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AN ALTERNATIVE PROOF THAT BING'S DOGBONE SPACE IS NOT TOPOLOGICALLY E^3

BY
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1. **Introduction.** R. H. Bing in [7] presented an example of an upper semicontinuous decomposition of E^3 into points and tame arcs, Bing's dogbone space, that is not topologically E^3. In this paper, a decomposition space resulting from a simpler construction than that of Bing's dogbone space will be proven to be topologically different from E^3 and the argument may be easily modified to apply to Bing's dogbone space.

This paper and [3] are from the author's doctoral dissertation. The author wishes to express his appreciation to Dr. B. G. Casler for his invaluable aid and assistance in preparing the dissertation and these papers. While writing this paper, the author was supported by NSF Grant 4073 at The University of Oklahoma, June through August, 1968.

It will be assumed where necessary or convenient that all embedded complexes are triangulated and polyhedral and any two are in relative general position and all homeomorphisms are piecewise linear.

The standard definitions and basic results employed will be those of Hocking and Young [9].

After Casler [8], if N is a positive integer, $N\alpha$ will denote a sequence of N positive integers $J(1), \ldots, J(N)$, and if r is a positive integer, the sequence $J(1), \ldots, J(N)$, r will be denoted by $N\alpha, r$. If $N=0$, $N\alpha=0$ and $N\alpha, r=r$. If N is a positive integer, $\{A_{N\alpha}\}$ will denote a collection of sets each with N subscripts and $\sum A_{N\alpha}$ will denote their sum.

If p is a positive integer, a p-od k is the union of the image sets of p homeomorphisms $\{f_i\}$ where the domain of each f_i is the unit interval $I=[0, 1]$ and for each pair $i, j, f_i(I)\cdot f_j(I)=f_i(0)=f_j(0)$. The center of k is $f_1(0)$ and the set of end-points of k is $\{f_i(1) : i=1, \ldots, p\}$.

The concept of linking of simple closed curves will be that of [6], namely two simple closed curves X_1 and X_2 link if and only if there is a two-complex Y_1 with boundary X_1 and X_2 intersects Y_1 an odd number of times.

2. **Construction of dogbone spaces.** To construct a dogbone space, of which Bing's dogbone space is an example, let A_0 be a solid double torus in E^3, as in

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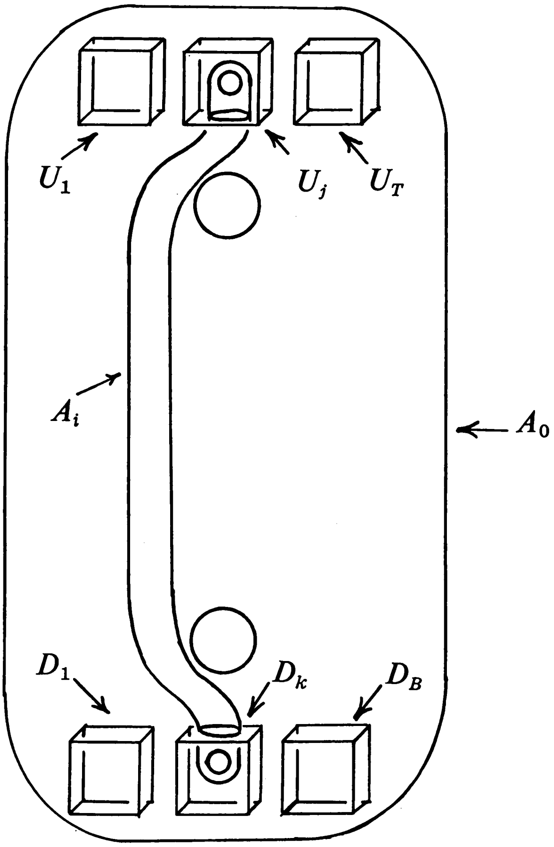

FIGURE 1

Figure 1. For fixed positive integers T and B, cubes $U_1, \ldots, U_T$ are embedded in the top of A_0 and cubes $D_1, \ldots, D_B$ are embedded in the bottom of A_0. Then, for a fixed positive integer K, solid double tori $A_1, \ldots, A_k$ are embedded in A_0 such that each A_i, $i=1, \ldots, K$, intersects exactly one cube in the top and one cube in the bottom of A_0, the intersection of any horizontal plane with Interior (A_i), $i=1, \ldots, K$, is either the interior of a disk or the union of the interiors of two disjoint disks, two solid double tori that intersect the same cube are linked in the interior of the cube, and if A_i intersects U_j and D_k, the closure of $A_i-(U_j+D_k)$ is a topological cube. To illustrate, the construction of Bing's dogbone space would correspond to the case where $T=B=1$ and $K=4$.

In each A_i, $i=1, \ldots, K$, cubes $U_{i,1}, \ldots, U_{i,T}$, $D_{i,1}, \ldots, D_{i,B}$ and solid double tori $A_{i,1}, \ldots, A_{i,K}$ are embedded such that there is a homeomorphism of E^3 onto itself which is the identity on the complement of some open set containing A_0 and takes A_0 onto A_i, U_j onto $U_{i,j}$, $j=1, \ldots, T$, D_k onto $D_{i,k}$, $k=1, \ldots, B$, and A_c onto $A_{i,c}$, $c=1, \ldots, K$. Let this process be continued; succeeding steps of the construction may be described inductively.

Let M denote $A_0\cdot\sum A_{1\alpha}\cdot\sum A_{2\alpha}\cdot\cdots$. Let G be the set whose elements are components of M and one-point subsets of E^3-M. Then, G is an upper semicontinuous decomposition of E^3 into tame arcs and one-point sets. Let E^3/G denote the associated decomposition space, a dogbone space.

Let C denote $\sum A_{1\alpha}+\sum U_{1\alpha}+\sum D_{1\alpha}$. We will be concerned only with cases where C is connected. Thus, C is a topological cube with handles. Let Γ_0 be a central curve of C consisting of points $u_1,\ldots,u_T$, $d_1,\ldots,d_B$ and arcs $a_1,\ldots,a_K$ where the end-points of a_i are u_j and d_k if A_i intersects U_j and D_K. Similarly for a fixed sequence $N\alpha$, $\sum A_{N\alpha,i}+\sum U_{N\alpha,i}+\sum D_{N\alpha,i}$ is a cube with handles with central curve $\Gamma_{N\alpha}$. The construction of a dogbone space may be conveniently represented by A_0 and Γ_0.

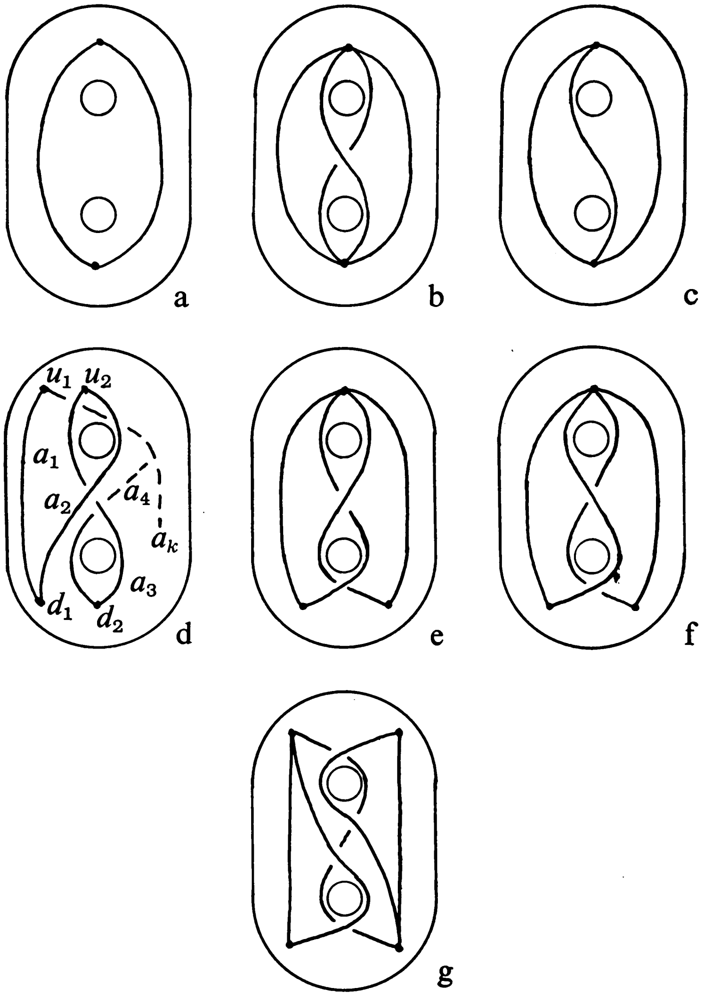

FIGURE 2

Some representations are shown in Figure 2. The construction by Bing in [5] could be done as in Figure 2a and was shown to be topologically E^3. Bing proved in [7] that Figure 2b represents a decomposition space, Bing's dogbone space, that is not topologically E^3. It was shown in [2] that Figure 2c represents a decomposition space that is topologically E^3. Figure 2d represents the decomposition space constructed by K solid double tori, K even, such that each solid double torus links exactly one solid double torus in the top of A_0 and exactly one solid double torus in the bottom of A_0; by suitable renumbering, A_1 links A_2 in the bottom of A_0, A_2 links A_3 in the top of $A_0, \ldots$ and A_K links A_1 in the top of A_0; the associated decomposition space was shown to be topologically E^3 in [2].

It will be shown in Theorem 5 that the decomposition space represented by Figure 2e is not topologically E^3 and the proof of Theorem 5 can be easily modified to show that Bing's dogbone space, represented by Figure 2b, is not topologically E^3. The proof of Theorem 5 cannot be easily modified and applied to the decomposition space represented by Figure 2f. The proof given by Bing in [7] for his dogbone space cannot be easily modified and applied to the decomposition spaces represented by Figure 2e, Figure 2f and Figure 2g. It is not known to the author whether or not the decomposition spaces represented by Figure 2f and Figure 2g are topologically E^3.

3. **The shrinking number.** Suppose E^3/G is a dogbone space. As in Figure 3, let P_1, P_2, P_3 and P_4 be disks such that for each i, $P_i \cdot \text{Boundary}\,(A_0) = \text{Boundary}\,(P_i)$ and let X_1, X_2 and X_3 be the closures of the components of $A_0 - \sum P_i$, $i = 1, \ldots, 4$. Each X_j is a topological 3-cell.

If E^3/G is topologically E^3, by Armentrout's result [4], if $\varepsilon > 0$ there is a homeomorphism f of E^3 onto E^3 which is the identity on the complement of Interior (A_0) and such that for each component m of M, the diameter of $f(m)$ is less than ε. The homeomorphism f is isotopic to the identity by an isotopy that is fixed on the complement of Interior (A_0). Thus, we have:

THEOREM 1. *If E^3/G is topologically E^3 and $\varepsilon > 0$, there is a homeomorphism f of E^3 onto E^3 which satisfies*

(i) *f is isotopic to the identity by an isotopy which is fixed on the complement of* Interior (A_0),

(ii) *if m is a component of M, the diameter of $f(m)$ is less than ε.*

We prove:

THEOREM 2. *If E^3/G is topologically E^3, there is a homeomorphism h of E^3 onto E^3 which satisfies*

(i) *h is isotopic to the identity by an isotopy which is fixed on the complement of* Interior (A_0),

(ii) *for some integer R, each $h(a_{R\alpha,i})$ in each $h(\Gamma_{R\alpha})$ intersects at most one of $P_1 + P_2$ and $P_3 + P_4$.*

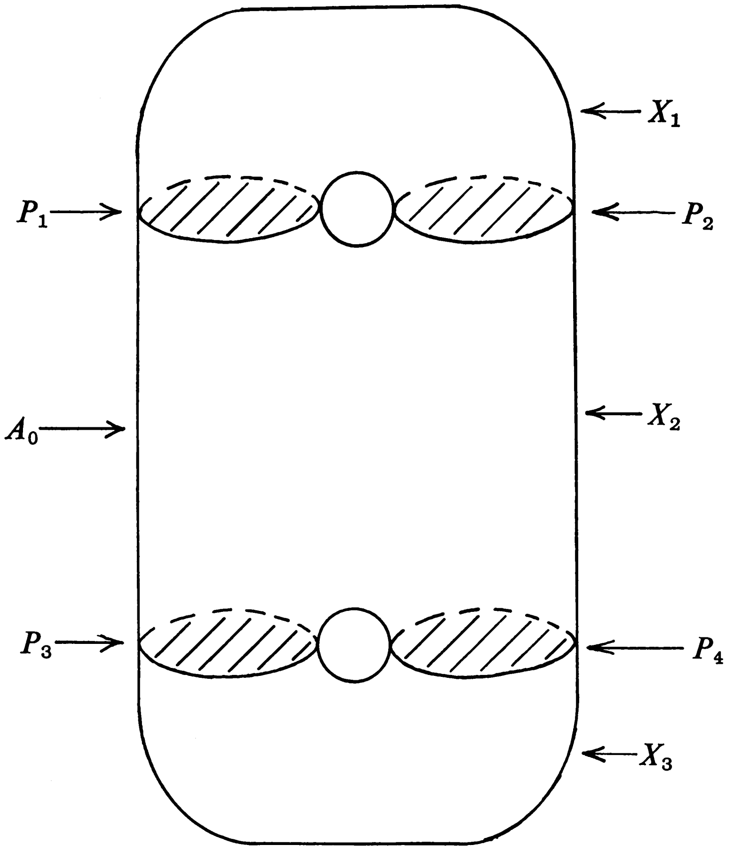

FIGURE 3

Proof. The distance from P_1+P_2 to P_3+P_4 is positive. By Theorem 1, there is a homeomorphism f of E^3 onto E^3 which is isotopic to the identity by an isotopy which is fixed on the complement of Interior (A_0) and such that if m is a component of M, $f(m)$ intersects at most one of P_1+P_2 and P_3+P_4. For each component m of M, there is a collection $A_{1\alpha}, A_{2\beta}, A_{3\delta}, \ldots$, such that $m=A_{1\alpha}\cdot A_{2\beta}\cdot A_{3\delta}\cdot \cdots$. Thus, for each component m of M, there is an integer $J(m)$ and a sequence $J(m)\alpha$ such that $f(m)\subset$ Interior $(f(A_{J(m)\alpha}))$ and $f((A_{J(m)\alpha}))$ intersects at most one of P_1+P_2 and P_3+P_4. Since $f(M)$ is compact and $\{$Interior $(f(A_{J(m)\alpha})) : m \in M\}$ is an open cover of $f(M)$, there is an integer R such that for each sequence $R\alpha$, $f(A_{R\alpha})$ intersects at most one of P_1+P_2 and P_3+P_4. Since for each sequence $R\alpha$, $f(a_{R\alpha,i})\subset f(\Gamma_{R\alpha})\subset f(A_{R\alpha})$, the proof is completed by letting $f=h$.

Theorem 2 allows the following:

DEFINITION. If E^3/G is topologically E^3, the first shrinking number $L(1)$ of E^3/G is the least integer such that there is a homeomorphism g of E^3 onto E^3 which satisfies

(i) g is isotopic to the identity by an isotopy which is fixed on the complement of Interior (A_0),

(ii) each $g(a_{L(1)\alpha,i})$ in each $g(\Gamma_{L(1)\alpha})$ intersects at most one of P_1+P_2 and P_3+P_4.

Let H be the restriction to Boundary $(A_0)+\sum P_i$, $i=1,\dots,4$, of the inverse of the homeomorphism h of Theorem 2. Then, we have:

THEOREM 3. *If E^3/G is topologically E^3, there is a homeomorphism H of* Boundary $(A_0)+\sum P_i$, $i=1,\dots,4$, *into A_0 such that*

(i) *H is the identity on* Boundary (A_0),

(ii) *for some integer R, each $a_{R\alpha,i}$ in each $\Gamma_{R\alpha}$ intersects at most one of $H(P_1+P_2)$ and $H(P_3+P_4)$.*

Theorem 3 allows the following:

DEFINITION. If E^3/G is topologically E^3, the second shrinking number $L(2)$ of E^3/G is the least integer such that there is a homeomorphism F of Boundary (A_0) $+\sum P_i$, $i=1,\dots,4$, into A_0 which satisfies

(i) F is the identity on Boundary (A_0),

(ii) each $a_{L(2)\alpha,i}$ in each $\Gamma_{L(2)\alpha}$ intersects at most one of $F(P_1+P_2)$ and $F(P_3+P_4)$.

If E^3/G is topologically E^3, the restriction to Boundary $(A_0)+\sum P_i$, $i=1,\dots,4$, of the inverse of a homeomorphism g of the definition of the first shrinking number $L(1)$ satisfies the requirements of the definition of the second shrinking number $L(2)$. Hence, $L(1)\geqq L(2)$. A homeomorphism F of the definition of the second shrinking number $L(2)$ may be extended to a homeomorphism, also denoted by F, of E^3 onto E^3 by defining F to be the identity on the complement of Interior (A_0) and by extending F from the boundary of each cube X_i, $i=1, 2, 3$, to X_i onto the closure of the bounded complementary domain of F(Boundary (X_i)), an extension justified by Alexander [1]. Since F^{-1} is a homeomorphism of E^3 onto E^3 and is the identity on the complement of Interior (A_0), F^{-1} is isotopic to the identity by an isotopy which is fixed on the complement of Interior (A_0) and each $F^{-1}(a_{L(2)\alpha,i})$ in each $F^{-1}(\Gamma_{L(2)\alpha})$ intersects at most one of P_1+P_2 and P_3+P_4. Thus, $L(2)\geqq L(1)$ and we have:

THEOREM 4. *If E^3/G is topologically E^3, $L(1)=L(2)$.*

Thus, we may speak of the shrinking number L of E^3/G if E^3/G is topologically E^3 and state the following:

CONJECTURE. *If E^3/G is topologically E^3, $L=0$.*

4. A new dogbone space that is not topologically E^3.

The principal result of this section is:

THEOREM 5. *The dogbone space represented by Figure* 2e *is not topologically E^3.*

Proof. Before proving Theorem 5, we prove two lemmas, both concerned with the decomposition space represented by Figure 2e. The notation for the statement of both lemmas will be that of Figure 4.

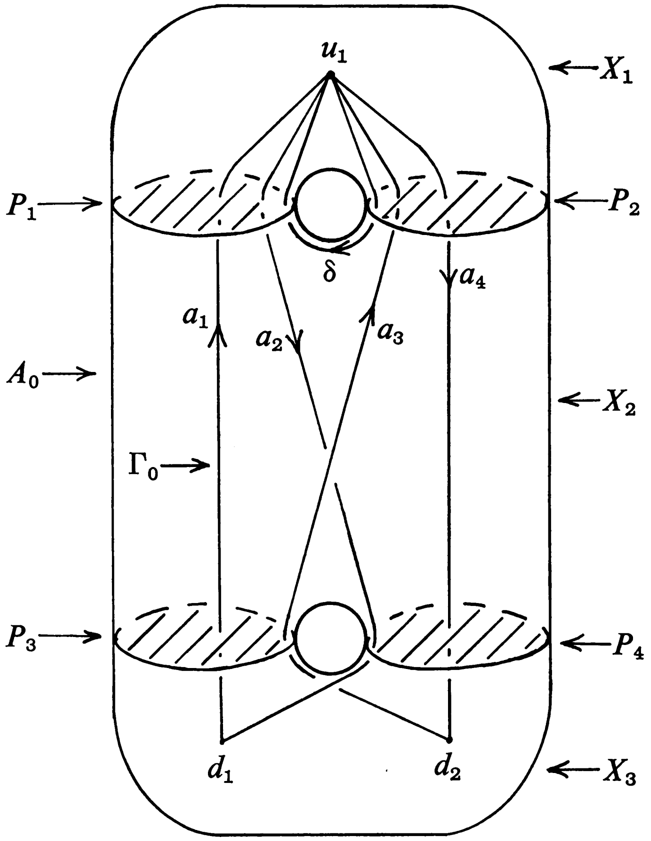

FIGURE 4

LEMMA 1. *Suppose g is a continuous function of A_0 into A_0 which is homotopic to the identity by a homotopy G which is fixed on* Boundary (A_0). *Then, for some i, $i=1,\dots,4$, $g(a_i)$ intersects both P_1+P_2 and P_3+P_4.*

Proof. As in Figure 4, let u_1 be a base-point for the fundamental group $\pi_1(A_0, u_1)$ and assign positive directions to the arcs a_i, $i=1,\dots,4$. Construct the simple closed curve δ with base-point u_1 and positive direction as shown. If r is a closed path in A_0 with basepoint u_1, $[r]$ will denote the element in $\pi_1(A_0, u_1)$ determined by r.

Suppose g is a continuous function satisfying the hypothesis of the lemma. Without loss of generality, it may be assumed that $g(u_1)=g(d_1)=g(d_2)$ is contained in X_2. For, if $g(u_1)$ is contained in X_1, $g(u_1)$ can be moved to X_2 by a homotopy which is fixed on some open set containing X_3+Boundary (A_0); thus, if for some $i=1,\ldots,4$, $g(a_i)\cdot(P_3+P_4)=\varnothing$, then the image of a_i under the homotopy will also not intersect P_3+P_4. A similar argument applies if $g(u_1)$ is contained in X_3 and also applies to $g(d_1)$ and $g(d_2)$. Thus, as in Figure 5, let $g(u_1)$ be a base-point for x and y, with positive directions as shown, generating elements of $\pi_1(A_0, g(u_1))$. If s is a closed path in A_0 with base-point $g(u_1)$, $\{s\}$ will denote the element in $\pi_1(A_0, g(u_1))$ determined by s. Construct the arc σ from $g(u_1)$ to u_1. Note that since $g(u_1)=g(d_1)=g(d_2)$, for each $i=1,\ldots,4$, $g(a_i)$ is a closed path with base-point $g(u_1)$.

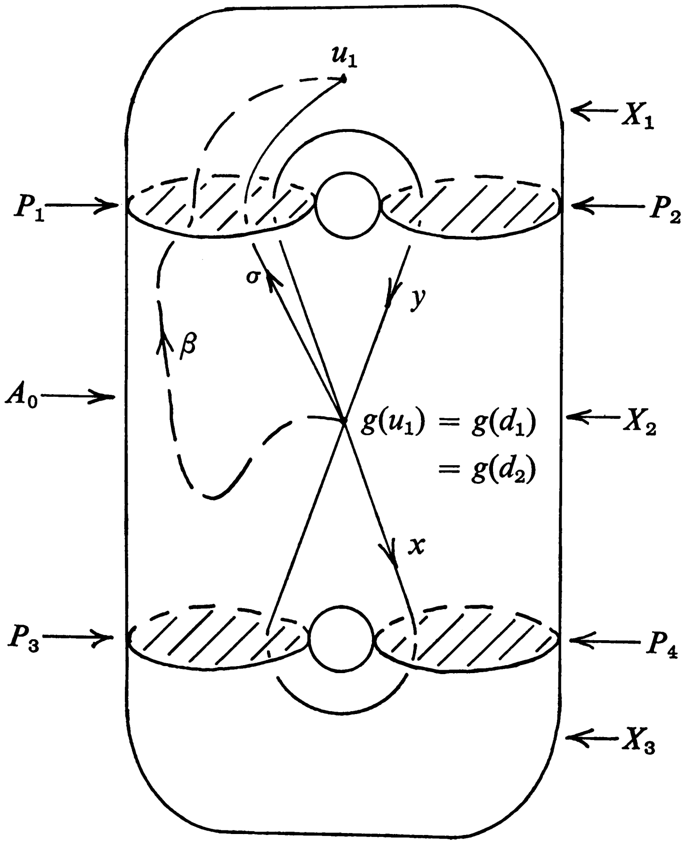

FIGURE 5

Critical to the proof is the observation that, by construction, for some $i=1,\ldots,4$, $g(a_i)$ intersects X_1. That is, $\{g(a_1)g(a_2)g(a_3)g(a_4)\}$ cannot be expressed in terms of x, a generating element of $\pi_1(A_0, g(u_1))$, alone.

The homotopy G determines an isomorphism between $\pi_1(A_0, u_1)$ and $\pi_1(A_0, g(u_1))$ which can be expressed by $\beta\pi_1(A_0, u_1)\beta^{-1}=\pi_1(A_0, g(u_1))$ for some arc β, shown schematically in Figure 5, from $g(u_1)$ to u_1.

Suppose the lemma to be false; that is, suppose g satisfies the hypotheses of the lemma but for each $i=1, \ldots, 4$, $g(a_i)$ intersects at most one of P_1+P_2 and P_3+P_4. Then, for each $i=1, \ldots, 4$, $g(a_i)$ intersects at most one of X_1 and X_3.

Since $g(a_2+a_1)\cdot X_3 \neq \varnothing$ and $g(a_4+a_3)\cdot X_3 \neq \varnothing$, there are four possible cases:

Case I. $g(a_2)\cdot X_3 \neq \varnothing$ and $g(a_4)\cdot X_3 \neq \varnothing$.

Case II. $g(a_2)\cdot X_3 \neq \varnothing$ and $g(a_3)\cdot X_3 \neq \varnothing$.

Case III. $g(a_1)\cdot X_3 \neq \varnothing$ and $g(a_4)\cdot X_3 = \varnothing$.

Case IV. $g(a_1)\cdot X_3 \neq \varnothing$ and $g(a_3)\cdot X_3 \neq \varnothing$.

We prove that Case I is impossible and note that the proof of the impossibility of the other cases is similar.

Suppose Case I holds, that is, $g(a_2)\cdot X_3 \neq \varnothing$ and $g(a_4)\cdot X_3 \neq \varnothing$. Then, there are integers c and m such that $\{g(a_2)\}=\{x^c\}$ and $\{g(a_4)\}=\{x^m\}$ since neither $g(a_2)$ or $g(a_4)$ intersect X_1. Then,

$$\{\beta a_2 a_1 a_4 a_3 \beta^{-1}\} = \{g(a_2)g(a_1)g(a_4)g(a_3)\} = \{x^c g(a_1) x^m g(a_3)\}.$$

By construction, for some $i=1, \ldots, 4$, $g(a_i)\cdot X_1 \neq \varnothing$. Therefore, either $\{g(a_1)\} =\{y^t\}$ or $\{g(a_3)\}=\{y^q\}$. If $\{g(a_1)\}=\{y^t\}$, then $\{g(a_2a_1)\}=\{x^c y^t\}$ and $[a_2a_1]=[\beta^{-1}x^c y^t \beta]$. By construction, $\{\sigma a_2 a_1 \sigma^{-1}\}=\{x\}$. Therefore, $\{\sigma\beta^{-1}x^c y^t \beta\sigma^{-1}\}=\{x\}$. On the left side of this expression, the sum of the exponents of x is c and the sum of the exponents of y is t. Equating these sums to those on the right, we have $c=1$ and $t=0$. Thus, $\{g(a_1)\}=\{1\}$ and we must have $\{g(a_3)\}=\{y^q\}$. Then, using $\{\sigma a_4 a_3 \sigma^{-1}\}= \{yxy^{-1}\}$, we have

$$\begin{aligned}\{x^m y^q\} &= \{g(a_4a_3)\} = \{\beta a_4 a_3 \beta^{-1}\} \\ &= \{\beta\sigma^{-1}\sigma a_4 a_3 \sigma^{-1}\sigma\beta^{-1}\} = \{\beta\sigma^{-1} yxy^{-1}\sigma\beta^{-1}\}.\end{aligned}$$

Again comparing sums of exponents of x and y on opposite sides of this equation, we have $m=1$ and $q=0$. Thus $\{g(a_4a_3)\}=\{x\}$.

Finally, $\{g(a_2a_1a_4a_3)\}=\{g(a_2a_1)g(a_4a_3)\}=\{x\}$. But, by construction $\{g(a_2a_1a_4a_3)\}$ cannot be expressed in terms of x alone. This contradiction shows that Case I is impossible, thus completing the proof of Lemma 1.

LEMMA 2. *Suppose E is a positive integer and F is a homeomorphism of* Boundary $(A_0)+\sum P_i$, $i=1, \ldots, 4$, *into A_0 which satisfies*

(i) *F is the identity on* Boundary (A_0),

(ii) *each $a_{E\alpha,i}$ in each $\Gamma_{E\alpha}$ intersects at most one of $F(P_1+P_2)$ and $F(P_3+P_4)$.*

Then, there is a homeomorphism h of Boundary $(A_0)+\sum P_i$, $i=1, \ldots, 4$, *into A_0 which satisfies*

(i) *h is the identity on* Boundary (A_0),

(ii) *each $a_{(E-1)\alpha,i}$ in each $\Gamma_{(E-1)\alpha}$ intersects at most one of $h(P_1+P_2)$ and $h(P_3+P_4)$.*

Proof. Suppose E is a positive integer and F a homeomorphism which satisfy the hypotheses of the lemma. Let $(E-1)\alpha$ be a fixed sequence. The solid double

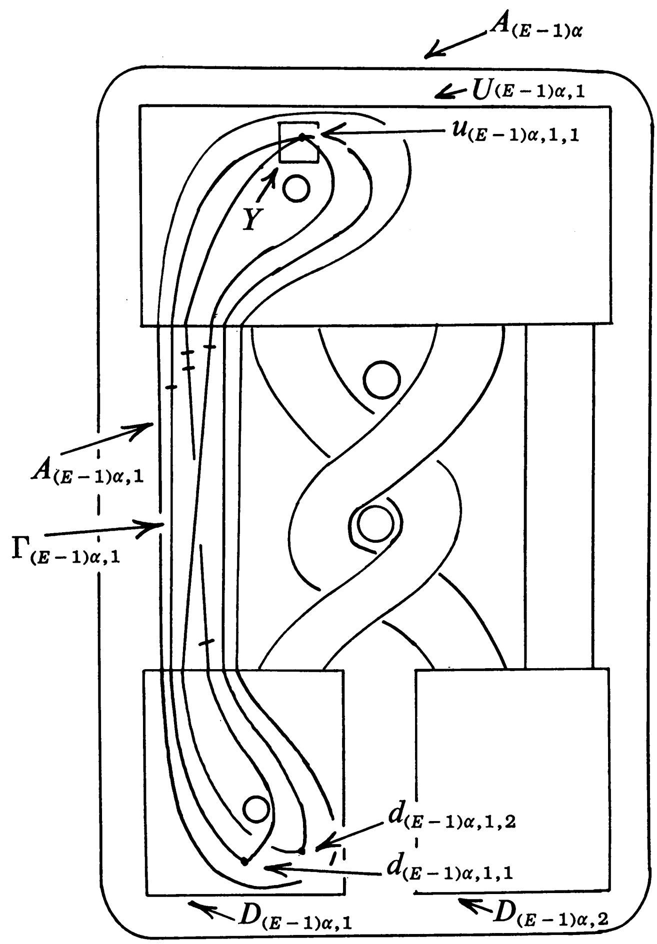

FIGURE 6

torus $A_{(E-1)\alpha}$ is shown in Figure 6. For clarity, only the details of $A_{(E-1)\alpha,1}$ and $\Gamma_{(E-1)\alpha,i}$ are shown and possible intersections of $F(P_1+P_2+P_3+P_4)$ with $a_{(E-1)\alpha,1,i}$, $i=1,\ldots,4$, are indicated. It may be assumed that $F(P_1+P_2+P_3+P_4)$ does not intersect $u_{(E-1)\alpha,1,1}+\sum d_{(E-1)\alpha,1,j}$ since $F(P_1+P_2+P_3+P_4)$ could be adjusted in a neighborhood of, say, $u_{(E-1)\alpha,1,1}$ without adding intersections to any arc $a_{(E-1)\alpha,1,i}$. Thus, a cube Y may be constructed in $A_{(E-1)\alpha,1}$ such that Y contains $u_{(E-1)\alpha,1,1}$, $Y\cdot\Gamma_{(E-1)\alpha,1}$ is a 4-od and Y does not intersect $F(P_1+P_2+P_3+P_4)$. Replace $Y\cdot\Gamma_{(E-1)\alpha,1}$ by two 3-ods with a single common end-point, expand Y by a homeomorphism h_1 of E^3 onto E^3 which is the identity on the complement of Interior $(A_{(E-1)\alpha,1})$ and arrive at the situation of Figure 7.

If a cube Y' similar to Y is constructed in $A_{(E-1)\alpha,2}$, $Y'\cdot\Gamma_{(E-1)\alpha,2}$ is replaced by two 3-ods and Y' is expanded by a homeomorphism h_2 of E^3 onto E^3 which is

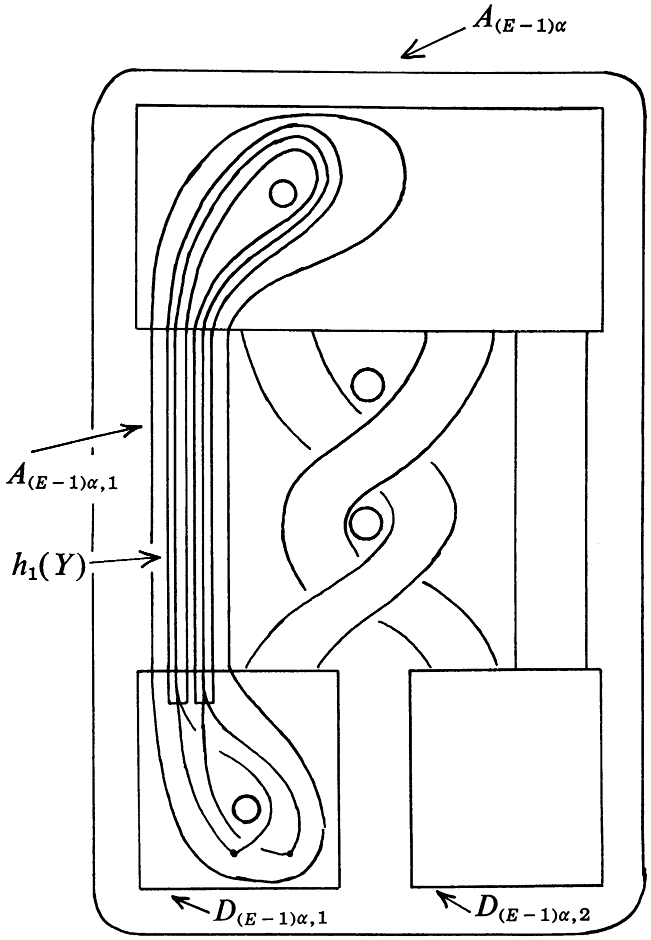

FIGURE 7

the identity on the complement of Interior $(A_{(E-1)\alpha,2})$, there results four simple closed curves which link in $D_{(E-1)\alpha,1}$ as in Figure 8. For $i=1, 2$ each pair of simple closed curves in $A_{(E-1)\alpha,i}$ is connected by an arc in $A_{(E-1)\alpha,i}$ which does not intersect $h_2h_1F(P_1+P_2+P_3+P_4)$. Further, each simple closed curve is the union of two arcs which intersect only at their end-points and each arc intersects at most one of $h_2h_1F(P_1+P_2)$ and $h_2h_1F(P_3+P_4)$. By Theorems 3 and 5 of [7], there is a component V of Interior $(D_{(E-1)\alpha,1})-h_2h_1F(P_1+P_2+P_3+P_4)$ which intersects each simple closed curve. In V, select a point and construct arcs in V from this point to each of the curves. Each arc may be extended along the associated curve and then along the connecting arc to Boundary $(D_{(E-1)\alpha,1})$ so as to intersect at most one of $h_2h_1F(P_1+P_2)$ and $h_2h_1F(P_3+P_4)$. The result is a 4-od k_1 contained in $D_{(E-1)\alpha,1}$ as in Figure 9. The 4-od k_1 is contained in a Figure 8, Φ_1, which may be regarded

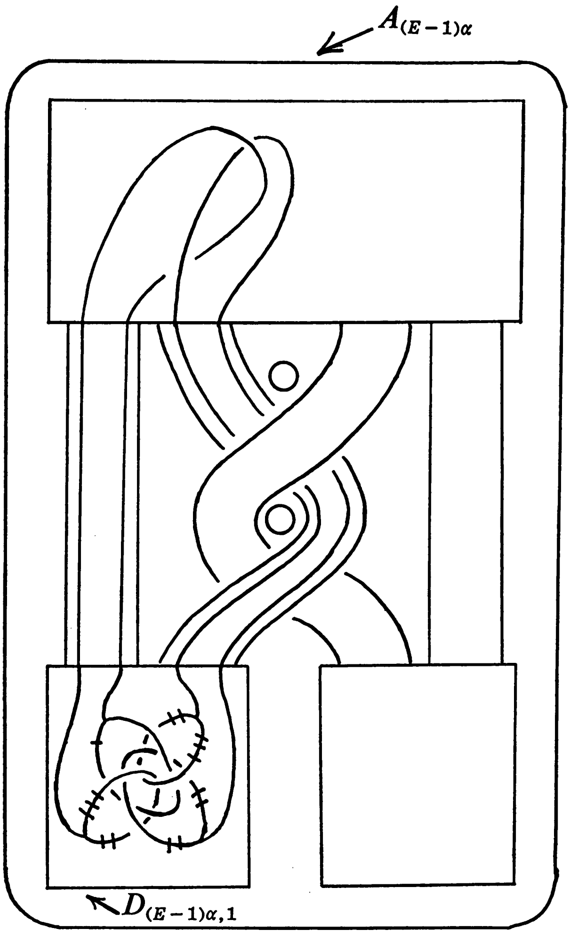

FIGURE 8

as the union of four arcs with at most end-points in common and each arc intersects at most one of $h_2h_1F(P_1+P_2)$ and $h_2h_1F(P_3+P_4)$. By a homeomorphism h_3 of E^3 onto E^3 which is the identity on the complement of a small neighborhood W_1 of $D_{(E-1)\alpha,1}$, each of $h_2h_1F(P_1+P_2)$ and $h_2h_1F(P_3+P_4)$ may be pushed along the arcs of Φ_1 they intersect to the complement of $D_{(E-1)\alpha,1}$ so that each of the four arcs intersect at most one of $h_3h_2h_1F(P_1+P_2)$ and $h_3h_2h_1F(P_3+P_4)$.

The 4-od k_1 is contained in the cube $D_{(E-1)\alpha,1}$ and has end-points only on Boundary $(D_{(E-1)\alpha,1})$. Let W_2 be a neighborhood of $D_{(E-1)\alpha,1}$ contained in W_1. The cutting and sewing process of [3] may be applied which results in a homeomorphism h_4 of $\sum P_i$, $i=1,\ldots,4$, into A_0 such that $h_4(\sum P_i)\cdot D_{(E-1)\alpha,1}=\varnothing$ and for each i, h_4 is the identity on Boundary (P_i), h_4 (Interior (P_i))$\subset$Interior (A_0), $h_4(P_i)-W_2\subset h_3h_2h_1F(P_i)$ and each arc in Φ_1 intersects at most one of $h_4(P_1+P_2)$ and

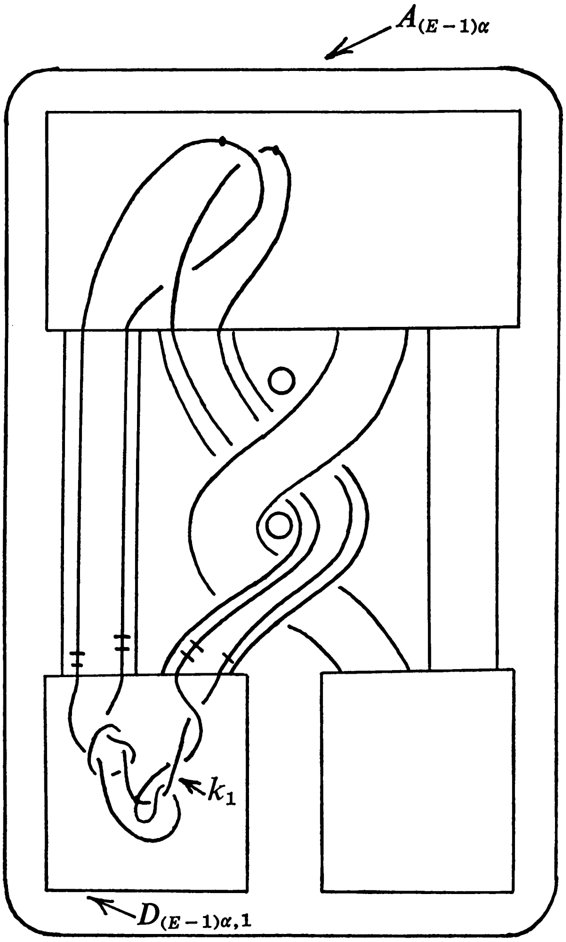

FIGURE 9

$h_4(P_3+P_4)$. An important point is that for each sequence $(E-1)\beta$, $j\neq(E-1)\alpha$, 1 or $(E-1)\alpha$, 2, $h_4(P_i)$, $i=1,\ldots,4$, intersects an arc $a_{(E-1)\beta,j,c}$ in $\Gamma_{(E-1)\beta,j}$ only if $F(P_i)$ intersects $a_{(E-1)\beta,j,c}$ since $h_4(P_i)-W_2\subset h_3h_2h_1F(P_i)$ and $h_3h_2h_1$ is the identity on the complement of $A_{(E-1)\alpha,1}+A_{(E-1)\alpha,2}$. Extend h_4 to a homeomorphism of Boundary $(A_0)+\sum P_i$, $i=1,\ldots,4$, by defining h_4 as the identity on Boundary (A_0).

Let h_5 be a homeomorphism of E^3 onto E^3 which is the identity on the complement of Interior $(A_{(E-1)\alpha,1}+A_{(E-1)\alpha,2}+D_{(E-1)\alpha,1})$ and, as shown in Figure 10, expands Interior $(D_{(E-1)\alpha,1})$ so that h_5 (Interior $(D_{(E-1)\alpha,1})$) contains $(a_{(E-1)\alpha,1}+a_{(E-1)\alpha,2})-U_{(E-1)\alpha,1}$. The closure of $h_5(\Phi_1-D_{(E-1)\alpha,1})$ consists of four arcs which intersect in pairs at one end-point. Each arc intersects at most one of $h_5h_4(P_1+P_2)$ and $h_5h_4(P_3+P_4)$ as indicated. Extend each pair of intersecting arcs

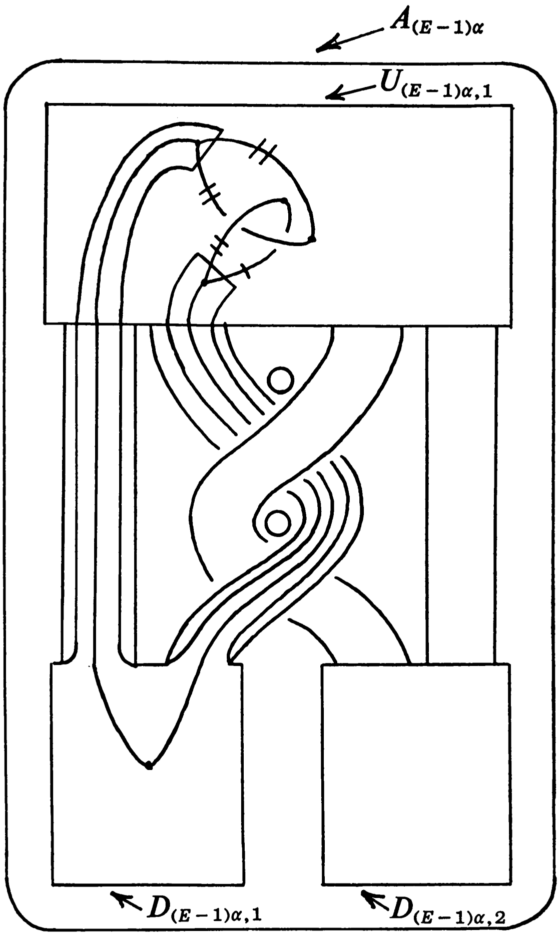

FIGURE 10

to a point in the interior of the component of $h_5(D_{(E-1)\alpha,1})\cdot U_{(E-1)\alpha,1}$ which they intersect and from this point construct an arc in $h_5(D_{(E-1)\alpha,1})\cdot U_{(E-1)\alpha,1}$ to $\Gamma_{(E-1)\alpha}\cdot \mathrm{Bd}\,(U_{(E-1)\alpha,1})$. Thus, a finite graph consisting of two simple closed curves joined by a connecting arc has been constructed in $A_{(E-1)\alpha,1}+A_{(E-1)\alpha,2}+D_{(E-1)\alpha,1}$. Each simple closed curve consists of two arcs which intersect only at their end-points and each arc intersects at most one of $h_5h_4(P_1+P_2)$ and $h_5h_4(P_3+P_4)$. The simple closed curves are linked and each links $A_{(E-1)\alpha,3}$ and $A_{(E-1)\alpha,4}$ in Interior $(U_{(E-1)\alpha,1})$. That part of the finite graph in the complement of $U_{(E-1)\alpha,1}$, which is also that part of the connecting arc in the complement of $U_{(E-1)\alpha,1}$, is $(a_{(E-1)\alpha,1}+a_{(E-1)\alpha,2})-U_{(E-1)\alpha,1}$. The connecting arc does not intersect $h_5h_4(P_1+P_2+P_3+P_4)$. The restriction of h_5h_4 to Boundary $(A_0)+\sum P_i$, $i=1,\ldots,4$, is a homeomorphism into A_0 which is the identity on Boundary (A_0). If $(E-1)\beta, j\neq(E-1)\alpha$, 1 or $(E-1)\alpha$, 2,

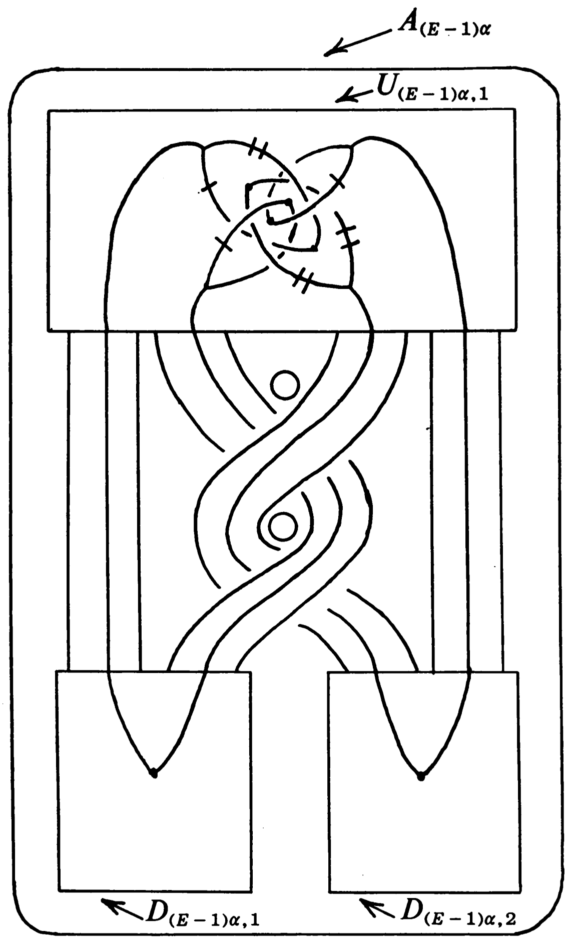

FIGURE 11

$h_5h_4(P_i)$ intersects an arc $a_{(E-1)\beta,j,c}$ only if $F(P_i)$ intersects $a_{(E-1)\beta,j,c}$ since h_5 is the identity on the complement of Interior $(A_{(E-1)\alpha,1}+A_{(E-1)\alpha,2}+D_{(E-1)\alpha,1})$.

Thus far, the definition of homeomorphisms and construction has been done relative to $A_{(E-1)\alpha,1}$, $A_{(E-1)\alpha,2}$ and $D_{(E-1)\alpha,1}$. A similar definition of homeomorphisms and construction is to be done relative to $A_{(E-1)\alpha,3}$, $A_{(E-1)\alpha,4}$ and $D_{(E-1)\alpha,2}$ resulting, as shown in Figure 11, in a homeomorphism h_6 of Boundary $(A_0)+\sum P_i$, $i=1,\dots,4$, into A_0 which is the identity on Boundary (A_0), and a finite graph in $A_{(E-1)\alpha,3}+A_{(E-1)\alpha,4}+D_{(E-1)\alpha,2}$. In the complement of $A_{(E-1)\alpha,3}+A_{(E-1)\alpha,4}+D_{(E-1)\alpha,2}$, for each i, $h_6(P_i)$ is contained in $h_5h_4(P_i)$. Thus, for $(E-1)\beta$, $j\neq (E-1)\alpha$, n, $n=1,\dots,4$, each arc $a_{(E-1)\beta,j,c}$ intersects $h_6(P_i)$ only if $F(P_i)$ intersects $a_{(E-1)\beta,j,c}$. The finite graph in $A_{(E-1)\alpha,3}+A_{(E-1)\alpha,4}+D_{(E-1)\alpha,2}$, like the finite graph in $A_{(E-1)\alpha,1}+A_{(E-1)\alpha,2}+D_{(E-1)\alpha,1}$, consists of two simple closed curves connected by an arc. Each simple closed curve consists of two arcs which intersect only at their end-points and each arc intersects at most one of $h_6(P_1+P_2)$ and $h_6(P_3+P_4)$. The connecting arc does not intersect $h_6(\sum P_i)$, $i=1,\dots,4$. All four simple closed curves in both graphs are linked in Interior $(U_{(E-1)\alpha,1})$. The sum of the finite graphs in the complement of $U_{(E-1)\alpha,1}$ which is also the sum of the connecting arcs in the complement of $U_{(E-1)\alpha,1}$, is $\Gamma_{(E-1)\alpha}-U_{(E-1)\alpha,1}$.

Since the four simple closed curves link in Interior $(U_{(E-1)\alpha,1})$, by Theorems 3 and 5 of [7], there is a component V' of Interior $(U_{(E-1)\alpha,1})-h_6(\sum P_i)$, $i=1,\dots,4$, which intersects each simple closed curve. In V', select a point and construct arcs in V' from this point to each of the simple closed curves. Each arc may be extended along the associated curve and then along the connecting arc to Boundary $(U_{(E-1)\alpha,1})$ so as to intersect at most one of $h_6(P_1+P_2)$ and $h_6(P_3+P_4)$. The sum of the four arcs is a 4-od k_2, similar to the 4-od k_1 of Figure 9, contained in $U_{(E-1)\alpha,1}$ with end-points only on Boundary $(U_{(E-1)\alpha,1})$ and $k_2\cdot$Boundary $(U_{(E-1)\alpha,1})$ $=\Gamma_{(E-1)\alpha}\cdot$Boundary $(U_{(E-1)\alpha,1})$. The sum of k_2 and that part of $\Gamma_{(E-1)\alpha}$ in the complement of $U_{(E-1)\alpha,1}$ is a figure 8, Φ_2, consisting of four arcs with at most end-points in common and each arc intersects at most one of $h_6(P_1+P_2)$ and $h_6(P_3+P_4)$.

By a homeomorphism h_7 of E^3 onto E^3 which is the identity on the complement of a small neighborhood W_3 of $U_{(E-1)\alpha,1}$, each of $h_6(P_1+P_2)$ and $h_6(P_3+P_4)$ may be pushed along the arcs of Φ_2 they intersect to the complement of $U_{(E-1)\alpha,1}$ so that each of the four arcs in Φ_2 intersect at most one of $h_7(P_1+P_2)$ and $h_7(P_3+P_4)$. Let W_4 be a neighborhood of $U_{(E-1)\alpha,1}$ contained in W_3. The cutting and sewing process of [3] may be applied which results in a homeomorphism h_8 of $\sum P_i$, $i=1,\dots,4$, into A_0 such that $h_8(\sum P_i)\cdot U_{(E-1)\alpha,1}=\varnothing$ and for each i, h_8 is the identity on Boundary (P_i), h_8 (Interior $(P_i))\subset$Interior (A_0), $h_8(P_i)-W_4\subset h_7h_6(P_i)$, and each arc in Φ_2 intersects at most one of $h_8(P_1+P_2)$ and $h_8(P_3+P_4)$. Since $h_8(\sum P_i)$, $i=1,\dots,4$, does not intersect $U_{(E-1)\alpha,1}$, each arc $a_{(E-1)\alpha,j}$ in $\Gamma_{(E-1)\alpha}$ intersects at most one of $h_8(P_1+P_2)$ and $h_8(P_3+P_4)$, as shown in Figure 12. An important point is that for each sequence $(E-1)\beta\neq(E-1)\alpha$, $h_8(P_i)$, $i=1,\dots,4$, intersects an arc $a_{(E-1)\beta,j,c}$ in $\Gamma_{(E-1)\beta,j}$ only if $F(P_i)$ intersects $a_{(E-1)\beta,j,c}$ since $h_8(P_i)-W_4\subset h_7h_6(P_i)$ and h_7 is the identity on the complement of $A_{(E-1)\alpha}$. Extend h_8 to a homeomorphism of Boundary $(A_0)+\sum P_i$, $i=1,\dots,4$, by defining h_8 as the identity on Boundary (A_0).

Thus far, the construction and definition of homeomorphisms has resulted in a homeomorphism h_8 of Boundary $(A_0)+\sum P_i$, $i=1,\dots,4$, into A_0 which is the identity on Boundary (A_0), each arc $a_{(E-1)\alpha,j}$ in $\Gamma_{(E-1)\alpha}$ intersects at most one of $h_8(P_1+P_2)$ and $h_8(P_3+P_4)$ and for each $i=1,\dots,4$, $h_8(P_i)-A_{(E-1)\alpha}\subset F(P_i)$. Let $(E-1)\delta\neq(E-1)\alpha$ be a fixed sequence. Then, do a similar construction and definition of homeomorphisms relative to $A_{(E-1)\delta}$ as has been done for $A_{(E-1)\alpha}$. The result is homeomorphism h_9 of Boundary $(A_0)+\sum P_i$, $i=1,\dots,4$, into A_0 which is the identity on Boundary (A_0), each arc $a_{(E-1)\delta,j}$ in $\Gamma_{(E-1)\delta}$ intersects at most one

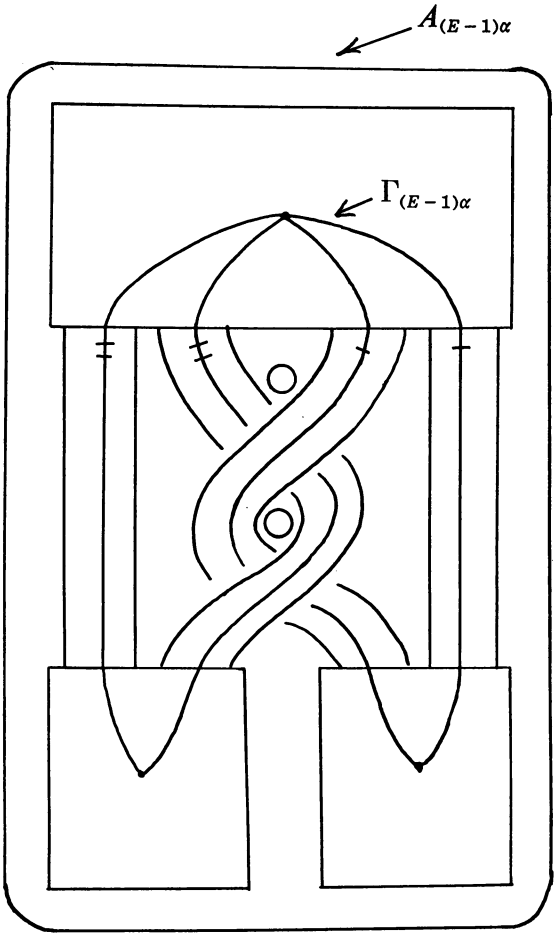

FIGURE 12

of $h_9(P_1+P_2)$ and $h_9(P_3+P_4)$ and for each $i=1,\ldots,4$,

$$h_9(P_i)-A_{(E-1)\delta}\subset h_8(P_i)$$

and

$$h_9(P_i)-(A_{(E-1)\alpha}+A_{(E-1)\delta})\subset h_8(P_i)-A_{(E-1)\alpha}\subset F(P_i).$$

An important point is that for each sequence $(E-1)\beta\neq(E-1)\alpha$, $(E-1)\delta$, $h_9(P_i)$, $i=1,\ldots,4$, intersects an arc $a_{(E-1)\beta,j,c}$ in $\Gamma_{(E-1)\beta,j}$ only if $F(P_i)$ intersects $a_{(E-1)\beta,j,c}$ since $h_9(P_i)-(A_{(E-1)\alpha}+A_{(E-1)\beta})\subset F(P_i)$. Another important point is that for each $i=1,\ldots,4$, $h_9(P_i)$ intersects an arc $a_{(E-1)\alpha,j}$ in $\Gamma_{(E-1)\alpha}$ only if $h_8(P_i)$ intersects $a_{(E-1)\alpha,j}$ since $h_9(P_i)-A_{(E-1)\delta}\subset h_8(P_i)$.

A continuation of the construction and definition of homeomorphisms through the remaining finite set of all solid double tori whose subscripts are sequences with

$E-1$ terms results in the desired homeomorphism h of Boundary $(A_0)+\sum P_i$, $i=1,\ldots,4$, into A_0 such that h is the identity on Boundary (A_0) and each arc $a_{(E-1)r,j}$ in each $\Gamma_{(E-1)r}$ intersects at most one of $h(P_1+P_2)$ and $h(P_3+P_4)$. Thus, the proof of Lemma 2 is completed.

We now complete the proof of Theorem 5 by contradiction.

Suppose the dogbone space E^3/G represented by Figure 2e is topologically E^3 and L is the shrinking number. Then, by the definition of the first shrinking number and Lemma 1, L is not zero since there is no homeomorphism g of E^3 onto E^3 which is isotopic to the identity by an isotopy which is fixed on the complement of Interior (A_0) and such that for each $i=1,\ldots,4$, $g(a_i)$ intersects at most one of P_1+P_2 and P_3+P_4. By definition of the second shrinking number and Lemma 2, L cannot be greater than zero since if L is greater than zero, there is a homeomorphism h of Boundary $(A_0)+\sum P_i$, $i=1,\ldots,4$, into A_0 which is the identity on Boundary (A_0) and such that each $a_{(L-1)\alpha,i}$ in each $\Gamma_{(L-1)\alpha}$ intersects at most one of $h(P_1+P_2)$ and $h(P_3+P_4)$. The contradiction that L is not zero nor greater than zero completes the proof of Theorem 5.

A slight modification of the proof of Theorem 5 gives:

THEOREM 6. *Bing's dogbone space, represented by Figure* 2b, *is not topologically* E^3.

5. **Remarks on other dogbone spaces.** Figure 2e and Figure 2f are so similar that a technique similar to that of Theorem 5 might be applied to the dogbone space represented by Figure 2f. However, such is not the case. Suppose the dogbone space represented by Figure 2f is topologically E^3 and Q is the shrinking number. The proof that $Q\neq 0$ follows that of Lemma 1 for Theorem 5. Thus, we assume $Q\geqq 1$ and attempt a construction similar to that done in the proof of Lemma 2 for Theorem 5. Since $Q\geqq 1$, there is a homeomorphism h_{10} of Boundary (A_0) $+\sum P_i$, $i=1,\ldots,4$, into A_0 such that h_{10} is the identity on Boundary (A_0) and each $a_{Q\alpha,j}$ in each $\Gamma_{Q\alpha}$ intersects at most one of $h_{10}(P_1+P_2)$ and $h_{10}(P_3+P_4)$. Let $(Q-1)\alpha$ be a fixed sequence. In $A_{(Q-1)\alpha,1}$, construct a cube Y about $u_{(Q-1)\alpha,1,1}$, as was done in Figure 6 for $A_{(L-1)\alpha,1}$ and $u_{(L-1)\alpha,1,1}$, and expand Y by a homeomorphism h_{11} of E^3 onto E^3 which is the identity on the complement of Interior $(A_{(Q-1)\alpha,1})$. A similar construction and expansion in $A_{(Q-1)\alpha,2}$ by a homeomorphism h_{12} of E^3 onto E^3 which is the identity on the complement of Interior $(A_{(Q-1)\alpha,2})$ results in the situation shown in Figure 13. Since the four simple closed curves are not mutually linked in $D_{(Q-1)\alpha,1}$, there is no assurance that there is a component V'' of Interior $(D_{(Q-1)\alpha,1})-h_{12}h_{11}h_{10}(\sum P_i)$, $i=1,\ldots,4$, which intersects each simple closed curve. Thus, the proof cannot be continued along the line of the proof of Theorem 5.

Bing's proof of [7] which applies to the dogbone space represented by Figure 2b cannot be easily modified to apply to the dogbone spaces represented by Figure 2e, Figure 2f and Figure 2g. We demonstrate this using Figure 2g and paraphrase Bing's definitions as follows:

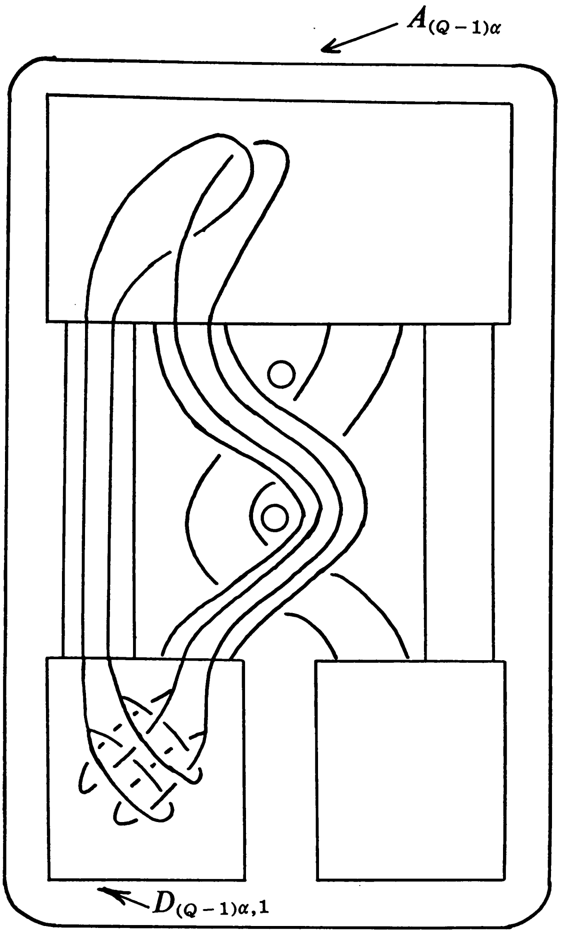

FIGURE 13

PROPERTY *P*. A topological figure 8 has Property P if it contains two points p and q in opposite loops such that any arc from p to q in it intersects both P_1+P_2 and P_3+P_4.

PROPERTY *Q*. An image of A_0 under a homeomorphism h has Property Q if each figure 8 in it homotopic to its center has Property P.

Using these definitions, Bing shows

THEOREM 10 OF [1]. *If a continuum B in A_0 is the image of A_0 under a homeomorphism h of E^3 onto itself and B has Property Q, then one of $h(A_1)$, $h(A_2)$, $h(A_3)$, $h(A_4)$ has Property Q.*

A similar theorem for the dogbone space represented by Figure 2g is false, as shown in Figure 14. In Figure 14, h_{13} is a homeomorphism of E^3 onto itself. The intersections of $h_{13}(A_0)$ with P_1 and P_3 are as shown and $h_{13}(A_0)$ does not

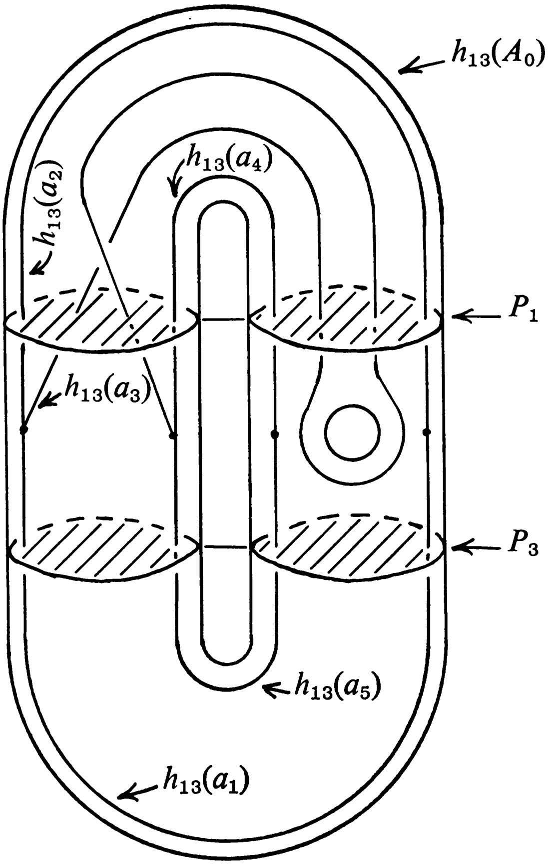

FIGURE 14

intersect either P_2 or P_4. For each $i=1, \ldots, 5$, the distance from $h_{13}(a_i)$ to the complement of $h_{13}(A_i)$ is so small that $h_{13}(A_i)$ intersects P_1 or P_3 only if $h_{13}(a_i)$ also intersects P_1 or P_3. Any figure 8 in $h_{13}(A_0)$ homotopic to the center of $h_{13}(A_0)$ has Property P since the figure 8 must contain a point p in one loop above P_1 and a point q in the other loop below P_3 and any arc in the figure 8 from p to q intersects both P_1 and P_3. Thus, $h_{13}(A_0)$ has Property Q. But, for $i=1, \ldots, 5$, $h_{13}(A_i)$ does not have Property Q since $h_{13}(A_i)$ intersects at most one of P_1 and P_3 and neither of P_2 or P_4 and the demonstration for the dogbone space represented by Figure 2g is complete. A like demonstration can be made for the dogbone spaces represented by Figure 2e and Figure 2f.

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