwise. In this case, if $B\colon X \to X$ is a bounded linear operator, ess (B) is nonempty, since otherwise $\lambda - B$ would be Fredholm of index zero for all $\lambda \in \boldsymbol{C}$, whence by Theorem 3.2 of [2], X would be finite dimensional.

THEOREM 1. *Let B be a bounded linear operator of a Banach space X into itself. Let f be an analytic function defined on an open neighborhood of $\sigma(B)$. Then* ess $(f(B)) = f(\text{ess}\,(B))$.

Proof. Let G be the open neighborhood of $\sigma(B)$ on which f is defined and let Γ be a Jordan curve (not necessarily connected) contained in G and enclosing $\sigma(B)$. We can select $\varepsilon_0 > 0$ such that Γ encloses $N_{\varepsilon_0}(\sigma(B)) = \{\lambda \mid d(\lambda, \sigma(B)) < \varepsilon_0\}$.

First, let us show that $\text{ess}\,(f(B)) \subset f(\text{ess}\,(B))$. To prove this it suffices to show that $\text{ess}\,(f(B)) \subset f(N_\varepsilon(\text{ess}\,B))$ for all ε, $\varepsilon_0 \geqq \varepsilon > 0$. For suppose $\mu_0 \in f(N_\varepsilon(\text{ess}\,B))$ for all ε, $\varepsilon_0 \geqq \varepsilon > 0$. Then we can find a sequence of points ν_n such that $d(\nu_n, \text{ess}\,B) \to 0$ and $f(\nu_n) = \mu_0$. Since ess B is compact, we can find a convergent subsequence $\nu_{n_i} \to \nu_0 \in \text{ess}\,(B)$, and $f(\nu_0) = \mu_0$.

Thus, select ε, $\varepsilon_0 \geqq \varepsilon > 0$. Since $\sigma(B) - N_\varepsilon(\text{ess}\,B)$ is a compact set consisting of isolated points, let $\sigma(B) - N_\varepsilon(\text{ess}\,B) = \{\nu_1, \ldots, \nu_m\}$. Let $C_1, \ldots, C_m$ be disjoint circles

about $\nu_1, \ldots, \nu_m$ and such that the only point of $\sigma(B)$ enclosed in C_i is ν_i. Write $P=(1/2\pi i)\sum_{j=1}^{m}\int_{C_j}(\nu-B)^{-1}\,d\nu$ and recall that P is a finite dimensional projection commuting with B (see, for instance, [2] or [4]). Since we can assume ess (B) nonempty, let us select $\nu_0 \in \operatorname{ess}(B)$ and set $B_1=B-BP+\nu_0 P$. We claim that $\sigma(B_1)\subset N_\varepsilon(\operatorname{ess} B)$. To see this, set $M=(I-P)(X)$ and note that $B_1|M=B|M$ (B clearly maps M into itself). By the functional calculus for bounded linear operators, it is known that $\mu-B|M$ is one-to-one and onto M for μ inside one of the C_i, and since the only other points in $\sigma(B)$ lie in $N_\varepsilon(\operatorname{ess} B)$, $\mu-B_1|M$ is one-to-one and onto M for $\mu \notin N_\varepsilon(\operatorname{ess} B)$. On the other hand, if we write $V=PX$,

$$\mu-B_1|V = (\mu-\nu_0)|V,$$

so $\mu-B_1$ is one-to-one and onto V for $\mu\neq\nu_0$. It follows that $\mu-B_1$ is one-to-one and onto X for $\mu \notin N_\varepsilon(\operatorname{ess} B)$.

Our next claim is that $\operatorname{ess}(f(B))=\operatorname{ess}(f(B_1))$. (Notice that $f(B_1)$ is defined since $\sigma(B_1)\subset N_\varepsilon(\operatorname{ess} B)\subset G$, and in fact Γ encloses $\sigma(B_1)$.) Our claim will follow from Lay's theorem if we can show that $K=f(B_1)-f(B)$ is compact and commutes with $f(B)$. Commutativity is clear. To see compactness notice that for $\mu\in\Gamma$ (so $\mu\in\rho(B)\cap\rho(B_1)$), $\mu-B$ and $\mu-B_1$ take M into M and V into V. Since $X=M\oplus V$, it follows that $(\mu-B)^{-1}$ and $(\mu-B_1)^{-1}$ take M into M and $(\mu-B)^{-1}x=(\mu-B_1)^{-1}x$ for $x\in M$. Thus, for $x\in X$, $(\mu-B_1)^{-1}x-(\mu-B)^{-1}x=(\mu-B_1)^{-1}(Px)-(\mu-B_1)^{-1}(Px)$, and we see finally that

$$f(B_1)x-f(B)x = \frac{1}{2\pi i}\int_\Gamma f(\mu)(\mu-B)^{-1}x\,d\mu = (f(B_1)-f(B))(Px),$$

a finite dimensional operator.

With the above observation we can complete the first half of the proof. We have $\operatorname{ess}(f(B))=\operatorname{ess}(f(B_1))\subset\sigma(f(B_1))$. But by the ordinary spectral mapping theorem, $\sigma(f(B_1))=f(\sigma(B_1))\subset f(N_\varepsilon(\operatorname{ess} B))$.

Conversely, we want to show that $\operatorname{ess}(f(B))\supset f(\operatorname{ess} B)$. Assume not, so that there exists $\mu_0\in f(\operatorname{ess} B)-\operatorname{ess} f(B)$. Since $\mu_0\notin\operatorname{ess} f(B)$, take a circle C about μ_0 which contains no other points of $\sigma(f(B))$. Let $P_1=(1/2\pi i)\int_C(\mu-f(B))^{-1}\,d\mu$, a finite dimensional projection, and let $M_1=(I-P_1)X$. It follows as above that for $\nu\in\Gamma$, $(\nu-B)^{-1}\colon M_1\to M_1$ and thus $f(B)|M_1=f(B|M_1)$. Also as above, we know that $\mu_0-f(B)|M_1$ is a one-to-one map of M_1 onto M_1. Thus if $f(\nu_0)=\mu_0$ for $\nu_0\in\operatorname{ess}(B)$, $\nu_0\notin\sigma(B|M_1)$, since if it were, we would have

$$\mu_0\in\sigma(f(B|M_1)) = \sigma(f(B)|M_1).$$

Now we can show that $\nu_0\notin\operatorname{ess}(B)$, which gives a contradiction. We use Browder's original definition. If we set $V_1=P_1X$, the range of ν_0-B is $M_1\oplus(\nu_0-B)V_1$, the span of a closed subspace and a finite dimensional subspace, hence closed. Since V_1 is finite dimensional $(\nu-B)|V_1$ must be one-to-one and onto V_1 for

$0<|\nu-\nu_0|<\delta$, and since $\nu_0-B|M_1$ is one-to-one and onto M_1, $\nu-B|M_1$ is one-to-one and onto M_1 for $|\nu-\nu_0|<r$, $r>0$. Thus $\nu-B$ is one-to-one and onto for $0<|\nu-\nu_0|<\min(r,\delta)$ and ν_0 is an isolated point of $\sigma(B)$. Finally, we clearly have $\dim(\bigcup_{j>0} N(\nu_0-B)^j)\leqq \dim V_1$. Q.E.D.

REMARK 1. Theorem 1 was proved by T. T. West [8] for the case that ess $(B)=\{0\}$ (the so-called "Riesz operators") and $f(0)=0$.

If we try to generalize Theorem 1 to unbounded linear operators T, we have the problem of defining $f(T)$. If f is a polynomial, however, there is basically no problem and we can obtain an analogue of Theorem 1. First we need a lemma. This result is not difficult; a proof is given by Goldberg [3, pp. 107–108].

LEMMA 1 (GOLDBERG). *Let T be a closed, densely defined linear operator such that $\sigma(T)\neq \mathbf{C}$. Let p be a polynomial of degree n; then $p(T)$, with domain $D(T^n)$, is a closed, densely defined linear operator and $\sigma(p(T))=p(\sigma(T))$.*

LEMMA 2. *Let T be a closed, densely defined linear operator on X such that $\sigma(T)\neq \mathbf{C}$. Let F be a bounded, finite dimensional linear operator such that $(\nu-T)^{-1}F=F(\nu-T)^{-1}$ for $\nu\in\rho(T)$. Then* (a) *$F(X)\subset\bigcap_{j>0} D(T^j)$;* (b) *$(FT)x=(TF)x$ for $x\in D(T)$;* (c) *for all $j\geqq 1$, $(F+T)^jx-T^jx=E_jx$ for $x\in D(T^j)$, E_j a bounded, finite dimensional linear operator such that E_j: $D(T^j)\to D(T^j)$ and $T^jE_jx=E_jT^jx$ for $x\in D(T^j)$;* (d) *for any polynomial p,* ess $(p(T+F))=$ ess $(p(T))$.

Proof. Select $\nu\in\rho(T)$. Since $D(T)=(\nu-T)^{-1}X$ and since $(\nu-T)^{-1}F=F(\nu-T)^{-1}$, F: $D(T)\to D(T)$. Showing $(FT)x=(TF)x$ for $x\in D(T)$ is equivalent to showing that $F(\nu-T)x=(\nu-T)Fx$ for $x\in D(T)$, and to prove this we simply apply $(\nu-T)^{-1}$ to both sides.

If we set $V=F(X)$, by assumption, V is finite dimensional. Since $D(T)$ is dense in X and F is bounded, $F(D(T))$ is a dense subspace of V and hence $F(D(T))=V$, so that $V\subset D(T)$. Because $(FT)x=(TF)x$ for $x\in D(T)$, T: $V\to V$ and thus $\nu-T$: $V\to V$. However, V is finite dimensional, so we have $(\nu-T)^{-1}$: $V\to V$ and in fact $V=(\nu-T)^{-1}V$. It follows that $V=(\nu-T)^{-j}V$ for all $j\geqq 1$, so $V\subset D(T^j)$ for all $j\geqq 1$.

To prove (c), note that because $(TF)x=(FT)x$ for $x\in D(T)$, we can write $(T+F)^jx-T^jx=A_jFx$ for $x\in D(T^j)$, A_j a linear operator defined on $D(T^j)$. However, by our above work, $V\subset D(T^j)$ and since V is finite dimensional, $\|A_jv\|\leqq M\|v\|$ for $v\in V$. Thus we see that for $x\in X$, $\|A_jFx\|\leqq M\|Fx\|\leqq M\|F\|\,\|x\|$, so that $E_j=A_jF$ is a bounded linear operator. By using the commutativity of F and T, we see that for $v\in D(T^j)$,

$$A_jv=\sum_{i=0}^{j-1}\binom{j}{i}T^iF^{j-i-1}v$$

and since $Fx\in D(T^{2j})$ for any x, it follows immediately that $A_jFx\in D(T^j)$ for any x. Finally, since $T^kFx=FT^kx$ for $x\in D(T^k)$ (by repeatedly using $(FT)x=(TF)x$ for $x\in D(T)$), it is easy to see that $T^jE_jx=E_jT^jx$ for $x\in D(T^j)$.

By using (c), we see that $p(T+F)x-p(T)x=Ex$ for $x \in D(T^n)$ ($n=$degree of p), E a finite dimensional, bounded linear operator such that E: $D(T^n) \to D(T^n)$ and $p(T)Ex=Ep(T)x$ for $x \in D(T^n)$. It follows by Lay's theorem that

$$\text{ess}\,(p(T+F)) = \text{ess}\,(p(T)).$$

THEOREM 2. *Let T be a closed, densely defined linear operator on a Banach space X and assume that $\sigma(T) \neq C$. Then if p is a polynomial, $p(T)$ is closed and* ess $(p(T))$ $=p(\text{ess}\,(T))$.

Proof. The proof follows the basic outlines of Theorem 1, so we shall be sketchy.

Let $V_R(0)$ denote the closed ball of radius R about 0 in C and let $V_R'(0)$ denote the complement of $V_R(0)$. We want to show that ess $(p(T)) \subset p(\text{ess}\,T)$ and in order to prove this it suffices to prove that

$$\text{ess}\,(p(T)) \subset p(N_\varepsilon(\text{ess}\,T) \cup V_R'(0)) \quad \text{for all } \varepsilon > 0,\ R > 0.$$

For suppose we have established the above inclusion and $\lambda_0 \in \text{ess}\,(p(T))$. Unless p is constant, in which case the theorem is trivial, there exists an R so large that $p(V_R'(0)) \subset V_{|\lambda_0|}'(0)$, so that $\lambda_0 \in p(N_\varepsilon(\text{ess}\,T) \cap V_R(0))$. If we select $\varepsilon_n \to 0$ and $\mu_n \in N_{\varepsilon_n}(\text{ess}\,T) \cap V_R(0)$ such that $p(\mu_n)=\lambda_0$, then since $\mu_n \in V_R(0)$, a compact set, we can assume $\mu_n \to \mu \in \text{ess}\,(T)$ and $\lambda_0=p(\mu)$.

Consider

$$\begin{aligned}&(\sigma(T)-N_\varepsilon(\text{ess}\,T)) \cap V_R(0)\\ &\qquad = \{\mu_1, \ldots, \mu_m\}\ (\sigma(T)-N_\varepsilon(\text{ess}\,T) = \{\lambda \mid \lambda \in \sigma(T),\ \lambda \notin N_\varepsilon(\text{ess}\,T)\}).\end{aligned}$$

If we let $C_1, \ldots, C_m$ be disjoint circles such that μ_j is the only point of $\sigma(T)$ contained in C_j, and if we set $P=(1/2\pi i) \sum_{j=1}^m \int_{C_j} (\lambda-T)^{-1}\, d\lambda$, then just as before P is a finite dimensional projection and P: $X \to D(T)$. If we select μ_0 with $|\mu_0| > R$ and if we set $F=-TP+\mu_0 P$, the same proof as before shows that $\sigma(T+F) \subset N_\varepsilon(\text{ess}\,T) \cup V_R'(0)$. Since P is finite dimensional, F is finite dimensional; and since P is continuous as a map from X to $D(T)$ with the graph topology and T is continuous from $D(T)$ with the graph topology to X, F is bounded. It is also clear that for $\nu \in \rho(T)$, $(\nu-T)^{-1}F=F(\nu-T)^{-1}$. It follows by Lemma 2 that

$$\text{ess}\,(p(T)) = \text{ess}\,(p(T+F)) \subset \sigma(p(T+F)) = p(\sigma(T+F)) \subset p(N_\varepsilon(\text{ess}\,T) \cup V_R'(0)).$$

The proof that $p(\text{ess}\,T) \subset \text{ess}\,(p(T))$ is essentially the same as in Theorem 1, and we omit it. We just note that a little more care must be exercised since T is not everywhere defined. Q.E.D.

2. **Perturbation theorems for the essential spectrum.** To begin this section let us establish some notation. We shall say that linear operators T and B satisfy hypothesis one (H_1) if T is a closed densely defined linear operator with domain $D(T)$, B is a bounded linear operator such that B: $D(T) \to D(T)$, and $(TB)(x) =(BT)(x)$ for $x \in D(T^2)$. Condition H_1 is Lay's definition of "T commutes with

B" [5]. We shall say that linear operators T and B satisfy H_2 if T and B are the same as above, but we strengthen the commutativity condition to $(TB)(x)=(BT)(x)$ for $x \in D(T)$.

Our next lemma is concerned with the behavior of $\sigma(T+B)$ when T and B satisfy H_1. In the case that T is bounded, this lemma is well known and follows by Banach algebra techniques. In the general case the lemma has been proved by J. T. Marti [6].

LEMMA 3 (MARTI). *Let T and B be linear operators satisfying H_1. Then $T+B$ is closed and $\sigma(T+B)\subset\sigma(T)+\sigma(B)=\{\mu+\nu \mid \mu\in\sigma(B),\ \nu\in\sigma(T)\}$. If $\sigma(T)$ is empty, we interpret $\sigma(T)+\sigma(B)$ as the empty set.*

The following lemma is also proved by Marti [6]. If T is bounded, it follows by Banach algebra techniques without the assumption that $0\notin\sigma(B)$.

LEMMA 4 (MARTI). *Let T and B be linear operators satisfying H_2. Assume that $0\notin\sigma(B)$. Then BT is a closed linear operator, and*

$$\sigma(BT) \subset \sigma(B)\sigma(T) = \{\mu\nu \mid \mu\in\sigma(B),\ \nu\in\sigma(T)\}.$$

If $\sigma(T)$ is empty, $\sigma(B)\sigma(T)$ is interpreted as empty.

REMARK 2. Marti actually proves Lemma 3 under the assumption that B: $D(T)\to D(T)$ and $(TB)x=(BT)x$ for $x\in D(T)$. However, if $\sigma(T)=\boldsymbol{C}$, Lemma 3 is immediate. If $\sigma(T)\neq\boldsymbol{C}$ and $\nu\in\rho(T)$, H_1 implies that $B(\nu-T)^{-1}x=(\nu-T)^{-1}Bx$ for $x\in D(T)$ (apply $(\nu-T)$ to both sides), whence by continuity $B(\nu-T)^{-1}x=(\nu-T)^{-1}Bx$ for all x. If $u\in D(T)$, $u=(\nu-T)^{-1}x$, so that

$$(\nu-T)Bu = (\nu-T)B(\nu-T)^{-1}x = Bx = B(\nu-T)u,$$

and Marti's commutativity condition holds.

REMARK 3. Marti considers closed, densely defined linear operators T and S such that $D(T)\subset D(S)$, S: $D(T)\to D(T)$, and $(TS)x=(ST)x$ for all $x\in D(T)$ such that $Tx\in D(S)$. This generality is illusory, however. For suppose $\rho(T)\neq\varnothing$ and select $\nu\in\rho(T)$. For $x\in D(S)$, $(\nu-T)S(\nu-T)^{-1}x=Sx$, by the commutativity condition. If we let Y denote the Banach space $D(T)$ with the graph topology ($\|x\|_Y=\|x\|+\|Tx\|$), it is easy to see that S is a closed linear operator as a map from Y to Y. Thus S is bounded as a map from Y to Y. Since $(\nu-T)^{-1}$: $X\to Y$ is bounded and $(\nu-T)$: $Y\to X$ is bounded, $\|(\nu-T)S(\nu-T)^{-1}x\|\leqq M\|x\|$. It follows that $\|Sx\|\leqq M\|x\|$ for $x\in D(S)$, and since S was assumed closed, $D(S)=X$ and S is bounded.

The same remark applies to Lay's article [5].

THEOREM 3. *Let T and B be linear operators satisfying H_1. Then $T+B$ is closed and* $\operatorname{ess}(T+B)\subset\operatorname{ess}(T)+\operatorname{ess}(B)=\{\nu+\mu \mid \nu\in\operatorname{ess}(T),\ \mu\in\operatorname{ess}(B)\}$. *If* $\operatorname{ess}(T)$ *is empty, this means that* $\operatorname{ess}(T+B)$ *is empty.*

Proof. If $\sigma(T)=C$, ess $(T)=C$ and we are done. Thus we assume $\sigma(T)\neq C$. Now we establish some simple commutativity relationships. By Remark 2 we know that $(TB)x=(BT)x$ for $x\in D(T)$. By the proof of Remark 2 we have $(\nu-T)^{-1}B=B(\nu-T)^{-1}$ for $\nu\in\rho(T)$, and it follows (by applying $\mu-B$ to both sides of the equation) that $(\mu-B)^{-1}(\nu-T)^{-1}=(\nu-T)^{-1}(\mu-B)^{-1}$ for $\mu\in\rho(B)$, $\nu\in\rho(T)$. The latter equation implies that $(\mu-B)^{-1}$ takes $D(T)$ into $D(T)$, so that we see (by applying $\mu-B$ to both sides of the equation) that $(\mu-B)^{-1}Tx=T(\mu-B)^{-1}x$ for $x\in D(T)$.

Select $\lambda_0\in$ ess $(T+B)$. We want to show that $\lambda_0\in$ ess $(T)+$ess (B). Suppose we can show that for every $\varepsilon>0$, $\lambda_0\in N_\varepsilon(\text{ess } T)+N_\varepsilon(\text{ess } B)$. This will imply that for every $\varepsilon>0$, $\lambda_0\in N_{2\varepsilon}(\text{ess } T+\text{ess } B)$. Since ess T is closed and ess B is compact, ess $T+$ess B is closed, and it will follow that $\lambda_0\in$ ess $T+$ess B.

Thus take $\lambda_0\in$ ess $(B+T)$ and select $\varepsilon>0$. Let R be a constant such that $|\lambda_0|<R$ and $|\mu|<R$ for $\mu\in N_\varepsilon(\sigma(B))$. As in §1, let $V_{2R}(0)=\{x \mid \|x\|\leqq 2R\}$. Since spectral points which are not in the essential spectrum are isolated, we know that $\sigma(T)\cap(V_{2R}(0)-N_\varepsilon(\text{ess } T))=\{\nu_1,\ldots,\nu_n\}$ and $\sigma(B)-N_\varepsilon(\text{ess } B)=\{\mu_1,\ldots,\mu_m\}$. Let $D_1,\ldots,D_n$ be disjoint circles about $\nu_1,\ldots,\nu_n$, each D_j containing no points of $\sigma(T)$ except ν_j. Similarly, let $C_1,\ldots,C_m$ be disjoint circles about $\mu_1,\ldots,\mu_m$ such that each C_j contains no point of $\sigma(B)$ except μ_j. As in §1 we know that

$$Q=\sum_{j=1}^{m}\frac{1}{2\pi i}\int_{C_j}(\mu-B)^{-1}\,d\mu$$

is a bounded, finite dimensional projection; similarly

$$P=\sum_{j=1}^{n}\frac{1}{2\pi i}\int_{D_j}(\nu-T)^{-1}\,d\nu$$

is a bounded, finite dimensional projection, and in fact $P\colon X\to D(T)$ is continuous from X to $D(T)$ with the graph topology.

We are now in a position to complete the proof. Take ν_0, $|\nu_0|>2R$, and select $\mu_0\in$ ess (B) (since we always assume X infinite dimensional, ess $(B)\neq\varnothing$). Consider the bounded, finite dimensional linear operator $F=\nu_0P-TP+\mu_0Q-BQ$. By using the commutativity relations of the first paragraph of the proof, it is easy to see that for $\nu\in\rho(T)$, $(\nu-T)^{-1}F=F(\nu-T)^{-1}$. It follows by Lemma 2 that $F(X)\subset\bigcap_{j\geqq 1}D(T^j)$ and $TFx=FTx$ for $x\in D(T)$. It is also clear that $FB=BF$, so that $(T+B)Fx=F(T+B)x$ for $x\in D(T+B)=D(T)$. It follows by Lay's theorem that ess $(T+B+F)=$ess $(T+B)$.

The next step is to make use of the form of F. We have

$$\text{ess }(B+T+F)\subset\sigma((B+\mu_0Q-BQ)+(T+\nu_0P-TP))=\sigma(B_1+T_1).$$

It is not hard to see that B_1 and T_1 satisfy H_1. For instance, since

$$(\mu_0Q-BQ)(\nu-T)^{-1}=(\nu-T)^{-1}(\mu_0Q-BQ),$$

it follows by Lemma 2 that $\mu_0 Q - BQ$: $D(T) \to D(T)$ and $(\mu_0 Q - BQ)Tx = T(\mu_0 Q - BQ)x$ for $x \in D(T)$. Thus by Marti's theorem $\sigma(B_1 + T_1) \subset \sigma(B_1) + \sigma(T_1)$. However, by the same arguments used in the first section, $\sigma(T_1) \subset N_\varepsilon(\text{ess } T) \cup V'_{2R}(0)$ and $\sigma(B_1) \subset N_\varepsilon(\text{ess } B)$. Putting all this together, $\lambda_0 \in (N_\varepsilon(\text{ess } T) \cup V'_{2R}(0)) + N_\varepsilon(\text{ess } B)$. However, it is impossible that $\lambda_0 = \nu + \mu$ for $\nu \in V'_{2R}(0)$ and $\mu \in N_\varepsilon(\text{ess } B)$, since $\nu + \mu > 2R - R = R$ and $|\lambda_0| \leqq R$. Thus $\lambda_0 \in N_\varepsilon(\text{ess } T) + N_\varepsilon(\text{ess } B)$. Since λ_0 was an arbitrary element of ess $(T+B)$, this shows that ess $(T+B) \subset N_\varepsilon(\text{ess } T) + N_\varepsilon(\text{ess } B)$. Q.E.D.

Notice that the above proof shows ess $(T+B)$ is empty if ess (T) is empty.

Before we prove the corresponding theorem for ess (BT), we need another lemma.

LEMMA 5. *Let B and T be linear operators satisfying H_2. Assume that $\sigma(T) \neq \boldsymbol{C}$ and $0 \notin$ ess (B). Then BT is a closed linear operator. Furthermore, there exists a bounded, finite dimensional linear operator F such that $B_1 = B - F$ is invertible, B_1 and T satisfy H_2,* ess $(B_1) =$ ess (B), *and* ess $(B_1 T) =$ ess (BT).

Proof. If B is invertible, the lemma is immediate (even without the assumption that $\sigma(T) \neq \boldsymbol{C}$). Thus assume $0 \in \sigma(B) -$ ess (B). Let C be a circle about 0 which contains no other points of $\sigma(B)$, and let $Q = (1/2\pi i) \int_C (\mu - B)^{-1}\, d\mu$, a bounded, finite dimensional projection. We define $F = BQ - Q$, and we have to show F satisfies the claims of the lemma.

The usual argument shows that $0 \notin \sigma(B - F)$, so that B_1 is invertible. It is easy to show that $(\nu - T)^{-1} F = F(\nu - T)^{-1}$ for $\nu \in \rho(T)$, so that by Lemma 2 we know that $F(X) \subset \bigcap_{j \geqq 1} D(T^j)$ and $(TF)x = (FT)x$ for $x \in D(T)$. It follows in particular that since $F(X)$ is a finite dimensional subspace of $D(T)$, TF is actually a bounded, finite dimensional linear operator. Since B_1 is invertible, $B_1 T$ is closed and thus $BT = B_1 T + TF$, a sum of a closed and a bounded linear operator, is closed.

To see that B_1 and T satisfy H_2, it suffices to show F and T satisfy H_2, and we have already seen that $(FT)x = (TF)x$ for $x \in D(T)$. Trivially, $BF = FB$, so that by Lay's theorem, ess $(B_1) =$ ess (B). Finally, to show that ess $(B_1 T) =$ ess (BT), it suffices, by Lay's theorem, to show that $(BT)(TF)x = (TF)x(BT)x$ for $x \in D(T)$. However, $Fx \in \bigcap_{j \geqq 1} D(T^j)$ for any x, so that $(TF)x \in D(T)$, and we can write $(BT)(TF)x = (TB)(TF)x$. If $x \in D(T)$, we also have $(TB)(TF)x = (TB)(FT)x = (TF)(BT)x$. Q.E.D.

THEOREM 4. *Let B and T satisfy H_2. Assume that $0 \notin$ ess (B) and that BT is closed (under the hypotheses, Lemma 5 implies that BT is closed if $\sigma(T) \neq \boldsymbol{C}$). Then* ess $(BT) \subset$ ess (B) ess $(T) = \{\nu\mu \mid \mu \in \text{ess } (B),\ \nu \in \text{ess } (T)\}$. *If* ess (T) *is empty,* ess (B) ess (T) *is interpreted as being empty.*

Proof. Since ess $(B) \neq \varnothing$ and $0 \notin$ ess (B) (recall that we assume X infinite dimensional), we can select $\mu_0 \in$ ess (B), $\mu_0 \neq 0$. If $\sigma(T) = \boldsymbol{C}$, then we have ess (T) ess $(B) \supset \{\lambda\mu_0 \mid \lambda \in \boldsymbol{C}\} \supset \boldsymbol{C}$, so that ess $(BT) \subset$ ess (B) ess (T). Thus we can

assume that $\sigma(T)\neq C$. By Lemma 5 we can find a bounded, invertible operator B_1 such that B_1 and T satisfy H_2, ess $(B_1)=$ess (B) and ess $(B_1T)=$ess (BT). Thus it suffices to show that ess $(B_1T)\subset$ess (B_1) ess (T). For notational convenience, we shall write B instead of B_1, but now we can also assume B invertible.

Select $\lambda_0\in$ ess (BT). Notice that to show $\lambda_0\in$ ess (B) ess (T), it suffices to show that $\lambda_0\in N_\varepsilon(\text{ess } B)N_\varepsilon(\text{ess } T)$ for all $\varepsilon>0$. For suppose $\lambda_0\in N_\varepsilon(\text{ess } B)N_\varepsilon(\text{ess } T)$ for all $\varepsilon>0$. Since ess (B) is closed and $0\notin$ ess (B), we can assume that $|\mu|\geqq\delta>0$ for $\mu\in N_\varepsilon(\text{ess } B)$ and $0<\varepsilon<\varepsilon_0$. For $0<\varepsilon<\varepsilon_0$, it follows that if $\lambda_0=\mu\nu$, $\mu\in N_\varepsilon(\text{ess } B)$ and $\nu\in N_\varepsilon(\text{ess } T)$, then $|\nu|\leqq M=|\lambda_0|/\delta$. Thus (for n large enough), if we select $\mu_n\in N_{1/n}(\text{ess } B)$ and $\nu_n\in N_{1/n}(\text{ess } T)$ such that $\mu_n\nu_n=\lambda_0$, μ_n and ν_n lie in compact sets, and by taking subsequences we can assume $\mu_n\to\mu\in$ ess (B) and $\nu_n\to\nu\in$ ess (T). Thus we see that $\lambda=\mu\nu\in$ ess (B) ess (T).

We now proceed as in Theorem 3, though we shall omit many details. Take $\lambda_0\in$ ess (BT). Select $\varepsilon>0$ small enough so that $|\mu|\geqq\delta>0$ for $\mu\in N_\varepsilon(\text{ess } B)$. Select a constant $R>\lambda_0/\delta$ and a number ν_0 with $|\nu_0|\geqq R$. Select $\mu_0\in$ ess (B) as before. Just as in Theorem 3, we can find finite dimensional projections P and Q which correspond respectively to the eigenvalues $\{\nu_1,\ldots,\nu_n\}=V_R(0)\cap(\sigma(T)-N_\varepsilon \text{ ess } (T))$ and $\{\mu_1,\ldots,\mu_m\}=\sigma(B)-N_\varepsilon(\text{ess } B)$. We want to use Lay's theorem to show that

$$\text{ess } (BT) = \text{ess } ((B+\mu_0Q-BQ)(T+\nu_0P-TP)).$$

We write $F_1=\mu_0Q-BQ$ and $F_2=\nu_0P-TP$. It is easy to show that $(\nu-T)^{-1}F_1=F_1(\nu-T)^{-1}$ and $(\nu-T)^{-1}F_2=F_2(\nu-T)^{-1}$ for $\nu\in\rho(T)$. It follows by Lemma 2 that $F_i(X)\subset\bigcap_{j\geqq1}D(T^j)$, $(TF_i)x=(F_iT)x$ for $x\in D(T)$, and TF_i is a bounded, finite dimensional linear operator, $i=1,2$. Using these results, we see that if we write $(B+F_1)(T+F_2)=BT+F$, F is a bounded, finite dimensional linear operator and $(\nu-T)^{-1}F=F(\nu-T)^{-1}$ for $\nu\in\rho(T)$. Applying Lemma 2 again shows $F(X)\subset D(T)$ and $(TF)x=(FT)x$ for $x\in D(T)$. Since it is obvious that $FB=BF$, Lay's theorem now implies ess $(BT+F)=$ess (BT).

Using the above results, we see that $B_2=B+F_1$ and $T_2=T+F_2$ satisfy the hypotheses of Lemma 4, so ess $(B_2T_2)\subset\sigma(B_2T_2)\subset\sigma(B_2)\sigma(T_2)$. Just as before, $\sigma(B_2)\subset N_\varepsilon(\text{ess } B)$ (so, in particular, B_2 is invertible), and $\sigma(T_2)\subset N_\varepsilon(\text{ess } T)\cup V'_R(0)$. By the selection of R, $\lambda_0\neq\mu\nu$ for $\mu\in N_\varepsilon(\text{ess } B)$ and $\nu\in V'_R(0)$, so that $\lambda_0\in N_\varepsilon(\text{ess } T)N_\varepsilon(\text{ess } T)$. As we have remarked, this establishes our theorem. Q.E.D.

Our next goal is to establish a perturbation theorem when B may not be bounded. To compensate for the unboundedness of B, we have to strengthen our assumptions on T.

THEOREM 5. *Let T and B be closed, densely defined linear operators with $D(T)\subset D(B)$. Let $D=\{x\in D(T)\mid Tx\in D(B)\}$ and assume $B\colon D\to D(T)$ and $(TB)x=(BT)x$ for $x\in D$. Assume that $T+B$ is closed and $\rho(T+B)$ is nonempty. Finally, suppose* ess (T) *is compact. Then there exists a closed subspace X_1 of X and a bounded linear operator $B_1\colon X_1\to X_1$ such that* ess $(B_1)\subset$ess (B) *and* ess $(T+B)\subset$ess $(T)+$ess (B_1).

Proof. For $\nu \in \rho(T)$ and $x \in D(B)$, we have $B(\nu-T)^{-1}x=(\nu-T)^{-1}Bx$, because $(\nu-T)^{-1}x \in D$ for $x \in D(B)$ and then $(\nu-T)B(\nu-T)^{-1}x=B(\nu-T)(\nu-T)^{-1}x$ $=(\nu-T)(\nu-T)^{-1}Bx$. Since ess (T) is compact, let C be a simple Jordan curve enclosing ess (T) and lying in $\rho(T)$. Define $P=(1/2\pi i)\int_C (\nu-T)^{-1}\,d\nu$ and set $X_1=PX$, $X_2=(I-P)X$. As usual, $P\colon X \to D(T)$, and in fact since P is a projection, $P\colon X \to \bigcap_{j\geq 1} D(T^j)$. Since $(\nu-T)^{-1}Bx=B(\nu-T)^{-1}x$ for $x \in D(B)$, we find easily that $(BP)x=(PB)x$ for $x \in D(B)$. It follows that for $u \in X_1$, $Bu=BP^2u=PBPu \in X_1$. Thus we can define $B_1\colon X_1 \to X_1$ by $B_1u=Bu$, $u \in X_1$. Similarly, we define $T_1\colon X_1 \to X_1$. Since both B_1 and T_1 are closed and defined on all of X_1, they are bounded. Similarly, we note that $B\colon X_2 \cap D(B) \to X_2$ and $T\colon X_2 \cap D(T) \to X_2$, so we define $B_2=B|X_2 \cap D(B)$ (viewed as a map into X_2) and $T_2=T|X_2 \cap D(T)$ (viewed as a map into X_2).

Standard results now imply that $\sigma(B)=\sigma(B_1) \cup \sigma(B_2)$, $\sigma(T)=\sigma(T_1) \cup \sigma(T_2)$ and $\sigma(T+B)=\sigma(T_1+B_1) \cup \sigma(T_2+B_2)$. Using these results and Browder's original definition of the essential spectrum, it is not hard to see that ess $(B)=$ess (B_1) $\cup$ ess (B_2), ess $(T)=$ess $(T_1) \cup$ ess (T_2) and ess $(T+B)=$ess $(T_1+B_1) \cup$ ess (T_2+B_2). We omit the proof. However, we know as usual that $\sigma(T_2)$ lies outside C, so that ess (T_2) lies outside C. Since C encloses ess (T), ess $(T_1)=$ess (T) and ess (T_2) is empty. Note also that $B_1T_1=T_1B_1$. To show this, it suffices to show that $(BT)(Px)$ $=(TB)(Px)$ for $x \in X$, and this is true since $Px \in D(T^2) \subset D$. It follows by Theorem 3 that ess $(T_1+B_1) \subset$ess $(T_1)+$ess $(B_1)=$ess $(T)+$ess (B_1). Thus to complete the proof it suffices to show that ess (T_2+B_2) is empty.

Since we assume $\rho(T+B)$ is nonempty, take $\mu_0 \in \rho(T+B)$, so that certainly $\mu_0 \in \rho(T_2+B_2)$. Similarly, take $\nu_0 \in \rho(T)$, so that $\nu_0 \in \rho(T_2)$. Then we can write

$$\begin{aligned}\text{ess}\,(T_2+B_2) &= \mu_0+\text{ess}\,(-\mu_0+T_2+B_2)\\ &= \text{ess}\,((-\mu_0+T_2+B_2)(\nu_0-T_2)^{-1}(\nu_0-T_2))+\mu_0.\end{aligned}$$

If we set $B_3=(-\mu_0+T_2+B_2)(\nu_0-T_2)^{-1}$, B_3 is a bounded, invertible operator. It is also easy to check that $B_3\colon D(T_2) \to D(T_2)$ and $B_3(\nu_0-T_2)x=(\nu_0-T_2)B_3x$ for $x \in D(T_2)$. It follows by Theorem 4 that ess $(B_3(\nu_0-T_2)) \subset$ess (B_3) ess (ν_0-T_2). However, ess $(\nu_0-T_2)=\nu_0-$ess (T_2) is empty, so that ess $(B_3(\nu_0-T_2))$ is empty and ess (T_2+B_2) is empty. Q.E.D.

COROLLARY 1. *Let T and B be closed, densely defined linear operators with $D(T) \subset D(B)$. If $D=\{x \in D(T) \mid Tx \in D(B)\}$, assume that $B\colon D \to D(T)$ and $(TB)x=(BT)x$ for $x \in D$. Assume that* ess (T) *is compact. Suppose that there exists a sequence $\lambda_n \in \rho(T)$, $|\lambda_n| \to \infty$, such that $\|(\lambda_n-T)^{-1}\| \leqq C/|\lambda_n|$, C a constant. Finally, assume that for every $\varepsilon>0$, there is a constant K_ε such that $\|Bx\| \leqq \varepsilon\|Tx\|$ $+K_\varepsilon\|x\|$ for $x \in D(T)$. Then the conclusion of Theorem 5 holds.*

Proof. It is known (see [4]) that if $\|Bx\| \leqq c\|Tx\|+K\|x\|$ for $x \in D(T)$ with $c<1$, then $T+B$ is closed. Thus $T+B$ is certainly closed. To apply Theorem 5, it only remains to show that $\rho(T+B)$ is nonempty. We claim that $\lambda_n \in \rho(T+B)$ for n

large enough, and since $(\lambda_n - T - B) = (I - B(\lambda_n - T)^{-1})(\lambda_n - T)$, it suffices to show that $\|B(\lambda_n - T)^{-1}\| < 1$ for n large enough. But by our hypothesis,

$$\begin{aligned}\|B(\lambda_n - T)^{-1}x\| &\leqq \varepsilon\|T(\lambda_n - T)^{-1}x\| + K_\varepsilon\|(\lambda_n - T)^{-1}x\| \\ &= \varepsilon\|\lambda_n(\lambda_n - T)^{-1}x - x\| + K_\varepsilon\|(\lambda_n - T)^{-1}x\| \\ &\leqq \varepsilon\|x\| + \varepsilon C\|x\| + K_\varepsilon/|\lambda_n|\ \|x\|.\end{aligned}$$

Thus we merely select ε such that $\varepsilon + \varepsilon C < 1$ and then select n so large that

$$\varepsilon + \varepsilon C + K_\varepsilon/|\lambda_n| < 1. \qquad \text{Q.E.D}$$

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