THE STRICT TOPOLOGY FOR DOUBLE CENTRALIZER ALGEBRAS

BY
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Abstract. Sufficient conditions are given for a double centralizer algebra under the strict topology to be a Mackey space.

0. Introduction. Let $C(S)$ be the $B^*$-algebra of all bounded complex valued continuous functions on a locally compact Hausdorff space $S$; let $C_0(S)$ be the algebra of all functions in $C(S)$ that vanish at infinity, and let $C(S)_\beta$ denote $C(S)$ under the $\beta$ or strict topology. In 1958, R. C. Buck [3] proved that the strict dual of $C(S)$ under the strong topology is isometrically isomorphic to the norm dual of $C_0(S)$ and then raised the following question: Is it in fact true that the strict topology $\beta$ coincides with the Mackey topology? In 1967, J. B. Conway [6] answered this question for the most part. He showed that if $S$ is paracompact, then indeed the strict topology is the Mackey topology and he also gave examples of locally compact spaces $S$ where the strict topology for $C(S)$ is not the Mackey topology.

More recently, R. C. Busby [4] in his study of double centralizers of $B^*$-algebras introduced a generalized notion of the strict topology. Specifically, if $A$ is a $B^*$-algebra and $M(A)$ is its double centralizer algebra, then the strict topology $\beta$ for $M(A)$ is defined to be that locally convex topology generated by the seminorms $(\lambda_a)_{a \in A}$ and $(\rho_a)_{a \in A}$, where $\lambda_a(x) = \|ax\|$ and $\rho_a(x) = \|xa\|$, and we let $M(A)_\beta$ denote $M(A)$ under the strict topology. Although Busby investigated some of the properties of the strict topology in this setting, no mention was made of the strict dual of $M(A)$. Thus, the questions under consideration are the following: (1) Is the strict dual of $M(A)$ under the strong topology a Banach space that is isometrically isomorphic to the norm dual of $A$? (2) What are some sufficient conditions for the strict topology for $M(A)$ to be the Mackey topology? The answer to question (1) is yes and to answer question (2) we prove the following two theorems:

**Theorem I.** Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of $B^*$-algebras and let $A=(\Sigma A_\lambda)_0$. Then $M(A)_\beta$ is a Mackey space if, and only if, for each $\lambda \in \Lambda$, $M(A_\lambda)_\beta$ is a Mackey space.

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Theorem II. Let $A$ be a $B^*$-algebra and suppose one of the following conditions holds:

1. $M(A)$ is isometrically $*$-isomorphic to the bidual of $A$.
2. $A$ has a countable approximate identity.

Then $M(A)_\beta$ is a Mackey space.

If $S$ is a locally compact paracompact Hausdorff space, then by [2, p. 107] $S$ can be expressed as the union of a collection $\{Y_\lambda : \lambda \in \Lambda\}$ of pairwise disjoint open and closed $\sigma$-compact subsets of $S$. For each $\lambda \in \Lambda$ set $A_\lambda = C_0(Y_\lambda)$ and observe that $A_\lambda$ has a countable approximate identity. Since $A$ and $M(A)$ are isometrically $*$-isomorphic to $C_0(S)$ and $C(S)$ respectively, where $A = (\sum A_\lambda)_0$, it follows that Theorem II, together with Theorem I, generalizes Conway’s result [6, Theorem 2.6, p. 478] as well as a result of LeCam [11, Proposition 3, p. 220].

Furthermore Theorem II, together with the fact that the strict dual of $M(A)$ under the strong topology is isometrically isomorphic to the norm dual of $A$, gives for a special case a characterization of the Mackey topology of $W^*$-algebras (see [1]).

1. Notation and preliminaries. Let $A$ be a $B^*$-algebra. By a double centralizer on $A$, we mean a pair $(R, S)$ of functions from $A$ to $A$ such that $aR(b) = S(a)b$ for $a, b$ in $A$, and we will denote the set of all double centralizers on $A$ by $M(A)$. If $(R, S) \in M(A)$, then $R$ and $S$ are continuous linear operators on $A$ and $\|R\| = \|S\|$, so $M(A)$ under the usual operations of addition and multiplication is a Banach algebra, where $R, S) = \|R\|$. Furthermore, if we define $(R, S)^* = (S^*, R^*)$, where $R^*(a) = (R(a^*))^*$ and $S^*(a) = (S(a^*))^*$ for all $a \in A$, then $(R, S)^* \in M(A)$ and this implies that $M(A)$ is a $B^*$-algebra. If we define a map $\mu_0 : A \to M(A)$ by the formula $\mu_0(a) = (L_a, R_a)$, where $L_a(b) = ab$ and $R_a(b) = ba$ for all $b \in A$, then $\mu_0$ is an isometric $*$-isomorphism from $A$ into $M(A)$ and $\mu_0(A)$ is a closed two sided ideal in $M(A)$. Hence throughout this paper we will view $A$ as a closed two sided ideal in $M(A)$. If $A$ is commutative, then $M(A)$ is isometrically $*$-isomorphic to the algebra of multipliers as studied by Wang [17]. If $\{A_\lambda\}$ is a family of $B^*$-algebras, then $\sum A_\lambda$ and $(\sum A_\lambda)_0$ are defined as in [12]. It is clear that $\sum A_\lambda$ and $(\sum A_\lambda)_0$ are $B^*$-algebras. For a more detailed account of the theory of double centralizers on a $B^*$-algebra, we refer the reader to [4], and for definitions and concepts in general, we refer the reader to [10] and [12].

2. The dual of $M(A)_\beta$. In this section we prove that the strict dual of $M(A)$ under the strong topology is isometrically isomorphic to the norm dual of $A$ and furthermore, we characterize the $\beta$-equicontinuous subsets of the strict dual of $M(A)$.

Theorem 2.1. Let $A$ be a $B^*$-algebra and let $A^*$ denote the dual of $A$. Then $A^* = \{a \cdot f : a \in A$ and $f \in A^*\} = \{f \cdot a : a \in A$ and $f \in A^*\}$, where $a \cdot f(b) = f(ba)$ and $f \cdot a(b) = f(ab)$ for all $b \in A$. 

Proof. Let $f$ be a positive linear functional in $A^*$. By virtue of [12, Theorem 4.5.14, p. 219] $f$ is representable; that is, there exists a Hilbert space $H$, a continuous \*$-representation $a \rightarrow T_a$ of $A$ on $H$, and a topologically cyclic vector $h_0$ in $H$ such that $f(a) = (T_a h_0, h_0)$ for all $a \in A$. Let $\{e_i\}$ be an approximate identity for $A$. Since $h_0 = \lim T_{e_i} h_0$ for some sequence $\{a_i\}$ of elements in $A$, we can easily show that $\lim T_{\alpha} h_0 = h_0$. Due to the fact that $H$ is an $A$-module in the sense of [9, Definition 2.1, p. 147], we have by the Cohen-Hewitt factorization theorem [9, Theorem 2.5, p. 151] that $h_0 = T_{\alpha} h_1$ for some $a \in A$ and $h_1 \in H$. Define $g$ on $A$ by the formula $g(b) = (T_b h_1, h_1)$ for each $b \in A$ and note that $g \in A^*$ and $f = a \cdot g \cdot a^*$.

Now assume that $f$ is any element of $A^*$. Since $f$ can be expressed as a finite linear combination of positive functionals on $A$ [14, Theorem 1, p. 439], we see that $\lim e_i \cdot f = \lim f \cdot e_i = f$. Hence, by [9, Theorem 2.5, p. 151], there exist elements $a$ and $b$ in $A$ and linear functionals $g_1$ and $g_2$ in $A^*$ such that $f = a \cdot g_1 = g_2 \cdot b$ and our proof is complete.

**Corollary 2.2.** If $A$ is a B*-algebra, then $M(A)^* = \{a \cdot f : a \in A$ and $f \in M(A)^*\} = \{f \cdot a : a \in A$ and $f \in M(A)^*\}$, where $a \cdot f(x) = f(xa)$ and $f \cdot a(x) = f(ax)$ for all $x \in M(A)$.

**Proof.** Due to the fact that the strict topology is weaker than the norm topology, we have that $M(A)^* \subseteq M(A)^*$. Now let $f \in M(A)^*$ and let $\phi f$ denote the restriction of $f$ to $A$. By Theorem 2.1 there exists an $a \in A$ and a $g \in A^*$ such that $\phi f = a \cdot g$. By the Hahn-Banach theorem there exists an $h \in M(A)^*$ such that $g = \phi h$. Now let $\{e_i\}$ be an approximate identity for $A$ and let $x \in M(A)$. Since $e_i x + x e_i - e_i x e_i$ converges to $x$ in the strict topology and $A$ is a closed two sided ideal in $M(A)$, we have that

$$f(x) = \lim f(e_i x + x e_i - e_i x e_i) = \lim a \cdot g(e_i x + x e_i - e_i x e_i)$$

$$= g(xa) = h(xa) = a \cdot h(x).$$

Hence $f = a \cdot h$ and similarly there is a $b \in A$ and an $h_1 \in M(A)^*$ such that $f = h_1 \cdot b$. Since it is easy to show that $a \cdot f$ and $f \cdot a$ are strictly continuous for each $a \in A$ and $f \in M(A)^*$, our proof is complete.

The strong topology for $M(A)^*$ is defined to be the topology of uniform convergence on the $\beta$-bounded subsets of $M(A)^*$.

**Corollary 2.3.** If $A$ is a B*-algebra, then $M(A)^*$ under the strong topology is a Banach space that is isometrically isomorphic to $A^*$.

**Proof.** By virtue of the uniform boundedness principle, it is straightforward to show that the $\beta$-bounded subsets of $M(A)$ are norm bounded. Therefore, the strong topology for $M(A)^*$ is the usual topology generated by the norm of $M(A)^*$. Since $A$ is strictly dense in $M(A)_\beta$, we have by Theorem 2.1 and Corollary 2.2 that the restriction map $\phi$ is an isomorphism of $M(A)^*$ onto $A^*$. Therefore, to complete the proof we need to show that $\phi$ is an isometry. But this follows from the fact
that \( f(x) = \lim f(xe_A) \) for each \( f \in M(A)^* \) and \( x \in M(A) \), where \( \{e_A\} \) is an approximate identity for \( A \).

**Lemma 2.4.** Let \( A \) be a \( B^* \)-algebra and let \( \{d_n\} \) be a sequence of elements of \( A \), \( \|d_n\| < 1 \), that converges to zero. Then there exist sequences \( \{b_n\} \) and \( \{c_n\} \) of elements of \( A \) and a hermitian element \( a \) of \( A \) such that

1. \( d_n = ab_n = c_na; \)
2. \( \|a\| \leq 1, \|b_n\|^2, \|c_n\|^2 \).

**Proof.** Let \( A_1 \) be the \( B^* \)-algebra obtained by adjoining the identity, let \( \{e_A\} \) be an approximate identity for \( A \) consisting of hermitian elements, and let \( Z = \{x \in A : x = d_n, x = d^*_n, x = (d_n d_n^*)^{1/4}, \text{or } x = (d_n^* d_n)^{1/4}\} \). Since \( e_A x \to x \) uniformly on \( Z \), we may define by induction a sequence \( \{e_{A_k}\} \) of elements in the unit ball of \( A \) such that \( \|x - e_{A_k} x\| < \delta/2^n + 1, x \in Z \), and \( \|e_{A_k} - e_{A_{k+1}} e_{A_k}\| < \delta/32^n + 1, k = 1, 2, \ldots, n \), where \( \delta = \min \{1 - \|d_n\|^{-1} : n = 1, 2, 3, \ldots\} \). Now set

\[
 a_n = \sum_{k=1}^{n} \nu(1 - \nu)^{k-1} e_{A_k} + (1 - \nu)^n, \quad \text{where } 0 < \nu < 1/4.
\]

It follows, as in the proof of [16, Theorem 2.1], that \( a_n^{-1} \) exists, \( \|a_n^{-1}\| \leq 4^n \), and \( a_n^{-1} - a_{n+1}^{-1} = r(1 - e_{A_{n+1}}) + s \), where \( \|r\| \leq 4^n \) and \( \|s\| \leq \delta/2^n + 2 \). These facts together with the fact that \( a_n^{-1} \) is hermitian gives us, as in the proof of [16, Theorem 2.1], that \( \lim a_n^{-1} x \) and \( \lim a_n x^{-1} \) exist for each \( x \in Z \) and that \( \|a_n^{-1} x\| \leq \|x\| + \delta \).

So, by setting \( b_n = \lim_{p \to \infty} a_p^{-1} d_n \), \( c_n = \lim_{p \to \infty} d_n a_p^{-1} \), and \( a = \lim a_p \), we see that (1) holds. We now wish to show that (2) holds. But

\[
\|b_n\|^2 = \|b_n b_n^*\| = \lim_{p \to \infty} \|a_p^{-1} d_n d_n^* a_p^{-1}\| = \lim_{p \to \infty} \|a_p^{-1}(d_n d_n^*)^{1/4}(d_n d_n^*)^{1/2}(d_n d_n^*)^{1/4} a_p^{-1}\| \\
\leq (\|d_n d_n^*\|^{1/4} + \delta)^2 \|d_n\| \leq \|d_n\|.
\]

Similarly \( \|c_n\|^2 \leq \|d_n\| \) and (2) holds.

**Lemma 2.5.** Let \( A \) be a \( B^* \)-algebra. The collection of all sets

\[
 V_a = \{x \in M(A) : \|ax\| \leq 1, \|xa\| \leq 1\}
\]

for \( a \in A \) is a base at 0 in \( M(A) \) for the strict topology.

**Proof.** The proof follows from a straightforward application of Lemma 2.4.

**Theorem 2.6.** Let \( A \) be a \( B^* \)-algebra and let \( \{e_\lambda : \lambda \in \Lambda\} \) be an approximate identity for \( A \). If \( H \) is a subset of \( M(A)^* \), then the following statements are equivalent:

1. \( H \) is \( \beta \)-equicontinuous.
2. \( H \) is uniformly bounded and \( e_\lambda f + f \cdot e_\lambda - e_\lambda \cdot f \to f \) uniformly on \( H \), where \( e_\lambda \cdot f(x) = f(xe_\lambda) \) and \( f \cdot e_\lambda(x) = f(e_\lambda x) \) for all \( x \in M(A) \).

**Proof.** Assume (1) holds. Then \( H \) is contained in the polar of some basic neighborhood \( V_a = \{x \in M(A) : \|ax\| \leq 1, \|xa\| \leq 1\} \) of 0. Since the \( \beta \)-topology is weaker than the norm topology, it follows that \( H \) is uniformly bounded. Now for
each $x \in M(A)$ and $\epsilon > 0$ the element $x/(\|ax\| + \|xa\| + \epsilon)$ belongs to $V_\sigma$. So for $f \in H$

$$|f(x)| = |(\|ax\| + \|xa\| + \epsilon)f(x/(\|ax\| + \|xa\| + \epsilon))| < \|ax\| + \|xa\| + \epsilon.$$ 

Since $\epsilon$ was picked arbitrarily, it follows that $|f(x)| \leq \|ax\| + \|xa\|$. Hence

$$|(f-e_\lambda f-f \cdot e_\lambda + e_\lambda f \cdot e_\lambda)(x)| = |f((1-e_\lambda)x(1-e_\lambda))| \leq 2(\|ae_\lambda - a\| + \|ea_\lambda - a\|)\|x\|$$

for each $f \in H$ and $x \in M(A)$. So for each $f \in H$

$$\|f-(e_\lambda f + f \cdot e_\lambda - e_\lambda \cdot f \cdot e_\lambda)\| \leq 2 \|ae_\lambda - a\| + 2 \|ea_\lambda - a\|$$

and therefore it is clear that (2) holds.

Now assume (2) holds and that $H$ is uniformly bounded by 1. To prove that $H$ is $\beta$-equicontinuous, it will suffice to show that $H$ is contained in the polar of some basic neighborhood of 0 in $M(A)_\epsilon$. For each $\lambda \in \Lambda$ set $R_\lambda f = e_\lambda f + f \cdot e_\lambda - e_\lambda \cdot f \cdot e_\lambda$ for each $f \in M(A)_\epsilon$ and set $S_\lambda x = e_\lambda x + xe_\lambda - e_\lambda xe_\lambda$ for each $x \in M(A)_\sigma$. Now choose a sequence $\{e_\lambda(n)\}$ of elements from our approximate identity such that for each positive integer $n$ we have $\lambda_{n+1} > \lambda_n$, $\|R_{\lambda_{n+1}} f - R_{\lambda_n} f\| \leq 1/4^n + 1$ for each $f \in H$, $\|e_\lambda - e_\lambda e_{\lambda + 1}\| \leq 1/9 \cdot 4^n$ for $k = 1, 2, \ldots, n$, and $\|e_\lambda - e_{\lambda + 1} e_\lambda\| < 1/9 \cdot 4^n$ for $k = 1, 2, \ldots, n$. Let $\{d_k\}$ be a sequence of elements in $A$ defined by $d_{3k-4} = (3/2k+1)e_{\lambda_k}$, $d_{3k-3} = e_\lambda - e_\lambda e_{\lambda + 1}$, $d_{3k-2} = e_\lambda - e_{\lambda + 1} e_\lambda$, $d_{3k-1} = e_\lambda - e_\lambda e_{\lambda + 2}$, and $d_{3k} = e_\lambda - e_{\lambda + 2} e_\lambda$. It is clear that $d_k \to 0$ uniformly and $\|d_k\| < 1$. Therefore, by Lemma 2.4, there exist sequences $\{b_k\}$ and $\{c_k\}$ of elements in $A$ and a hermitian element $a \in A$, $\|a\| \leq 1$, such that $d_k = ab_k = c_k a$ and max $\{\|b_k\|^2, \|c_k\|^2\} \leq \|d_k\|$. Set $a_1 = 8a$. We now wish to show that $H \subset V_{a_1}^0$, where $V_{a_1}^0$ is the polar of

$$V_{a_1} = \{x \in M(A) : \|a_1 x\| \leq 1 \text{ and } \|xa_1\| \leq 1\}$$

in $M(A)_\epsilon$. Since $d_{3k-4} = ab_{3k-4} = c_{3k-4} a$, we have for each $x \in V_{a_1}$ that $\|xe_\lambda x\| = (2^{k+1}/3)\|x\|e_{\lambda-4} x\| \leq 2^{k+1}/3 \cdot 8$ and similarly $\|e_\lambda e_\lambda x\| \leq 2^{k+1}/3 \cdot 8$. It follows, by straightforward computations, that for each $f \in H$ and $x \in V_{a_1}$ that

$$|R_{\lambda_k} f(x - S_{\lambda_k} x)| \leq 1/2^{k+3}, \quad |R_{\lambda_k+1} f(x - S_{\lambda_k+2} x)| \leq 1/2^{k+3},$$

and

$$\|S_{\lambda_k} x\| \leq 2^{k+1}/8.$$ 

These inequalities and the fact that $f = R_{\lambda_1} f + \sum_{k=1}^\infty (R_{\lambda_k+1} f - R_{\lambda_k} f)$ for each $f \in H$ imply that

$$|f(x)| \leq |f(S_{\lambda_1}(x))| + \sum_{k=1}^\infty |(R_{\lambda_k+1} f - R_{\lambda_k} f)(x - S_{\lambda_k+2} x + S_{\lambda_k+2} x)| < 1$$

whenever $f \in H$ and $x \in V_{a_1}$. Hence $H \subset V_{a_1}^o$ and our proof is complete.
We will now generalize a result due to L. LeCam [11, Proposition 2, p. 217] and J. R. Dorroh [8] that concerns the $\beta'$ or bounded strict topology. The $\beta'$ topology is the strongest locally convex topology for $M(A)$ that agrees with the $\beta$ topology on norm bounded sets. For a proof of existence, we refer the reader to [5] where an explicit neighborhood base is given. Another generalization of this theorem exists. F. D. Sentilles proved a similar result [15] in a Banach module setting and though we use the same technique his result does not seem to subsume our

**Corollary 2.7.** If $A$ is a $B^*$-algebra, then the $\beta$ and $\beta'$ topologies for $M(A)$ give the same dual. Consequently, $\beta = \beta'$.

**Proof.** By virtue of Theorem 2.1, the proof that the $\beta'$ dual of $M(A)$ is $M(A)_{\beta'}^*$ is similar to the one given for Corollary 2.2. Therefore, it remains to be shown that $\beta = \beta'$. Let $W$ be an absolutely convex $\beta'$-closed $\beta'$-neighborhood of 0. Then there exists a sequence $\{a_n\}$ of elements in $A$ such that $\mathcal{B}_n \cap V_{a_n} \subseteq \mathcal{B}_n \cap W$, where $V_{a_n} = \{x \in M(A) : \|a_n x\| \leq 1 \text{ and } \|x_{a_n}\| \leq 1\}$ and $\mathcal{B}_n = \{x \in M(A) : \|x\| \leq n\}$. Set $\mathcal{D}_n = \mathcal{B}_n \cap V_{a_n}$ and $W'$ equal the $\beta'$-closed absolutely convex hull of $\bigcup \mathcal{D}_n$. Then $W' \subseteq W$, and $(W')^0 = \bigcap (D_n)^0$, where $(W')^0$ and $(D_n)^0$ are the polars of $W'$ and $D_n$ respectively in $M(A)^\#$. We will show that $(W')^0$ is $\beta$-equicontinuous which implies that the $\beta$-closure of $W'$ is a $\beta$-neighborhood. To this end, we will show that $e_{\lambda} \cdot f + f \cdot e_{\lambda} - e_{\lambda} \cdot f \cdot e_{\lambda} \to f$ uniformly on $(W')^0$, where $\{e_{\lambda}\}$ is an approximate identity for $A$ consisting of positive elements. Let $\varepsilon > 0$. Choose a positive integer $n$ so that $1/n < \varepsilon$ and then choose a $\lambda_0$ so that for $\lambda \leq \lambda_0$, $\| (1 - e_{\lambda}) a_n \| < 1/n$ and $\| a_n (1 - e_{\lambda}) \| < 1/n$. Hence $\{n(1 - e_{\lambda}) x (1 - e_{\lambda}) : x \in B_1 \} \subseteq D_n$ for $\lambda \geq \lambda_0$. Therefore for $f \in (W')^0$, $x \in B_1$, and $\lambda \geq \lambda_0$

$$|(f - e_{\lambda} \cdot f - f \cdot e_{\lambda} + e_{\lambda} \cdot f \cdot e_{\lambda}) (x)| = |f((1 - e_{\lambda}) x (1 - e_{\lambda}))| < 1/n < \varepsilon.$$ 

In other words, $\|f - e_{\lambda} \cdot f - f \cdot e_{\lambda} + e_{\lambda} \cdot f \cdot e_{\lambda}\| < \varepsilon$ for all $f \in (W')^0$ and $\lambda \geq \lambda_0$. Thus, by Theorem 2.6, $(W')^0$ is a $\beta$-equicontinuous and our proof is complete.

It is well known that the bidual $A^{**}$ of a $B^*$-algebra $A$ is a $W^*$-algebra, and when $A$ is canonically imbedded into $A^{**}$, $A$ is a $*$-subalgebra of $A^{**}$. We will now consider the case when $M(A)$ is isometrically $*$-isomorphic to $A^{**}$. For example, if $A$ is also an annihilator algebra, then this is true.

**Corollary 2.8.** Let $A$ be a $B^*$-algebra such that $M(A)$ is isometrically $*$-isomorphic to $A^{**}$. Then $M(A)_\beta$ is a Mackey space.

**Proof.** The proof follows from Corollary 2.2, Corollary 2.3, Corollary 2.7, and [1, Theorem II.7, p. 292].

3. **Proof of Theorem I and Theorem II.**

**Lemma 3.1.** Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of $B^*$-algebras and let $A = (\sum A_\lambda)_0$. Then $M(A)$ is isometrically $*$-isomorphic to $\sum M(A_\lambda)$.
Proof. Let \((R, S) \in M(A)\) and let \(\lambda \in \Lambda\). Define \(R_\lambda\) and \(S_\lambda\) on \(A_\lambda\) by the formula 
\[R_\lambda(a(\lambda)) = (R(a))(\lambda)\] 
and similarly, \(S_\lambda(a(\lambda)) = 0\). It is straightforward to show that \((R_\lambda, S_\lambda) \in M(A_\lambda)\) and that 
\[\|R_\lambda, S_\lambda\| \leq \|(R, S)\|,\] 
so define the map \(\mu : M(A) \to \sum M(A_\lambda)\) by the formula \(\mu((R, S))(\lambda) = (R_\lambda, S_\lambda)\). It is clear that \(\mu\) is a *-isomorphism from \(M(A)\) into \(\sum M(A_\lambda)\) and that \(\|\mu((R, S))\| \leq \|(R, S)\|\) for all \((R, S) \in M(A)\). Now for \((R, S) \in M(A)\) and \(a \in A, \|a\| \leq 1,\) 
\[\|R(a)\| = \sup \{\|R(\lambda)\| : \lambda \in \Lambda\} = \sup \{\|R_\lambda(a(\lambda))\| : \lambda \in \Lambda\}\] 
\[\leq \sup \{\|R_\lambda\| : \lambda \in \Lambda\} = \sup \{\|(R_\lambda, S_\lambda)\| : \lambda \in \Lambda\} = \|\mu((R, S))\|.\] 
In other words, \(\|(R, S)\| = \|\mu((R, S))\|\). Therefore to complete the proof we need to show that \(\mu\) is onto. Let \((R_\lambda, S_\lambda) \in \sum M(A_\lambda)\) and define \((R(a))(\lambda) = R_\lambda(a(\lambda))\) and \((S(a))(\lambda) = S_\lambda(a(\lambda))\) for each \(a \in A\) and \(\lambda \in \Lambda\). But it is clear that \((R, S) \in M(A)\) and \(\mu((R, S)) = \sum (R_\lambda, S_\lambda)\). Hence \(\mu\) is onto and our proof is complete.

Lemma 3.2. Let \(\{A_\lambda : \lambda \in \Lambda\}\) be a family of B*-algebras. Then the following statements are equivalent:

(1) If \(A = (\sum_{\lambda \in \Lambda} A_\lambda)_0\), then \(M(A)_\beta\) is a Mackey space.

(2) If \(\Lambda_0\) is a countable subset of \(\Lambda\) and \(A_0 = (\sum_{\lambda \in \Lambda_0} A_\lambda)_0\), then \(M(A_0)_\beta\) is a Mackey space.

Proof. By virtue of Theorem 2.6, Lemma 3.1, and [10, p. 173], it is easy to show that (1) implies (2). Now let \(H\) be a \(\beta\)-weak* compact convex circled subset of \(M(A)_\beta^*\) and let \(\phi_\lambda\) denote the restriction map from \(M(A)\) onto \(M(A_\lambda)\), where \(M(A_\lambda)\) is now viewed as a subspace of \(M(A)\). Set \(\Lambda_0 = \{\lambda \in \Lambda : \|\phi_\lambda f\| > 0\text{ for some } f \in H\}\). If \(\Lambda_0\) is countable, then (2), together with Theorem 2.6, Lemma 3.1, and [10, p. 173], implies that \(H\) is \(\beta\)-equicontinuous and therefore, by [10, p. 173], (2) implies (1). Hence, it remains to be shown that \(\Lambda_0\) is countable.

For each \(\lambda \in \Lambda_0\) choose an \(x_\lambda \in M(A_\lambda), \|x_\lambda\| \leq 1,\) so that for some \(f \in H\) we have \(f(x_\lambda) \neq 0\). Now define \(x \in M(A)\) by the formula 
\[x(\lambda) = x_\lambda \quad \text{if } \lambda \in \Lambda_0,\] 
\[= 0 \quad \text{if } \lambda \notin \Lambda_0,\] 
where we now view \(M(A)\) as \(\sum_{\lambda \in \Lambda} M(A_\lambda)\), and then define the map 
\(T : C(\Lambda)_\beta \to M(A)_\beta\) 
by the formula \(T(\alpha)(\lambda) = a(\lambda)x(\lambda)\) for each \(\alpha \in C(\Lambda)\) and \(\lambda \in \Lambda\). Here the topology
for \( \Lambda \) is the discrete topology. Let \( \{a_i\} \) be a norm bounded net in \( C(\Lambda) \) that converges to zero in the strict topology. It is straightforward to show that the net \( \{T(a_i)\} \) in \( M(A) \) converges to zero in the strict topology and therefore, by virtue of Corollary 2.7, \( T \) is \( \beta \)-continuous. This implies that \( T \) has a well-defined adjoint map \( T^*: M(A)_\beta^* \to C(\Lambda)_\beta^* \), which is continuous when both range and domain have their \( \beta \)-weak* topologies. It follows that \( T^*(H) \) is \( \beta \)-weak* compact and therefore, by virtue of [6, Theorem 2.6, p. 478] and [6, Theorem 2.2, p. 476], \( \Lambda_0 \) is countable. Hence our proof is complete.

**Lemma 3.3.** Let \( A \) be a \( B^* \)-algebra and let \( a \) and \( b \) be hermitian elements in \( A \) such that \( 1 \geq \|a\| \geq \|b\| \). Then \( \|a + b\| \leq 1 + 2\|ab\| \).

**Proof.** Let \( \sigma \) be the smallest number such that
\[
\|c + d\| \leq 1 + \sigma
\]
for all hermitian elements \( c \) and \( d \) in \( A \), where \( 1 \geq \|c\| \geq \|d\| \) and \( \|cd\| \leq \|ab\| \). It is clear that such a number exists. Now if \( \sigma > 2\|ab\| \), then \( \|c + d\|^2 = \|c^2 + d^2 + 2\|cd\| + 2\|ab\| \leq 1 + \sigma + 2\|ab\| < (1 + \sigma)^2 \) for all hermitian elements \( c \) and \( d \) in \( A \), where \( 1 \geq \|c\| \geq \|d\| \) and \( \|cd\| \leq \|ab\| \). But this contradicts (3.1), so \( \sigma \leq 2\|ab\| \) and our proof is complete.

The author would like to thank Professor L. Eifler for the suggestion of the argument given for Lemma 3.3. This argument eliminated a longer proof.

**Remark.** It follows immediately from Lemma 3.3 that for each pair of hermitian elements \( a, b \) in a \( B^* \)-algebra \( A \) the inequality \( \|a + b\| \leq \|a\| + 2\|ab\|/\|a\| \) holds whenever \( \|a\| \geq \|b\| \) and \( \|a\| \neq 0 \). In fact, there is a smallest number \( k \) such that \( \|a + b\| \leq \|a\| + k\|ab\|/\|a\| \) and this, in a sense, is a generalization of the triangle inequality for \( B^* \)-algebras. But \( k = 1 \) when the \( B^* \)-algebra \( A \) is commutative, and this fact suggests the following question: Is it true that \( k = 1 \) only if \( A \) is commutative?

**Proof of Theorem I.** Let \( \{A_k\}_{k=1}^\infty \) be a sequence of \( B^* \)-algebras such that \( M(A_k)_\beta \) is a Mackey space for each positive integer \( k \). If we show that \( M(A)_\beta \) is a Mackey space, where \( A = (\bigcup_{k=1}^\infty A_k)_0 \), then by virtue of Lemma 3.2 the proof will be complete. To this end, it will suffice to show that each \( \beta \)-weak* compact circled convex subset of \( M(A)_\beta^* \) is \( \beta \)-equicontinuous. Now suppose that \( H \) is a \( \beta \)-weak* compact circled convex subset of \( M(A)_\beta^* \) that is not \( \beta \)-equicontinuous. Since \( H \) is \( \beta \)-weak* compact, \( H \) is uniformly bounded and we can assume, without loss of generality, that \( H \) is uniformly bounded by 1. Let \( \{e_\delta : \delta \in \Delta \} \) be an approximate identity for \( A \) consisting of positive elements. Then by virtue of Theorem 2.6 there exists an \( \epsilon > 0 \) such that for each \( \delta_0 \in \Delta \) we have
\[
\|f - e_\delta \cdot f - e_\delta \cdot e_\delta \| \geq 4\epsilon
\]
for some \( f \in H \) and \( \delta > \delta_0 \). We will now define by induction a sequence of triples \( \{(f_k, x_k, n_k)\}_{k=1}^\infty \) that satisfies the following conditions:

1. \( f_k \in H, x_k \in M(A), \) and \( n_k \) is a positive integer less than \( n_{k+1} \).
\( \|x_k\| \leq 1, x_k(q) = 0 \) for each positive integer \( q \leq n_k - 1 \) or \( q > n_k \), where \( M(A) \) is now viewed as \( \sum_{k=1}^{\infty} M(A_k) \).

By virtue of (3.2) there exists an \( f \) in \( H \), a \( \delta \) in \( \Delta \), and a \( y \) in the unit ball of \( M(A) \) such that \( |f_1((1 - e_\delta)y(1 - e_\delta))| \geq 3\varepsilon \). Since \( M(A)_\beta^* \) under the strong topology is isometrically isomorphic to \( A^* \) and \( A^* \) is isometrically isomorphic to the \( L^1 \) direct sum of \( \{A_k^*\}_{k=1}^{\infty} \), we can find a positive integer \( n_1 \) such that \( |f_1(x_1)| \geq \varepsilon \), where \( x_1 \) is the element in \( M(A) \) defined by \( x_1(q) = ((1 - e_\delta)y(1 - e_\delta))(q) \) for \( q = 1, 2, \ldots, n_1 \) and \( x_1(q) = 0 \) for \( q > n_1 \). It is clear that \( (f_1, x_1, n_1) \) satisfies conditions (1), (2), and (3).

Now assume that \( (f_k, x_k, n_k) \) has been defined for \( k = 1, 2, \ldots, p \). Let \( B_n^p \) be the subspace of \( A \) defined by \( B_n^p = \sum_{k=1}^p A_k \) and let \( \phi \) denote the restriction mapping from \( M(A) \) to \( M(B_n^p) = \sum_{k=1}^p M(A_k) \). It is straightforward to show, by using Theorem 2.6, that \( M(B_n^p)_\beta \) is a Mackey space and therefore, by virtue of Theorem 2.6, (10, p. 173), and (3.2), there exists an \( f_{p+1} \) in \( H \), a \( \delta \) in \( \Delta \), and a \( y \) in the unit ball of \( M(A) \) such that \( |f_{p+1}((1 - e_\delta)y(1 - e_\delta))| \geq 3\varepsilon \) and

\[ \| \phi(f_{p+1} - e_\delta \cdot f_{p+1} - f_{p+1} \cdot e_\delta + e_\delta \cdot f_{p+1} + e_\delta) \| < \varepsilon. \]

By virtue of (3.3) and the fact that \( M(A)_\beta^* \) under the strong topology is isometrically isomorphic to the \( L^1 \) direct sum of \( \{A_k^*\}_{k=1}^{\infty} \), we can find a positive integer \( n_{p+1} > n_p \) such that \( |f(x_{p+1})| \geq \varepsilon \), where \( x_{p+1} \) is the element in \( M(A) \) defined by \( x_{p+1}(q) = ((1 - e_\delta)y(1 - e_\delta))(q) \) for \( n_p < q \leq n_{p+1} \) and \( x_{p+1}(q) = 0 \) otherwise. It is clear that \( (f_{p+1}, x_{p+1}, n_{p+1}) \) satisfies conditions (1), (2), and (3), and our induction is complete. Now let \( x \) be the element in \( M(A) \) defined by \( x(q) = x_k(q) \) when \( n_k - 1 < q \leq n_k \). Then define the map \( T: (l^\infty, \beta) \to M(A)_\beta \) by the formula \( T(\alpha)(q) = \alpha(q)x(q) \) for each \( \alpha \in l^\infty \) and positive integer \( q \). By virtue of Corollary 2.7, it is straightforward to show that \( T \) is continuous. Hence \( T \) has a well-defined adjoint map \( T^*: M(A)_\beta \to l^1 \), which is continuous when both range and domain have the \( \beta \)-weak* topologies. Thus, \( T^*(H) \) is a \( \beta \)-weak* compact subset of \( l^1 \) and this implies, by virtue of [6, Theorem 2.4, p. 477], that \( T^*(H) \) is \( \beta \)-equicontinuous in \( l^1 \). Since \( \sum_{k=1}^{\infty} \alpha(k)x(k) \) converges in the strict topology to \( T(\alpha) \) as \( q \to \infty \) for each \( \alpha \in l^\infty \), we see that \( T^*(\alpha) = \sum_{k=1}^{\infty} \alpha(k)f(x(k)) \) for each \( \alpha \in l^\infty \) and \( f \in M(A)_\beta^* \). Since \( T^*(H) \) is \( \beta \)-equicontinuous, there exists, by virtue of [6, Theorem 2.2, p. 476], a positive integer \( N \) such that \( \sum_{k=N+1}^{\infty} |f(x(k))| < \varepsilon \) for each \( f \in H \). This implies that \( |f(x)| \leq \sum_{k=N+1}^{\infty} |f(x(k))| < \varepsilon \) for \( n_k > N \). This holds for all \( f \in H \) and in particular \( |f_\delta(x_\delta)| < \varepsilon \). But this contradicts (3). Hence \( H \) is \( \beta \)-equicontinuous and our proof is complete.

**Proof of Theorem II.** Let \( A \) be a \( B^* \)-algebra. If condition (1) holds, then it follows from Corollary 2.8 that \( M(A)_\beta \) is a Mackey space. Now assume that \( A \) has a countable approximate identity. To show that \( M(A)_\beta \) is a Mackey space it will suffice to show that every \( \beta \)-weak* compact subset of \( M(A)_\beta^* \) is \( \beta \)-equicontinuous [10, p. 173]. Suppose that \( H \) is a \( \beta \)-weak* compact subset of \( M(A)_\beta^* \) that is not \( \beta \)-equicontinuous. Since \( H \) is \( \beta \)-weak* compact, \( H \) is uniformly bounded, and
without loss of generality we can assume that $H$ is uniformly bounded by 1. Suppose $(d_k)_{k=1}^\infty$ is an approximate identity for $A$ consisting of positive elements. We may assume that for each positive integer $n$

$$\tag{3.4} \|d_{n+1}d_k - d_k\| < 1/n \cdot 2^{n+3}$$

for $k = 1, 2, \ldots, n$. Now because of Theorem 2.6 there exists an $\epsilon > 0$ such that for each positive integer $N$ the inequality

$$\tag{3.5} \|f - d_n \cdot f - f \cdot d_n + d_n \cdot f \cdot d_n\| \geq 5\epsilon$$

holds for some $f \in H$ and integer $n > N$. We will now define by induction a sequence of quadruples $(f_k, a_k, n_{2k-1}, n_{2k})_{k=1}^\infty$ that satisfies the following conditions:

(a) $f_k \in H$, $a_k$ is a hermitian element in the unit ball of $A$, and $n_{2k-1}$, $n_{2k}$ are positive integers such that $n_{2k-1} < n_{2k} < n_{2k+1}$.

(b) $|f_k| d_{n_{2k}}(1 - d_{n_{2k-1}})a_k(1 - d_{n_{2k-1}})d_{n_{2k}}| \geq \epsilon$.

By virtue of (3.5) and Corollary 2.3, it is straightforward to show that there exist an $f_1 \in H$, a hermitian element $a_1$ in the unit ball of $A$, and positive integers $n_1, n_2$ with $n_1 < n_2$ such that

$$|f_1 d_{n_2}(1 - d_{n_1})a_1(1 - d_{n_1})d_{n_2}| \geq \epsilon.$$

Thus $(f_1, a_1, n_1, n_2)$ satisfies (a) and (b). Now suppose the quadruple $(f_k, a_k, n_{2k-1}, n_{2k})$ has been defined for $k = 1, 2, \ldots, p$ so that conditions (a) and (b) have been satisfied. Again, by virtue of (3.5) and Corollary 2.3, it is straightforward to show that there exist an $f_{p+1} \in H$, a hermitian element $a_{p+1}$ in the unit ball of $A$, and positive integers $n_{2p+1}, n_{2p+2}$ with $n_{2p} < n_{2p+1} < n_{2p+2}$ such that

$$|f_{p+1} d_{n_{2p+1}}(1 - d_{n_{2p-1}})a_{p+1}(1 - d_{n_{2p-1}})d_{n_{2p+1}}| \geq \epsilon$$

and our induction is complete. Set $x_k = d_{n_{2k}}(1 - d_{n_{2k-1}})a_k(1 - d_{n_{2k-1}})d_{n_{2k}}$ and $e_k = d_{2k}$. Because of (3.4), (a), and (b), $(f_k, x_k, e_k)_{k=1}^\infty$ is a sequence of triples such that the following conditions hold:

(a)' $f_k \in H$, $x_k$ is an hermitian element in the unit ball of $A$, and $e_k \in A$.

(b)' $(e_k)$ is an approximate identity for $A$ consisting of positive elements.

(c)' For each positive integer $p$, $\|e_p x_k\| = \|x_k e_p\| < 1/2^k$ for $k = p+1, p+2, \ldots$ and $\|x_{p+1} x_k\| = \|x_k x_{p+1}\| < 1/p \cdot 2^{p+2}$ for $k = 1, 2, \ldots, p$.

(d)' $|f_k(x_k)| \geq \epsilon$.

Let $\alpha = \{a_k\}_{k=1}^\infty$ belong to $l^\infty$. By virtue of Lemma 3.3, it is straightforward to show that $\|\sum_{k=1}^\infty \alpha_k x_k\| \leq \|\alpha\|_\infty \sum_{k=1}^\infty 1/2^{k-1} \leq 2 \|\alpha\|_\infty$ for each positive integer $n$. This inequality and the fact that $\|e_p x_p\| = \|x_p e_p\| < 1/2^p$ for $p \geq n+1$, imply that the sequence of partial sums $\{\sum_{k=1}^p \alpha_k x_k\}_{n=1}^\infty$ is $\beta$-Cauchy. Since $M(A)_\beta$ is complete [4, Proposition 3.6, p. 83], we may define the map $T: (l^\infty, \beta) \to M(A)_\beta$ by the formula $T(\alpha) = \sum_{k=1}^\infty \alpha_k x_k$, where $\alpha = \{a_k\}_{k=1}^\infty$ and $\sum_{k=1}^\infty \alpha_k x_k$ is the $\beta$-limit of the partial sums. By virtue of Corollary 2.7, it is straightforward to show that $T$ is continuous and therefore $T$ has a well-defined adjoint map $T^*: M(A)_\beta^* \to l^1$,
which is continuous when both range and domain have the $\beta$-weak* topologies. Thus $T^*(H)$ is a $\beta$-weak* compact subset of $l^1$ and this implies, by virtue of [6, Theorem 2.4, p. 477], $T^*(H)$ is $\beta$-equicontinuous in $l^1$. Observe that $T^*f(\alpha) = f(T(\alpha)) = \sum_{k=1}^{\infty} \alpha_k f(x_k)$ for each $\alpha \in l^\infty$ and $f \in M(A)^\beta$, so that $T^*f = \{f(x_k)\}_{k=1}^{\infty}$. Since $T^*(H)$ is $\beta$-equicontinuous, there exists by virtue of [6, Theorem 2.2, p. 476] a positive integer $N$ such that $\sum_{k=N+1}^{\infty} |f(x_k)| < \epsilon$ for each $f \in H$. Thus, for $f \in H$ and $k > N$ we have $|f(x_k)| < \epsilon$ and in particular $|f_k(x_k)| < \epsilon$ for $k > N$. But this contradicts (d)'. Hence $H$ is $\beta$-equicontinuous and our proof is complete.

**References**


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