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INFINITE GENERAL LINEAR GROUPS OVER RINGS

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Abstract. We are interested in the normal subgroups of the group G of automorphisms of a free module of infinite type over a commutative ring A. To this end, we introduce a certain "elementary" subgroup E of G and find that the subgroups of G normalised by E are exactly those which lie in congruence layers determined by the ideals of A. The normal subgroups are thus to be found in such layers.

0. **Introduction.** Let A be a commutative ring and M a free A-module of infinite type. We are interested in the structure of Aut (M). As a first step towards the determination of its normal subgroups, we find here at least those subgroups which are normalised by a certain group E of "elementary automorphisms." The results are quite analogous to those of Bass [1], [2] in the "stable" finite case, except that no hypotheses on A are required other than commutativity. The latter, regrettably, seems essential for a key result (2.5), although our arguments would work under stringent hypotheses such as A being a division ring. As a corollary, we determine the normal subgroups of E modulo the calculation of certain abelian groups.

Our procedure is also inspired by that of Bass; in fact, infinity merely makes life easier. Finally, we note that the successful work of Rosenberg [3] in the division ring case offers some hope that one might determine the normal subgroups of Aut (M) starting with these results.

1. **Definitions.** Let A be a commutative ring and M the free A-module $A^{(I)}$ for some infinite set I. Let $\{e_i\}_{i\in I}$ be the canonical basis of M; submodules of M of the type $\bigoplus_{j\in J} e_jA$ for some $J\subset I$ will be called elementary. We shall be interested in the group of units of the ring $R=\mathrm{End}_A(M)$.

Suppose $\mathfrak{q}$ is an ideal of A. Then

$$\mathrm{End}_A(M/M\cdot\mathfrak{q}) \cong \mathrm{End}_{A/\mathfrak{q}}((A/\mathfrak{q})^{(I)}) \cong R/(\mathfrak{q}), \tag{1.1}$$

where $(\mathfrak{q})=\{r\in R \mid r(M)\subset M\cdot\mathfrak{q}\}$ is an ideal of R. The projection $R\to R/(\mathfrak{q})$ induces a group morphism $U(R)\to U(R/(\mathfrak{q}))$; we shall denote its kernel by $U(\mathfrak{q})$ and the inverse image of the center of $U(R/(\mathfrak{q}))$ by $U'(\mathfrak{q})$.

Let V be an elementary submodule of M, $e_i\notin V$, and $h\colon V\to e_iA$ any module morphism. Extend h to an element of R by letting it vanish on the elementary

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complement of V. Then $h^2=0$ so that $1+h \in U(R)$ with inverse $1-h$. Such an element of R will be called an elementary automorphism. If $h \in (\mathfrak{q})$, we call it $\mathfrak{q}$-elementary.

Define $E(R)$ to be the subgroup of $U(R)$ generated by all elementary automorphisms and $E(\mathfrak{q})$ to be the normal subgroup of $E(R)$ generated by all $\mathfrak{q}$-elementary automorphisms. It is easy to see that $E(\mathfrak{q})=E(\mathfrak{q}')$ only if $\mathfrak{q}=\mathfrak{q}'$.

2. Preliminary results.

(2.1) PROPOSITION. *The orbits of* $E(\mathfrak{q})$ *operating on the unimodular elements of* M *are the congruence classes* mod $M\cdot\mathfrak{q}$. *In particular,* $E(R)$ *operates transitively.*

Proof. Take any $e_i \in M$; we first show that e_i can be mapped to any unimodular element $x \equiv e_i \bmod M\cdot\mathfrak{q}$ by something in $E(\mathfrak{q})$. Suppose $x=e_i(1+q_i)+\sum_{j\neq i} e_j q_j$; the unimodularity of x gives a relation $a_i(1+q_i)+\sum_{j\neq i} a_j q_j=1$. Choose an index $k\neq i$ such that $q_k=0$ and define $1+h_1 \in E(R)$, $1+h_2$, $1+h_3 \in E(\mathfrak{q})$ by

$$\begin{aligned} h_1(e_k) &= -e_i, \\ h_2(e_i) &= x-e_i(1+q_i)+e_k q_i, \\ h_3(e_j) &= -e_k q_i a_j \qquad (j \neq k). \end{aligned}$$

Then $(1+h_3)(1+h_1)^{-1}(1+h_2)(1+h_1) \in E(\mathfrak{q})$ and maps e_i to x.

Now suppose x and y are unimodular elements of M such that $x\equiv y \bmod M\cdot\mathfrak{q}$. The first part of the argument, with $\mathfrak{q}=A$, shows that $\beta(x)=e_i$ for some $\beta \in E(R)$. Since $\beta(y)\equiv e_i \bmod M\cdot\mathfrak{q}$, there exists $\tau \in E(\mathfrak{q})$ such that $\tau(e_i)=\beta(y)$, whence $\beta^{-1}\tau\beta(x)=y$. ∎

(2.2) PROPOSITION. $E(\mathfrak{q})$ *is normal in* $U(R)$.

Proof. It suffices to show that the conjugate of a $\mathfrak{q}$-elementary automorphism $1+h$ by $\sigma \in U(R)$ is in $E(\mathfrak{q})$. Suppose $h\colon V \to e_iA$; by (2.1), there exists $\beta \in E(R)$ such that $\beta(e_i)=\sigma^{-1}(e_i)$. Replacing σ by $\sigma\beta$ allows us to assume $\sigma(e_i)=e_i$. An easy calculation now shows that $\sigma^{-1}(1+h)\sigma$ is $\mathfrak{q}$-elementary. ∎

(2.3) PROPOSITION. $E(\mathfrak{q})=[E(R), E(\mathfrak{q})]$.

Proof. It suffices to prove that a $\mathfrak{q}$-elementary automorphism $1+h$ is in $[E(R), E(\mathfrak{q})]$. Suppose $h\colon V \to e_iA$ with $h(e_j)=e_ih_j$ for $e_j \in V$.

Case (i). For some $k\neq i$, $e_k \notin V$.

Define $1+h_1 \in E(R)$, $1+h_2 \in E(\mathfrak{q})$ by

$$h_1(e_k) = e_i, \qquad h_2(e_j) = e_k h_j \quad (j \neq k, i).$$

Then $1+h=[1+h_1, 1+h_2]$.

Case (ii). V contains all e_k for $k\neq i$.

Choose any $e_k \in V$ and write $V=e_kA \oplus V'$, where V' is elementary. If h_1 and h_2 are restrictions of h to these direct summands, then $1+h_1$ and $1+h_2$ are $\mathfrak{q}$-elementary and covered by Case (i); however, $1+h=(1+h_1)(1+h_2)$. ∎

(2.4) PROPOSITION. $[E(R), U'(\mathfrak{q})] = E(\mathfrak{q})$.

Proof. We first show that $[E(R), U(\mathfrak{q})] \subset E(\mathfrak{q})$. Let $1+h$ be an elementary automorphism and $\sigma \in U(\mathfrak{q})$. Suppose $h\colon V \to e_iA$ with $h(e_j) = e_ih_j$ for $e_j \in V$. By (2.1), there exists $\tau \in E(\mathfrak{q})$ such that $\tau(e_i) = \sigma^{-1}(e_i)$; replacing σ by $\sigma\tau$, which is still in $U(\mathfrak{q})$, we may assume $\sigma(e_i) = e_i$. An easy calculation shows that $[1+h, \sigma]$ is $\mathfrak{q}$-elementary.

Reducing mod $\mathfrak{q}$, we see that $[E(R), U'(\mathfrak{q})] \subset U(\mathfrak{q})$; therefore

$$[E(R), [E(R), U'(\mathfrak{q})]] \subset E(\mathfrak{q}).$$

In view of (2.2), the "3 subgroups" theorem ([4, p. 59]) implies that

$$[[E(R), E(R)], U'(\mathfrak{q})] \subset E(\mathfrak{q}).$$

However, $[E(R), E(R)] = E(R)$ by (2.3) so that $[E(R), U'(\mathfrak{q})] \subset E(\mathfrak{q})$; the opposite inclusion follows from (2.3). ∎

(2.5) PROPOSITION. *If G is a normal subgroup of $E(R)$ containing an elementary automorphism $\sigma \neq 1$, then $G \supset E(\mathfrak{q})$ for some $\mathfrak{q} \neq 0$.*

Proof. We first show this in the special case $\sigma = 1+h_0$ where $h_0\colon e_jA \to e_iA$. Say $h_0(e_j) = e_iq$; to prove that $E(qA) \subset G$, it will suffice to show that every qA-elementary automorphism $1+h \in G$. Suppose $h\colon V \to e_mA$ with $h(e_k) = e_mqh_k$ for $e_k \in V$. The argument used in Case (ii) of (2.3) shows that it suffices to consider two cases:

(i) $e_i, e_j \notin V$, (ii) $V = e_iA$ or e_jA.

Case (i). Define $1+h_1 \in E(R)$ by $h_1(e_k) = e_jh_k$ for $e_k \in V$. If $m = i$, we have $1+h = [1+h_0, 1+h_1] \in G$. If $m \neq i$, define $1+h_2 \in E(R)$ by $h_2(e_i) = e_m$; then $1+h = [1+h_2, [1+h_0, 1+h_1]] \in G$.

Case (ii). Suppose $V = e_iA$. Choose an index $k \neq i, j, m$ and define $1+h_1$, $1+h_2 \in E(R)$ by $h_1(e_k) = e_mqh_i$ and $h_2(e_i) = e_k$. Then $1+h_1 \in G$ by Case (i) and so $1+h = [1+h_1, 1+h_2] \in G$. The argument for $V = e_jA$ is similar.

In general, suppose $\sigma = 1+h$, with $h\colon V \to e_iA$. Choose $e_j \in V$ for which $h(e_j) \neq 0$ and also some index $k \neq i, j$. Define $1+h_1 \in E(R)$ by $h_1(e_k) = e_j$; then $1 \neq \sigma' = [1+h_1, \sigma]$ is in G and falls under the special case considered above. ∎

3. **The main theorem.** We seek to characterise those subgroups of $U(R)$ which are normalised by $E(R)$. Among such, of course, are the normal subgroups of $U(R)$. We shall denote the center of $U(R)$ by Z.

(3.1) PROPOSITION. *Suppose $G \not\subset Z$ is a subgroup of $U(R)$ normalized by $E(R)$. Then $G \supset E(\mathfrak{q})$ for some $\mathfrak{q} \neq 0$.*

Proof. We may assume $G \supset Z$ since from $E(\mathfrak{q}) \subset G\cdot Z$ follows, by (2.3), $E(\mathfrak{q}) = [E(R), E(\mathfrak{q})] \subset [E(R), G\cdot Z] \subset G$. It suffices to prove that G contains an elementary automorphism $\neq 1$, for then $G \cap E(R)$, being a normal subgroup of $E(R)$, will contain some $E(\mathfrak{q})$ for $\mathfrak{q} \neq 0$ by (2.5).

Take any $\sigma \in G \backslash Z$ and any $e_i \in M$; choose an index $j \neq i$ such that e_j appears with coefficient zero in $\sigma(e_i)$. For any $k \neq j$ define $1+h \in E(R)$ by $h(e_j) = e_k$. Form $\tau = [1+h, \sigma] \in G$; then $\tau(e_i) = e_i$. Suppose $\tau = 1$ for all possible k; then $\sigma(e_k) = \sigma h(e_j) = h\sigma(e_j)$, which implies that for some $u \in U(A)$, $\sigma(e_k) = e_k u$ if $k \neq j$ while $\sigma(e_j) = e_j \cdot u + \cdots$. This means that $1 \neq u^{-1}\sigma \in G$ is an elementary automorphism.

If $\tau \neq 1$ for some choice of k, we may instead assume a priori that $\sigma(e_i) = e_i$. For any $k \neq i$, define $1+h \in E(R)$ by $h(e_k) = e_i$; then $\tau = [1+h, \sigma] \in G$ is an elementary automorphism. If $\tau \neq 1$, we are finished; if $\tau = 1$ for all possible k, an easy calculation shows that σ itself is an elementary automorphism. ∎

(3.2) THEOREM. *The following are equivalent*:

(i) *G is a subgroup of $U(R)$ normalised by $E(R)$.*

(ii) *There exists a unique ideal $\mathfrak{q}$ such that $E(\mathfrak{q}) \subset G \subset U'(\mathfrak{q})$.*

Proof. Choose $\mathfrak{q}$ maximal w.r.t. the property $E(\mathfrak{q}) \subset G$. Suppose $G \not\subset U'(\mathfrak{q})$; then the image $\bar{G}$ of G in $U(R/(\mathfrak{q}))$ will not be in the center. In view of (1.1), we may apply (3.1) to $A/\mathfrak{q}$ and conclude that $\bar{G} \supset E(\mathfrak{q}'/\mathfrak{q})$ for some $\mathfrak{q}' \supsetneq \mathfrak{q}$; lifting to A, we have $E(\mathfrak{q}') \subset U(\mathfrak{q}) \cdot G$. Now by (2.3) and (2.4), $E(\mathfrak{q}') = [E(R), E(\mathfrak{q}')] \subset [E(R), U(\mathfrak{q}) \cdot G] \subset G$, contradicting the maximality of $\mathfrak{q}$. Therefore $G \subset U'(\mathfrak{q})$.

If $E(\mathfrak{q}) \subset G \subset U'(\mathfrak{q})$, then by (2.3) and (2.4) we have

$$E(\mathfrak{q}) = [E(R), E(\mathfrak{q})] \subset [E(R), G] \subset [E(R), U'(\mathfrak{q})] \subset E(\mathfrak{q}) \subset G$$

so that $[E(R), G] = E(\mathfrak{q})$. This shows that $\mathfrak{q}$ is unique and (ii) $\Rightarrow$ (i). ∎

Before stating the next result, we remark that the center of $E(R)$ is 1. Indeed, the usual argument (when I is finite) shows that a central element is a homothety; however[1], it is also clear that an element of $E(R)$ is of the form

$$\begin{pmatrix} 1_V & 0 \\ * & * \end{pmatrix}$$

w.r.t. some decomposition $M = V \oplus W$ into elementary submodules, with W finitely generated, so that 1 is the only homothety in $E(R)$.

(3.3) COROLLARY. *The following are equivalent*:

(i) *G is a normal subgroup of $E(R)$.*

(ii) *There exists a unique ideal $\mathfrak{q}$ such that $E(\mathfrak{q}) \subset G \subset E(R) \cap U(\mathfrak{q})$.*

The groups $\delta(\mathfrak{q}) = E(R) \cap U(\mathfrak{q})/E(\mathfrak{q})$ are all abelian.

Proof. Suppose G is normal in $E(R)$; (3.2) provides a unique ideal $\mathfrak{q}$ such that $E(\mathfrak{q}) \subset G \subset E(R) \cap U'(\mathfrak{q})$; to show (i) $\Rightarrow$ (ii), it therefore suffices to prove

$$E(R) \cap U'(\mathfrak{q}) = E(R) \cap U(\mathfrak{q}). \tag{3.4}$$

[1] The author is indebted to the referee for this argument in place of the original fallacy.

The projection $A \to A/\mathfrak{q}$ induces a group morphism $E(R) \to E(R/(\mathfrak{q}))$; since the center of $E(R/(\mathfrak{q}))$ is 1, we have (3.4).

Both (ii) $\Rightarrow$ (i) and the commutativity of $\delta(\mathfrak{q})$ are implied by (2.4). ▮

(3.5) COROLLARY. *If $\mathfrak{q}$ is a maximal ideal of A, the group $E(R)/E(R) \cap U(\mathfrak{q})$ is simple.* ▮

BIBLIOGRAPHY

1. H. Bass, *K-theory and stable algebra*, Inst. Hautes Études Sci. Publ. Math. No. 22 (1964), 5–60. MR **30** #4805.

2. ———, *Algebraic K-theory*, Benjamin, New York, 1968.

3. A. Rosenberg, *The structure of the infinite general linear group*, Ann. of Math. (2) **68** (1958), 278–294. MR **21** #1319.

4. W. R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, N. J., 1964. MR **29** #4785.

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