SPECTRAL MAPPING THEOREMS AND PERTURBATION THEOREMS FOR BROWDER'S ESSENTIAL SPECTRUM

BY

ROGER D. NUSSBAUM

Abstract. If $T$ is a closed, densely defined linear operator in a Banach space, F. E. Browder has defined the essential spectrum of $T$, $\text{ess } (T)$ [1]. We derive below spectral mapping theorems and perturbation theorems for Browder's essential spectrum. If $T$ is a bounded linear operator and $f$ is a function analytic on a neighborhood of the spectrum of $T$, we prove that $f(\text{ess } (T)) = \text{ess } (f(T))$. If $T$ is a closed, densely defined linear operator with nonempty resolvent set and $f$ is a polynomial, the same theorem holds. For a closed, densely defined linear operator $T$ and a bounded linear operator $B$ which commutes with $T$, we prove that $\text{ess } (T + B) \subseteq \text{ess } (T) + \text{ess } (B)$. By making additional assumptions, we obtain an analogous theorem for $B$ unbounded.

Introduction. Let $T$ be a closed, densely defined linear operator on a Banach space $X$. F. E. Browder [1] defined the essential spectrum of $T$, $\text{ess } (T)$, to be the set of $\lambda \in \sigma (T)$, the spectrum of $T$, such that at least one of the following conditions holds: (1) $R(\lambda - T)$, the range of $\lambda - T$, is not closed; (2) $\lambda$ is a limit point of $\sigma (T)$; (3) $\bigcup_{r > 0} N[(\lambda - T)^{-1}]$ is infinite dimensional, where $N(A)$ denotes the null space of a linear operator $A$. Browder proved that $\lambda_0 \notin \text{ess } (T)$ if and only if $(\lambda - T)^{-1}$ is defined for $0 < |\lambda - \lambda_0| < \delta$ and the Laurent expansion of $(\lambda - T)^{-1}$ around $\lambda_0$ has only a finite number of nonzero coefficients with negative indices.

Recall that a closed, densely defined operator $T$ is called a Fredholm operator if and only if $\dim N(T) < \infty$, $R(T)$ is closed, and $\operatorname{codim} R(T) < \infty$. Let us write $G = \{\lambda \in C | \lambda - T$ is a Fredholm operator}. Since $G$ is known to be an open set let us denote by $G_i$, $i \in I$, the open connected components of $G$. Gohberg and Krein [2] proved (though they never formally defined the essential spectrum) that $\lambda_0 \notin \text{ess } (T)$ if and only if $\lambda_0$ belongs to an open connected component $G_i$ which has a nonempty intersection with the resolvent set of $T$.

There are several other characterizations of $\text{ess } (T)$. D. Lay [5] has shown that if $C$ is a compact, bounded linear operator such that $C: D(T) \to D(T)$ and $(CT)(x) = (TC)(x)$ for $x \in D(T^n)$, then $\text{ess } (T + C) = \text{ess } (T)$ and that conversely $\text{ess } (T)$ is the largest subset of $\sigma (T)$ which remains invariant under such perturbations. Other authors have described the essential spectrum of $T$ in terms of the ascent, descent, nullity and defect of $T$, but we shall not need these results.

Received by the editors July 7, 1969.

AMS Subject Classifications. Primary 4710, 4730, 4748.

Key Words and Phrases. Browder's essential spectrum, spectral mapping theorems, perturbation theorems, noncompact perturbations.

Copyright © 1970, American Mathematical Society

445
In this paper we shall find Lay’s result especially useful. In the first section we shall establish spectral mapping theorems. If \( B \) is a bounded linear operator and \( f \) is a function analytic on a neighborhood of \( \sigma(B) \), we shall prove that \( \text{ess } (f(B)) = f(\text{ess } (B)) \). If \( T \) is a closed, densely defined linear operator with nonempty resolvent and \( p \) is a polynomial, we shall show that \( \text{ess } (p(T)) = p(\text{ess } (T)) \).

In the second section we consider perturbation theorems for the essential spectrum. Let \( T \) be a closed, densely defined linear operator and \( B \) a bounded linear operator such that \( B: D(T) \to D(T) \) and \( (BT)(x) = (TB)x \) for \( x \in D(T) \). Then we shall prove that \( \text{ess } (T + B) \subseteq \text{ess } (T) + \text{ess } (B) = \{ \nu + \mu \mid \nu \in \text{ess } (T), \mu \in \text{ess } (B) \} \). If \( 0 \notin \sigma(B) \) or if \( 0 \notin \text{ess } (B) \) and \( \sigma(T) \neq \mathbb{C} \) (either hypothesis implies that \( BT \) is closed), then we shall show that \( \text{ess } (BT) \subseteq \text{ess } (B) \text{ ess } (T) = \{ \lambda \mu \mid \mu \in \text{ess } (B), \nu \in \text{ess } (T) \} \). Finally, we shall establish a perturbation theorem which makes no boundedness assumption on \( B \). Let \( T \) and \( B \) be closed, densely defined linear operators with \( D(T) \subseteq D(B) \). Let \( D = \{ x \in D(T) \mid Tx \in D(B) \} \) and assume that \( B: D \to D(T) \) and \( (TB)x = (BT)x \) for \( x \in D \). Assume that \( T + B \) is closed, \( \rho(T + B) \) is nonempty, and \( \text{ess } (T) \) is compact. Then there is a compact subset \( S \) of \( \text{ess } (B) \) such that \( \text{ess } (T + B) \subseteq \text{ess } (T) + S = \{ \nu + s \mid \nu \in \text{ess } (T), s \in S \} \).

I would like to thank Richard Beals for a number of helpful discussions about the last mentioned theorem. In particular he suggested a simplified proof of an early version of the theorem, and it is that proof which will be given here.

1. Spectral mapping theorems for the essential spectrum. We begin with some trivialities. It is clear from the definition of \( \text{ess } (B) \) that if \( X \) is a finite dimensional vector space and \( B: X \to X \) is a linear operator, then \( \text{ess } (B) \) is empty. Thus we shall always assume that \( X \) is an infinite dimensional Banach space, since our theorems are trivial otherwise. In this case, if \( B: X \to X \) is a bounded linear operator, \( \text{ess } (B) \) is nonempty, since otherwise \( \lambda - B \) would be Fredholm of index zero for all \( \lambda \in \mathbb{C} \), whence by Theorem 3.2 of [2], \( X \) would be finite dimensional.

**Theorem 1.** Let \( B \) be a bounded linear operator of a Banach space \( X \) into itself. Let \( f \) be an analytic function defined on an open neighborhood of \( \sigma(B) \). Then \( \text{ess } (f(B)) = f(\text{ess } (B)) \).

**Proof.** Let \( G \) be the open neighborhood of \( \sigma(B) \) on which \( f \) is defined and let \( \Gamma \) be a Jordan curve (not necessarily connected) contained in \( G \) and enclosing \( \sigma(B) \). We can select \( \epsilon_0 > 0 \) such that \( \Gamma \) encloses \( N_{\epsilon_0}(\sigma(B)) = \{ \lambda \mid d(\lambda, \sigma(B)) < \epsilon_0 \} \).

First, let us show that \( \text{ess } (f(B)) \subseteq f(\text{ess } (B)) \). To prove this it suffices to show that \( \text{ess } (f(B)) \subseteq f(N_{\epsilon}(\text{ess } B)) \) for all \( \epsilon, \epsilon_0 \geq \epsilon > 0 \). For suppose \( \mu_0 \in f(N_{\epsilon}(\text{ess } B)) \) for all \( \epsilon, \epsilon_0 \geq \epsilon > 0 \). Then we can find a sequence of points \( \nu_n \) such that \( d(\nu_n, \text{ess } B) \to 0 \) and \( f(\nu_n) = \mu_0 \). Since \( \text{ess } B \) is compact, we can find a convergent subsequence \( \nu_{n_i} \to \nu_0 \in \text{ess } (B) \), and \( f(\nu_0) = \mu_0 \).

Thus, select \( \epsilon, \epsilon_0 \geq \epsilon > 0 \). Since \( \sigma(B) - N_{\epsilon}(\text{ess } B) \) is a compact set consisting of isolated points, let \( \sigma(B) - N_{\epsilon}(\text{ess } B) = \{ \nu_1, \ldots, \nu_m \} \). Let \( C_1, \ldots, C_m \) be disjoint circles...
about $v_1, \ldots, v_m$ and such that the only point of $\sigma(B)$ enclosed in $C_i$ is $v_i$. Write $P = (1/2\pi i) \sum_{j=1}^m \int_{C_i} (v-B)^{-1} \, dv$ and recall that $P$ is a finite dimensional projection commuting with $B$ (see, for instance, [2] or [4]). Since we can assume $\text{ess}(B)$ nonempty, let us select $v_0 \in \text{ess}(B)$ and set $B_1 = B - BP + v_0P$. We claim that $\sigma(B_1) \subset N_\varepsilon(\text{ess} B)$. To see this, set $M = (I-P)(X)$ and note that $B_1|M = B|M$ ($B$ clearly maps $M$ into itself). By the functional calculus for bounded linear operators, it is known that $\mu - B|M$ is one-to-one and onto $M$ for $\mu$ inside one of the $C_i$, and since the only other points in $\sigma(B)$ lie in $N_\varepsilon(\text{ess} B)$, $\mu - B_1|M$ is one-to-one and onto $M$ for $\mu \notin N_\varepsilon(\text{ess} B)$. On the other hand, if we write $V = PX$,

$$\mu - B_1|V = (\mu - v_0)|V,$$

so $\mu - B_1$ is one-to-one and onto $V$ for $\mu \neq v_0$. It follows that $\mu - B_1$ is one-to-one and onto $X$ for $\mu \notin N_\varepsilon(\text{ess} B)$.

Our next claim is that $\text{ess} (f(B)) = \text{ess} (f(B_1))$. (Notice that $f(B_1)$ is defined since $\sigma(B_1) \subset N_\varepsilon(\text{ess} B) \subset G$, and in fact $\Gamma$ encloses $\sigma(B_1)$.) Our claim will follow from Lay's theorem if we can show that $K = f(B_1) - f(B)$ is compact and commutes with $f(B)$. Commutativity is clear. To see compactness notice that for $p \in \Gamma$ (so $p \in \rho(B) \cap \rho(B_1)$), $\mu - B$ and $\mu - B_1$ take $M$ into $M$ and $V$ into $V$. Since $X = M \oplus V$, it follows that $(\mu - B)^{-1}$ and $(\mu - B_1)^{-1}$ take $M$ into $M$ and $V$ into $V$. Thus, for $x \in X$, $(\mu - B_1)^{-1}x - (\mu - B)^{-1}x = (\mu - B_1)^{-1}(P_x) - (\mu - B)^{-1}(P_x)$, and we see finally that

$$f(B_1)x - f(B)x = \frac{1}{2\pi i} \int_{\Gamma} f(\mu)(\mu - B)^{-1}x \, d\mu = (f(B_1) - f(B))(P_x),$$

a finite dimensional operator.

With the above observation we can complete the first half of the proof. We have $\text{ess} (f(B)) = \text{ess} (f(B_1))$. But by the ordinary spectral mapping theorem, $\sigma(f(B_1)) = f(\text{ess}(B)) \subset f(N_\varepsilon(\text{ess} B))$.

Conversely, we want to show that $\text{ess} (f(B)) \supset f(\text{ess} B)$. Assume not, so that there exists $\mu_0 \in f(\text{ess} B) - \text{ess} f(B)$. Since $\mu_0 \notin \text{ess} f(B)$, take a circle $C$ about $\mu_0$ which contains no other points of $\sigma(f(B))$. Let $P = (1/2\pi i) \int_C (\mu - f(B))^{-1} \, d\mu$, a finite dimensional projection, and let $M_1 = (I - P_1)X$. It follows as above that for $\nu \in \Gamma$, $(\nu - B)^{-1} : M_1 \to M_1$ and thus $f(B)|M_1 = f(B|M_1)$. Also as above, we know that $\mu_0 - f(B)|M_1$ is a one-to-one map of $M_1$ onto $M_1$. Thus if $f(\nu_0) = \mu_0$ for $\nu_0 \in \text{ess} (B)$, $\nu_0 \notin \sigma(B|M_1)$, since if it were, we would have

$$\mu_0 \in \sigma(f(B|M_1)) = \text{ess}(f(B|M_1)).$$

Now we can show that $\nu_0 \notin \text{ess} (B)$, which gives a contradiction. We use Browder's original definition. If we set $V_1 = P_1X$, the range of $\nu_0 - B$ is $M_1 \oplus (\nu_0 - B)V_1$, the span of a closed subspace and a finite dimensional subspace, hence closed. Since $V_1$ is finite dimensional $(\nu - B)|V_1$ must be one-to-one and onto $V_1$ for
R. D. NUSSBAUM

August 0<|v-v_0|<\delta, and since \(v_0-B\)M_1 is one-to-one and onto \(M_1, v-B\)M_1 is one-to-one and onto \(M_1\) for \(|v-v_0|<r, r>0\). Thus \(v-B\) is one-to-one and onto for \(|v-v_0|<\min (r, \delta)\) and \(v_0\) is an isolated point of \(\sigma(B)\). Finally, we clearly have \(\dim ((\bigcup_{j>0}N(v_0-B_j)) \leq \dim V_1\). Q.E.D.

Remark 1. Theorem 1 was proved by T. T. West [8] for the case that \(ess (B) =\{0\}\) (the so-called “Riesz operators”) and \(f(0)=0\).

If we try to generalize Theorem 1 to unbounded linear operators \(T\), we have the problem of defining \(f(T)\). If \(f\) is a polynomial, however, there is basically no problem and we can obtain an analogue of Theorem 1. First we need a lemma. This result is not difficult; a proof is given by Goldberg [3, pp. 107-108].

Lemma 1 (Goldberg). Let \(T\) be a closed, densely defined linear operator such that \(\sigma(T) \neq C\). Let \(p\) be a polynomial of degree \(n\); then \(p(T)\), with domain \(D(T^n)\), is a closed, densely defined linear operator and \(\sigma(p(T)) = p(\sigma(T))\).

Lemma 2. Let \(T\) be a closed, densely defined linear operator on \(X\) such that \(\sigma(T) \neq C\). Let \(F\) be a bounded, finite dimensional linear operator such that \((\nu-T)^{-1}F = F(\nu-T)^{-1}x, \nu \in \rho(T)\). Then \(\nu - T\) is a closed, densely defined linear operator and \(\sigma(\nu - T) = \sigma(\nu - T)\).

Proof. Select \(\nu \in \rho(T)\). Since \(D(T) = (\nu-T)^{-1}X\) and since \((\nu-T)^{-1} = (\nu-T)^{-1}\), \(F: D(T) \to D(T)\). Showing \((FT)x = (TF)x\) for \(x \in D(T)\) is equivalent to showing that \(F(\nu-T)x = (\nu-T)Fx\) for \(x \in D(T)\), and to prove this we simply apply \((\nu-T)^{-1}\) to both sides.

If we set \(V = F(X)\), by assumption, \(V\) is finite dimensional. Since \(D(T)\) is dense in \(X\) and \(F\) is bounded, \(F(D(T))\) is a dense subspace of \(V\) and hence \(F(D(T)) = V\), so that \(V \subset D(T)\). Because \((FT)x = (TF)x\) for \(x \in D(T)\), \(T: V \to V\) and thus \(\nu - T: V \to V\). However, \(V\) is finite dimensional, so we have \((\nu-T)^{-1}: V \to V\) and in fact \(V = (\nu-T)^{-1}V\). It follows that \(V = (\nu-T)^{-1}V\) for all \(j \geq 1\), so \(V \subset D(T^j)\) for all \(j \geq 1\).

To prove (c), note that because \((TF)x = (FT)x\) for \(x \in D(T)\), we can write \((T+F)/x - T^j x = A_j Fx\) for \(x \in D(T^j)\), \(A_j\) a linear operator defined on \(D(T^j)\). However, by our above work, \(V \subset D(T^j)\) and since \(V\) is finite dimensional, \(\|A_j v\| \leq M \|v\|\) for \(v \in V\). Thus we see that for \(x \in X\), \(\|A_j Fx\| \leq M \|Fx\| \leq M \|F\| \|x\|\), so that \(E_j = A_j F\) is a bounded linear operator. By using the commutativity of \(F\) and \(T\), we see that for \(v \in D(T^j)\),

\[A_j v = \sum_{i=0}^{j-1} \binom{j}{i} T^i F^{j-i-1} v\]

and since \(Fx \in D(T^{2j})\) for any \(x\), it follows immediately that \(A_j Fx \in D(T^j)\) for any \(x\). Finally, since \(T^j Fx = T^j F x\) for \(x \in D(T^j)\) (by repeatedly using \((FT)x = (TF)x\) for \(x \in D(T)\)), it is easy to see that \(T^j E_j x = E_j T^j x\) for \(x \in D(T^j)\).
By using (c), we see that \(p(T+F)x - p(T)x = Ex\) for \(x \in D(T^n)\) \((n = \text{degree of } p)\), \(E\) a finite dimensional, bounded linear operator such that \(E: D(T^n) \to D(T^n)\) and \(p(T)Ex = Ep(T)x\) for \(x \in D(T^n)\). It follows by Lay's theorem that
\[
\text{ess } (p(T+F)) = \text{ess } (p(T)).
\]

**Theorem 2.** Let \(T\) be a closed, densely defined linear operator on a Banach space \(X\) and assume that \(\sigma(T) \neq \mathbb{C}\). Then if \(p\) is a polynomial, \(p(T)\) is closed and \(\text{ess } (p(T)) = p(\text{ess } T)\).

**Proof.** The proof follows the basic outlines of Theorem 1, so we shall be sketchy.

Let \(V_R(0)\) denote the closed ball of radius \(R\) about \(0\) in \(\mathbb{C}\) and let \(V_R'(0)\) denote the complement of \(V_R(0)\). We want to show that \(\text{ess } (p(T)) \subseteq p(\text{ess } T)\) and in order to prove this it suffices to prove that
\[
\text{ess } (p(T)) \subseteq p(N_{\varepsilon}(\text{ess } T) \cup V_R'(0)) \quad \text{for all } \varepsilon > 0, \ R > 0.
\]

For suppose we have established the above inclusion and \(\lambda_0 \in \text{ess } (p(T))\). Unless \(p\) is constant, in which case the theorem is trivial, there exists an \(R\) so large that \(p(V_R(0)) \subseteq V_{R_0}(0)\), so that \(\lambda_0 \in p(N_{\varepsilon}(\text{ess } T) \cap V_R(0))\). If we select \(\varepsilon_n \to 0\) and \(\mu_n \in N_{\varepsilon}(\text{ess } T) \cap V_R(0)\) such that \(p(\mu_n) = \lambda_0\), then since \(\mu_n \in V_R(0)\), a compact set, we can assume \(\mu_n \to \mu \in \text{ess } T\) and \(\lambda_0 = p(\mu)\).

Consider
\[
(\sigma(T) - N_{\varepsilon}(\text{ess } T)) \cap V_R(0) = \{\mu_1, \ldots, \mu_m\}
\]

\[
\text{ess } (\sigma(T) - N_{\varepsilon}(\text{ess } T)) = \{\lambda \mid \lambda \in \sigma(T), \ \lambda \notin N_{\varepsilon}(\text{ess } T)\}.
\]

If we let \(C_1, \ldots, C_m\) be disjoint circles such that \(\mu_j\) is the only point of \(\sigma(T)\) contained in \(C_j\), and if we set \(P = (1/2\pi i) \sum_{j=1}^m \int_{C_j} (\lambda - T)^{-1} d\lambda\), then just as before \(P\) is a finite dimensional projection and \(P: X \to D(T)\). If we select \(\mu_0\) with \(|\mu_0| > R\) and if we set \(F = -TP + \mu_0P\), the same proof as before shows that \(\sigma(T+F) \subseteq N_{\varepsilon}(\text{ess } T) \cup V_R(0)\). Since \(P\) is finite dimensional, \(F\) is finite dimensional; and since \(P\) is continuous as a map from \(X\) to \(D(T)\) with the graph topology and \(T\) is continuous from \(D(T)\) with the graph topology to \(X\), \(F\) is bounded. It is also clear that for \(\nu \in p(T)\), \((\nu - T)^{-1}F = F(\nu - T)^{-1}\). It follows by Lemma 2 that
\[
\text{ess } (p(T)) = \text{ess } (p(T+F)) \subseteq \sigma(p(T+F)) = p(\sigma(T+F)) \subseteq p(N_{\varepsilon}(\text{ess } T) \cup V_R'(0)).
\]

The proof that \(p(\text{ess } T) \subseteq \text{ess } (p(T))\) is essentially the same as in Theorem 1, and we omit it. We just note that a little more care must be exercised since \(T\) is not everywhere defined. Q.E.D.

2. **Perturbation theorems for the essential spectrum.** To begin this section let us establish some notation. We shall say that linear operators \(T\) and \(B\) satisfy hypothesis one \((H_1)\) if \(T\) is a closed densely defined linear operator with domain \(D(T)\), \(B\) is a bounded linear operator such that \(B: D(T) \to D(T)\), and \((TB)(x) = (BT)(x)\) for \(x \in D(T^2)\). Condition \(H_1\) is Lay's definition of "\(T\) commutes with..."
We shall say that linear operators $T$ and $B$ satisfy $H_2$ if $T$ and $B$ are the same as above, but we strengthen the commutativity condition to $(TB)(x) = (BT)(x)$ for $x \in D(T)$.

Our next lemma is concerned with the behavior of $\sigma(T+B)$ when $T$ and $B$ satisfy $H_1$. In the case that $T$ is bounded, this lemma is well known and follows by Banach algebra techniques. In the general case the lemma has been proved by J. T. Marti [6].

Lemma 3 (Marti). Let $T$ and $B$ be linear operators satisfying $H_1$. Then $T+B$ is closed and $\sigma(T+B) \subseteq \sigma(T) + \sigma(B) = \{ \mu + \nu \mid \mu \in \sigma(B), \nu \in \sigma(T) \}$. If $\sigma(T)$ is empty, we interpret $\sigma(T) + \sigma(B)$ as the empty set.

The following lemma is also proved by Marti [6]. If $T$ is bounded, it follows by Banach algebra techniques without the assumption that $0 \notin \sigma(B)$.

Lemma 4 (Marti). Let $T$ and $B$ be linear operators satisfying $H_2$. Assume that $0 \notin \sigma(B)$. Then $BT$ is a closed linear operator, and

$$\sigma(BT) \subseteq \sigma(B)\sigma(T) = \{ \mu \nu \mid \mu \in \sigma(B), \nu \in \sigma(T) \}.$$

If $\sigma(T)$ is empty, $\sigma(B)\sigma(T)$ is interpreted as empty.

Remark 2. Marti actually proves Lemma 3 under the assumption that $B : D(T) \to D(T)$ and $(TB)x = (BT)x$ for $x \in D(T)$. However, if $\sigma(T) = C$, Lemma 3 is immediate. If $\sigma(T) \neq C$ and $\nu \in \rho(T)$, $H_1$ implies that $B(\nu - T)^{-1}x = (\nu - T)^{-1}Bx$ for $x \in D(T)$ (apply $(\nu - T)$ to both sides), whence by continuity $B(\nu - T)^{-1}x = (\nu - T)^{-1}Bx$ for all $x$. If $u \in D(T)$, $u = (\nu - T)^{-1}x$, so that

$$(\nu - T)Bu = (\nu - T)B(\nu - T)^{-1}x = Bx = B(\nu - T)u,$$

and Marti's commutativity condition holds.

Remark 3. Marti considers closed, densely defined linear operators $T$ and $S$ such that $D(T) \subseteq D(S)$, $S : D(T) \to D(T)$, and $(TS)x = (ST)x$ for all $x \in D(T)$ such that $Tx \in D(S)$. This generality is illusory, however. For suppose $\rho(T) \neq \emptyset$ and select $\nu \in \rho(T)$. For $x \in D(S)$, $(\nu - T)S(\nu - T)^{-1}x = Sx$, by the commutativity condition. If we let $Y$ denote the Banach space $D(T)$ with the graph topology $(\|x\|_Y = \|x\| + \|Tx\|)$, it is easy to see that $S$ is a closed linear operator as a map from $Y$ to $Y$. Thus $S$ is bounded as a map from $Y$ to $Y$. Since $(\nu - T)^{-1} : X \to Y$ is bounded and $(\nu - T) : Y \to X$ is bounded, $\| (\nu - T)S(\nu - T)^{-1}x \| \leq M \|x\|$. It follows that $\|Sx\| \leq M \|x\|$ for $x \in D(S)$, and since $S$ was assumed closed, $D(S) = X$ and $S$ is bounded.

The same remark applies to Lay's article [5].

Theorem 3. Let $T$ and $B$ be linear operators satisfying $H_1$. Then $T+B$ is closed and $\text{ess } (T+B) \subseteq \text{ess } (T) + \text{ess } (B) = \{ \nu + \mu \mid \nu \in \text{ess } (T), \mu \in \text{ess } (B) \}$. If $\text{ess } (T)$ is empty, this means that $\text{ess } (T+B)$ is empty.
Proof. If \( \sigma(T) = C \), \( \text{ess } (T) = C \) and we are done. Thus we assume \( \sigma(T) \neq C \). Now we establish some simple commutativity relationships. By Remark 2 we know that \((TB)x = (BT)x \) for \( x \in D(T) \). By the proof of Remark 2 we have \((v-T)^{-1}B = B(v-T)^{-1} \) for \( v \in \rho(T) \), and it follows (by applying \( \mu - B \) to both sides of the equation) that \((\mu - B)^{-1}(v-T)^{-1} = (v-T)^{-1}(\mu - B)^{-1} \) for \( \mu \in \rho(B) \), \( v \in \rho(T) \). The latter equation implies that \((\mu - B)^{-1} \) takes \( D(T) \) into \( D(T) \), so that we see (by applying \( \mu - B \) to both sides of the equation) that \((\mu - B)^{-1}Tx = T(\mu - B)^{-1}x \) for \( x \in D(T) \).

Select \( \lambda_0 \in \text{ess } (T + B) \). We want to show that \( \lambda_0 \in \text{ess } (T) + \text{ess } (B) \). Suppose we can show that for every \( \varepsilon > 0 \), \( \lambda_0 \in N_\varepsilon(\text{ess } T) + N_\varepsilon(\text{ess } B) \). This will imply that for every \( \varepsilon > 0 \), \( \lambda_0 \in N_{2\varepsilon}(\text{ess } T + \text{ess } B) \). Since \( \text{ess } T \) is closed and \( \text{ess } B \) is compact, \( \text{ess } T + \text{ess } B \) is closed, and it will follow that \( \lambda_0 \in \text{ess } T + \text{ess } B \).

Thus take \( \lambda_0 \in \text{ess } (B+T) \) and select \( \varepsilon > 0 \). Let \( R \) be a constant such that \( |\lambda_0| < R \) and \( |\mu| < R \) for \( \mu \in N_\varepsilon(\text{ess } B) \). As in §1, let \( V_{2\varepsilon}(0) = \{x | \|x\| \leq 2R \} \). Since spectral points which are not in the essential spectrum are isolated, we know that \( \sigma(T) \cap (V_{2\varepsilon}(0) - N_\varepsilon(\text{ess } T)) = \{v_1, \ldots, v_n\} \) and \( \sigma(B) - N_\varepsilon(\text{ess } B) = \{\mu_1, \ldots, \mu_m\} \). Let \( D_1, \ldots, D_n \) be disjoint circles about \( v_1, \ldots, v_n \), each \( D_i \) containing no points of \( \sigma(T) \) except \( v_i \). Similarly, let \( C_1, \ldots, C_m \) be disjoint circles about \( \mu_1, \ldots, \mu_m \) such that each \( C_j \) contains no point of \( \sigma(B) \) except \( \mu_j \). As in §1 we know that

\[
Q = \sum_{j=1}^n \frac{1}{2\pi i} \int_{C_j} (\mu - B)^{-1} d\mu
\]

is a bounded, finite dimensional projection; similarly

\[
P = \sum_{j=1}^n \frac{1}{2\pi i} \int_{D_j} (v - T)^{-1} dv
\]

is a bounded, finite dimensional projection, and in fact \( P: X \to D(T) \) is continuous from \( X \) to \( D(T) \) with the graph topology.

We are now in a position to complete the proof. Take \( v_0, |v_0| > 2R \), and select \( \mu_0 \in \text{ess } (B) \) (since we always assume \( X \) infinite dimensional, \( \text{ess } (B) \neq \emptyset \)). Consider the bounded, finite dimensional linear operator \( F = v_0P - TP + \mu_0Q - BQ \). By using the commutativity relations of the first paragraph of the proof, it is easy to see that for \( v \in \rho(T) \), \((v-T)^{-1}F = F(v-T)^{-1} \). It follows by Lemma 2 that \( F(X) \subseteq \bigcap_{\rho(T)} D(T') \) and \( TFX = FTX \) for \( x \in D(T) \). It is also clear that \( FB = BF \), so that \((T+B)FX = F(T+B)x \) for \( x \in D(T+B) = D(T) \). It follows by Lay's theorem that \( \text{ess } (T + B + F) = \text{ess } (T + B) \).

The next step is to make use of the form of \( F \). We have

\[
\text{ess } (B+T+F) \subseteq \sigma(B+\mu_0Q-BQ+(T+v_0P-TP)) = \sigma(B_1+T_1).
\]

It is not hard to see that \( B_1 \) and \( T_1 \) satisfy \( H_1 \). For instance, since

\[
(\mu_0Q-BQ)(v-T)^{-1} = (v-T)^{-1}(\mu_0Q-BQ),
\]
it follows by Lemma 2 that \( \mu_0 Q - BQ : D(T) \to D(T) \) and \( (\mu_0 Q - BQ)Tx = T(\mu_0 Q - BQ)x \) for \( x \in D(T) \). Thus by Marti's theorem \( \sigma(B_1 + T_1) \subseteq \sigma(B_1 + \sigma(T_1)) \).

However, by the same arguments used in the first section, \( \sigma(T_1) \subseteq N_\nu(\text{ess } T) \cup V_{2\nu}(0) \) and \( \sigma(B_1) \subseteq N_\nu(\text{ess } B) \). Putting all this together, \( \lambda_0 \in (N_\nu(\text{ess } T) \cup V_{2\nu}(0)) + N_\nu(\text{ess } B) \)

However, it is possible that \( \lambda_0 = v + \mu \) for \( v \in V_{2\nu}(0) \) and \( \mu \in N_\nu(\text{ess } B) \), since \( v + \mu > 2R - R = R \) and \( |\lambda_0| \leq R \). Thus \( \lambda_0 \in N_\nu(\text{ess } T) + N_\nu(\text{ess } B) \). Since \( \lambda_0 \) was an arbitrary element of \( \text{ess } (T + B) \), this shows that \( \text{ess } (T + B) \subseteq N_\nu(\text{ess } T) + N_\nu(\text{ess } B) \).

Q.E.D.

Notice that the above proof shows \( \text{ess } (T + B) \) is empty if \( \text{ess } (T) \) is empty.

Before we prove the corresponding theorem for \( \text{ess } (BT) \), we need another lemma.

**Lemma 5.** Let \( B \) and \( T \) be linear operators satisfying \( H_2 \). Assume that \( \sigma(T) \neq C \) and \( 0 \notin \text{ess } (B) \). Then \( BT \) is a closed linear operator. Furthermore, there exists a bounded, finite dimensional linear operator \( F \) such that \( B_1 = B - F \) is invertible, \( B_1 \) and \( T \) satisfy \( H_2 \), \( \text{ess } (B_1) = \text{ess } (B) \), and \( \text{ess } (B_1T) = \text{ess } (BT) \).

**Proof.** If \( B \) is invertible, the lemma is immediate (even without the assumption that \( \sigma(T) \neq C \)). Thus assume \( 0 \in \sigma(B) - \text{ess } (B) \). Let \( C \) be a circle about 0 which contains no other points of \( \sigma(B) \), and let \( Q = (1/2\pi i) \int_{C} (\mu^{-1} - B^{-1}) d\mu \), a bounded, finite dimensional projection. We define \( F = BQ - Q \), and we have to show \( F \) satisfies the claims of the lemma.

The usual argument shows that \( 0 \notin \sigma(B - F) \), so that \( B_1 \) is invertible. It is easy to show that \( (\nu - T)^{-1}F = F(\nu - T)^{-1} \) for \( \nu \in \rho(T) \), so that by Lemma 2 we know that \( F(X) \subseteq \bigcap_{1 \geq 1} D(T^x) \) and \( (TF)x = (FT)x \) for \( x \in D(T) \). It follows in particular that since \( F(X) \) is a finite dimensional subspace of \( D(T) \), \( TF \) is actually a bounded, finite dimensional linear operator. Since \( B_1 \) is invertible, \( B_1T \) is closed and thus \( BT = B_1T + TF \), a sum of a closed and a bounded linear operator, is closed.

To see that \( B_1 \) and \( T \) satisfy \( H_2 \), it suffices to show \( F \) and \( T \) satisfy \( H_2 \), and we have already seen that \( (FT)x = (TF)x \) for \( x \in D(T) \). Trivially, \( BF = FB \), so that by Lay's theorem, \( \text{ess } (B_1) = \text{ess } (B) \). Finally, to show that \( \text{ess } (B_1T) = \text{ess } (BT) \), it suffices, by Lay's theorem, to show that \( (BT)(TF)x = (TF)x(BT)x \) for \( x \in D(T) \). However, \( FX \subseteq \bigcap_{1 \geq 1} D(T^x) \) for any \( x \), so that \( (TF)x \in D(T) \), and we can write \( (BT)(TF)x = (TB)(TF)x \).

Q.E.D.

**Theorem 4.** Let \( B \) and \( T \) satisfy \( H_2 \). Assume that \( 0 \notin \text{ess } (B) \) and that \( BT \) is closed (under the hypotheses, Lemma 5 implies that \( BT \) is closed if \( \sigma(T) \neq C \)). Then \( \text{ess } (BT) \subseteq \text{ess } (B) \text{ ess } (T) = \{ \nu \mu \mid \mu \in \text{ess } (B), \nu \in \text{ess } (T) \} \). If \( \text{ess } (T) \) is empty, \( \text{ess } (B) \text{ ess } (T) \) is interpreted as being empty.

**Proof.** Since \( \text{ess } (B) \neq \varnothing \) and \( 0 \notin \text{ess } (B) \) (recall that we assume \( X \) infinite dimensional), we can select \( \mu_0 \in \text{ess } (B), \mu_0 \neq 0 \). If \( \sigma(T) = C \), then we have \( \text{ess } (T) \text{ ess } (B) \supseteq \{ \lambda \mu_0 \mid \lambda \in C \} \supseteq C \), so that \( \text{ess } (BT) \subseteq \text{ess } (B) \text{ ess } (T) \). Thus we can
assume that \( \sigma(T) \neq C \). By Lemma 5 we can find a bounded, invertible operator \( B_1 \) such that \( B_1 \) and \( T \) satisfy \( H_2 \), \( \text{ess} (B_1) = \text{ess} (B) \) and \( \text{ess} (B_1T) = \text{ess} (BT) \). Thus it suffices to show that \( \text{ess} (B_1T) \subseteq \text{ess} (B_1) \text{ ess} (T) \). For notational convenience, we shall write \( B \) instead of \( B_1 \), but now we can also assume \( B \) invertible.

Select \( \lambda_0 \in \text{ess} (BT) \). Notice that to show \( \lambda_0 \in \text{ess} (B) \text{ ess} (T) \), it suffices to show that \( A_0 \in \text{Ne}(\text{ess} B) \text{ Ne}(\text{ess} T) \) for all \( s > 0 \). For suppose \( A_0 \in \text{Ne}(\text{ess} B) \text{ Ne}(\text{ess} T) \), then \( |v| \leq M - |\lambda_0|/\delta \). Thus (for \( n \) large enough), if we select \( \mu_n \in \text{N}(\text{ess} B) \) and \( v_n \in \text{N}(\text{ess} T) \) such that \( |\mu_n| = |\lambda_0| \), \( \mu_n \) and \( v_n \) lie in compact sets, and by taking subsequences we can assume \( \mu_n \to \mu \in \text{ess} (B) \) and \( v_n \to v \in \text{ess} (T) \). Thus we see that \( \lambda = \mu v \in \text{ess} (B) \text{ ess} (T) \).

We now proceed as in Theorem 3, though we shall omit many details. Take \( \lambda_0 \in \text{ess} (BT) \), Select \( \varepsilon > 0 \) small enough so that \( |\mu| \geq \delta > 0 \) for \( \mu \in \text{N}(\text{ess} B) \). Select a constant \( R > \lambda_0/\delta \) and a number \( v_0 \) with \( |v_0| \geq R \). Select \( \lambda_0 \in \text{ess} (B) \) as before. Just as in Theorem 3, we can find finite dimensional projections \( P \) and \( Q \) which correspond respectively to the eigenvalues \( \{\nu_1, \ldots, \nu_n\} = V_R(0) \cap (\sigma(T) - N_{\varepsilon} \text{ ess} (T)) \) and \( \{\mu_1, \ldots, \mu_m\} = \sigma(B) - N_{\delta} \text{ ess} (B) \). We want to use Lay’s theorem to show that
\[
\text{ess} (BT) = \text{ess} ((B + \mu_0 Q - BQ)(T + v_0 P - TP)).
\]

We write \( F_1 = \mu_0 Q - BQ \) and \( F_2 = \mu_0 P - TP \). It is easy to show that \( (v - T)^{-1}F_1 = F_1(v - T)^{-1} \) and \( (v - T)^{-1}F_2 = F_2(v - T)^{-1} \) for \( v \in \rho(T) \). It follows by Lemma 2 that \( F_i(X) \subseteq \bigcap_{x \in D(T)} (T^i)^x \) for \( x \in D(T) \), and \( TF_i \) is a bounded, finite dimensional linear operator, \( i = 1, 2 \). Using these results, we see that if we write \( (B + F_1)(T + F_2) = BT + F \), \( F \) is a bounded, finite dimensional linear operator and \( (v - T)^{-1}F = F(v - T)^{-1} \) for \( v \in \rho(T) \). Applying Lemma 2 again shows \( F(X) \subseteq D(T) \) and \( (TF)x = (FT)x \) for \( x \in D(T) \). Since it is obvious that \( FB = BF \), Lay’s theorem now implies \( \text{ess} (BT + F) = \text{ess} (B) \).

Using the above results, we see that \( B_2 = B + F_1 \) and \( T_2 = T + F_2 \) satisfy the hypotheses of Lemma 4, so \( \text{ess} (B_2T_2) \subseteq \sigma(B_2T_2) \subseteq \sigma(B_2) \sigma(T_2) \). Just as before, \( \sigma(B_2) \subseteq \text{N}(\text{ess} B) \) (so, in particular, \( B_2 \) is invertible), and \( \sigma(T_2) \subseteq \text{N}(\text{ess} T) \cup V_R(0) \). By the selection of \( R \), \( \lambda_0 \neq \mu v \) for \( \mu \in \text{N}(\text{ess} B) \) and \( v \in V_R(0) \), so that \( \lambda_0 \in \text{N}(\text{ess} T) \text{N}(\text{ess} T) \). As we have remarked, this establishes our theorem. Q.E.D.

Our next goal is to establish a perturbation theorem when \( B \) may not be bounded. To compensate for the unboundedness of \( B \), we have to strengthen our assumptions on \( T \).

**Theorem 5.** Let \( T \) and \( B \) be closed, densely defined linear operators with \( D(T) \subseteq D(B) \). Let \( D = \{x \in D(T) \mid Tx \in D(B)\} \) and assume \( B : D \to D(T) \) and \( (TB)x = (BT)x \) for \( x \in D \). Assume that \( T + B \) is closed and \( \rho(T + B) \) is nonempty. Finally, suppose \( \text{ess} (T) \) is compact. Then there exists a closed subspace \( X_1 \) of \( X \) and a bounded linear operator \( B_1 : X_1 \to X_1 \) such that \( \text{ess} (B_1) \subseteq \text{ess} (B) \) and \( \text{ess} (T + B) \subseteq \text{ess} (T) + \text{ess} (B_1) \).
Proof. For $\nu \in \rho(T)$ and $x \in D(B)$, we have $B(\nu - T)^{-1}x = (\nu - T)^{-1}Bx$, because $(\nu - T)^{-1}x \in D$ for $x \in D(B)$ and then $(\nu - T)B(\nu - T)^{-1}x = B(\nu - T)(\nu - T)^{-1}x = (\nu - T)(\nu - T)^{-1}Bx$. Since $\text{ess}(T)$ is compact, let $C$ be a simple Jordan curve enclosing $\text{ess}(T)$ and lying in $\rho(T)$. Define $P = \frac{1}{(2\pi i)} \int_{C} (\nu - T)^{-1} d\nu$ and set $X_{1} = PX$, $X_{2} = (1 - P)X$. As usual, $P : X \to D(T)$, and in fact since $P$ is a projection, $P : X \to \bigcap_{j \geq 1} D(T^{j})$. Since $(\nu - T)^{-1}Bx = B(\nu - T)^{-1}x$ for $x \in D(B)$, we find easily that $(BP)x = (PB)x$ for $x \in D(B)$. It follows that for $u \in X_{1}$, $Bu = BP_{2}u = PB_{2}u \in X_{1}$. Thus we can define $B_{1} : X_{1} \to X_{1}$ by $B_{1}u = Bu$, $u \in X_{1}$. Similarly, we define $T_{1} : X_{1} \to X_{1}$. Since both $B_{1}$ and $T_{1}$ are closed and defined on all of $X_{1}$, they are bounded. Similarly, we note that $B : X_{2} \cap D(B) \to X_{2}$ and $T : X_{2} \cap D(T) \to X_{2}$, so we define $B_{2} = B|X_{2} \cap D(B)$ (viewed as a map into $X_{2}$) and $T_{2} = T|X_{2} \cap D(T)$ (viewed as a map into $X_{2}$).

Standard results now imply that $\sigma(B) = \sigma(B_{1}) \cup \sigma(B_{2})$, $\sigma(T) = \sigma(T_{1}) \cup \sigma(T_{2})$ and $\sigma(T+B) = \sigma(T_{1} + B_{1}) \cup \sigma(T_{2} + B_{2})$. Using these results and Browder's original definition of the essential spectrum, it is not hard to see that $\text{ess}(B) = \text{ess}(B_{1}) \cup \text{ess}(B_{2})$, $\text{ess}(T) = \text{ess}(T_{1}) \cup \text{ess}(T_{2})$ and $\text{ess}(T+B) = \text{ess}(T_{1} + B_{1}) \cup \text{ess}(T_{2} + B_{2})$. We omit the proof. However, we know as usual that $\sigma(T_{2})$ lies outside $C$, so that $\text{ess}(T_{2})$ lies outside $C$. Since $C$ encloses $\text{ess}(T)$, $\text{ess}(T_{1}) = \text{ess}(T)$ and $\text{ess}(T_{2})$ is empty. Note also that $B_{1}T_{1} = T_{1}B_{1}$. To show this, it suffices to show that $(BT)(Px) = (TB)(Px)$ for $x \in X$, and this is true since $Px \in D(T^{2}) \subset D$. It follows by Theorem 3 that $\text{ess}(T_{1} + B_{1}) \subset \text{ess}(T_{1}) + \text{ess}(B_{1}) = \text{ess}(T) + \text{ess}(B_{1})$. Thus to complete the proof it suffices to show that $\text{ess}(T_{2} + B_{2})$ is empty.

Since we assume $\rho(T+B)$ is nonempty, take $\mu_{0} \in \rho(T+B)$, so that certainly $\mu_{0} \in \rho(T_{2} + B_{2})$. Similarly, take $\nu_{0} \in \rho(T)$, so that $\nu_{0} \in \rho(T_{2})$. Then we can write

$\text{ess}(T_{2} + B_{2}) = \mu_{0} + \text{ess}(-\mu_{0} + T_{2} + B_{2})$

$= \text{ess}((-\mu_{0} + T_{2} + B_{2})(\nu_{0} - T_{2})^{-1}(\nu_{0} - T_{2})) + \mu_{0}$.

If we set $B_{3} = (-\mu_{0} + T_{2} + B_{2})(\nu_{0} - T_{2})^{-1}$, $B_{3}$ is a bounded, invertible operator. It is also easy to check that $B_{3} : D(T_{2}) \to D(T_{2})$ and $B_{3}(\nu_{0} - T_{2})x = (\nu_{0} - T_{2})B_{3}x$ for $x \in D(T_{2})$. It follows by Theorem 4 that $\text{ess}(B_{3}(\nu_{0} - T_{2})) \subset \text{ess}(B_{3}) \subset \text{ess}(\nu_{0} - T_{2})$. However, $\text{ess}(\nu_{0} - T_{2}) = \nu_{0} - \text{ess}(T_{2})$ is empty, so that $\text{ess}(B_{3}(\nu_{0} - T_{2}))$ is empty and $\text{ess}(T_{2} + B_{2})$ is empty. Q.E.D.

Corollary 1. Let $T$ and $B$ be closed, densely defined linear operators with $D(T) \subset D(B)$. If $D = \{x \in D(T) \mid Tx \in D(B)\}$, assume that $B : D \to D(T)$ and $(TB)x = (BT)x$ for $x \in D$. Assume that $\text{ess}(T)$ is compact. Suppose that there exists a sequence $\lambda_{n} \in \rho(T)$, $|\lambda_{n}| \to \infty$, such that $\|\lambda_{n} - T\|^{-1} \leq C \|\lambda_{n}\|$, $C$ a constant. Finally, assume that for every $\epsilon > 0$, there is a constant $K_{\epsilon}$ such that $\|Bx\| \leq \epsilon\|Tx\| + K_{\epsilon}\|x\|$ for $x \in D(T)$. Then the conclusion of Theorem 5 holds.

Proof. It is known (see [4]) that if $\|Bx\| \leq c\|Tx\| + K\|x\|$ for $x \in D(T)$ with $c < 1$, then $T+B$ is closed. Thus $T+B$ is certainly closed. To apply Theorem 5, it only remains to show that $\rho(T+B)$ is nonempty. We claim that $\lambda_{n} \in \rho(T+B)$ for $n$
large enough, and since \((\lambda_n - T - B) = (I - B(\lambda_n - T)^{-1}) (\lambda_n - T)\), it suffices to show that \(\|B(\lambda_n - T)^{-1}\| < 1\) for \(n\) large enough. But by our hypothesis,

\[
\|B(\lambda_n - T)^{-1}x\| \leq \epsilon \|T(\lambda_n - T)^{-1}x\| + K_\epsilon \|T(\lambda_n - T)^{-1}x\|
\]

\[
= \epsilon \|\lambda_n(\lambda_n - T)^{-1}x - x\| + K_\epsilon \|\lambda_n(\lambda_n - T)^{-1}x\|
\]

\[
\leq \epsilon \|x\| + \epsilon C \|x\| + K_\epsilon/|\lambda_n| \|x\|.
\]

Thus we merely select \(\epsilon\) such that \(\epsilon + \epsilon C < 1\) and then select \(n\) so large that

\[
e + \epsilon C + K_\epsilon/|\lambda_n| < 1.
\]

Q.E.D

REFERENCES


Rutgers University,
New Brunswick, New Jersey 08903

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use