

## ON THE GEOMETRIC MEANS OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES<sup>(1)</sup>

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**Abstract.** Let  $f(z_1, \dots, z_n)$  be an entire function of the  $n (\geq 2)$  complex variables  $z_1, \dots, z_n$ , holomorphic for  $|z_t| \leq r_t, t=1, \dots, n$ . We have considered the case of only two complex variables for simplicity. Recently many authors have defined the arithmetic means of the function  $|f(z_1, z_2)|$  and have investigated their properties. In the present paper, the geometric means of the function  $|f(z_1, z_2)|$  have been defined and the asymptotic behavior of certain growth indicators for entire functions of several complex variables have been studied and the results are given in the form of theorems.

1. Let

$$f(z_1, \dots, z_n) = \sum_{k_1, \dots, k_n \geq 0} a_{k_1 \dots k_n} z_1^{k_1} \cdots z_n^{k_n}$$

be an entire function of the  $n (\geq 2)$  complex variables  $z_1, \dots, z_n$ , holomorphic for  $|z_t| \leq r_t, t=1, \dots, n$ . Let us denote the maximum modulus of the function  $f(z_1, \dots, z_n)$  as

$$M(r_1, \dots, r_n) = \max_{|z_t| \leq r_t} |f(z_1, \dots, z_n)| \quad (t = 1, \dots, n).$$

Here we consider the case of only two complex variables for simplicity. The results can easily be extended to several complex variables.

The geometric mean of  $|f(z_1, z_2)|$  for  $|z_t| \leq r_t (t=1, 2)$  has been defined as [4]

$$(1.1) \quad G(r_1, r_2) = \exp \left\{ \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| d\theta_1 d\theta_2 \right\}.$$

Further, let us define

$$(1.2) \quad g_k(r_1, r_2) = \exp \left\{ \frac{(k+1)^2}{(r_1 r_2)^{k+1}} \int_0^{r_1} \int_0^{r_2} (x_1 x_2)^k \log G(x_1, x_2) dx_1 dx_2 \right\},$$

where  $k$  is any positive number.

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The finite order  $\rho$  of an entire function  $f(z_1, z_2)$  is defined as [2, p. 219]

$$\limsup_{r_1, r_2 \rightarrow \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} = \rho.$$

Similarly, we can define the lower order  $\lambda$  as

$$\liminf_{r_1, r_2 \rightarrow \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} = \lambda,$$

where  $\lambda = \min (\lambda_1, \lambda_2)$  and

$$\lambda_1 = \liminf_{r_2 \rightarrow \infty} \liminf_{r_1 \rightarrow \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)},$$

$$\lambda_2 = \liminf_{r_1 \rightarrow \infty} \liminf_{r_2 \rightarrow \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)}.$$

In this paper we have investigated a few properties of the above defined mean values  $G(r_1, r_2)$  and  $g_k(r_1, r_2)$ ; the results are given in the form of theorems.

2. THEOREM 1. *Let  $f(z_1, z_2)$  be an entire function of finite order  $\rho$  and lower order  $\lambda$  (nonintegral with respect to  $z_1$  and  $z_2$ ), then*

$$(2.1) \quad \lim_{r_1, r_2 \rightarrow \infty} \left\{ \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log g_k(r_1, r_2)}{\log (r_1 r_2)} \right\} = \begin{cases} \rho. \\ \lambda \end{cases}$$

**Proof.** From (1.1), it follows that  $G(r_1, r_2)$  is an increasing function of (i)  $r_1$  for given  $r_2$ , (ii)  $r_2$  for a given  $r_1$ , and (iii)  $r_1$  and  $r_2$  both increasing.

Next from (1.1) we have

$$(2.2) \quad \int_0^{r_1} \int_0^{r_2} (x_1 x_2)^k \log G(x_1, x_2) dx_1 dx_2 \leq \frac{(r_1 r_2)^{k+1}}{(k+1)^2} \log M(r_1, r_2).$$

Hence

$$(2.3) \quad \lim_{r_1, r_2 \rightarrow \infty} \left\{ \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log g_k(r_1, r_2)}{\log (r_1 r_2)} \right\} \leq \lim_{r_1, r_2 \rightarrow \infty} \left\{ \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} \right\} = \begin{cases} \rho. \\ \lambda \end{cases}$$

Further, if  $f(z_1, z_2)$  is analytic in  $|z_1| \leq R_1, |z_2| \leq R_2, z_j = r_j e^{i\theta_j}$ , and  $\zeta_j = R_j e^{i\phi_j}$ , ( $j=1, 2$ ), then

$$(2.4) \quad \log |f(z_1, z_2)| \leq \int_0^{2\pi} \int_0^{2\pi} P(z_1, \zeta_1) P(z_2, \zeta_2) \log |f(\zeta_1, \zeta_2)| d\phi_1 d\phi_2,$$

where

$$P(z_s, \zeta_s) = \frac{1}{2\pi} \left\{ \frac{R_s^2 - r_s^2}{R_s^2 - 2r_s R_s \cos(\phi_s - \theta_s) + r_s^2} \right\} \quad (s = 1, 2).$$

Using (2.4) for an entire function  $f(z_1, z_2)$ , we get

$$(2.5) \quad \log |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| \leq \frac{R_1 + r_1}{R_1 - r_1} \frac{R_2 + r_2}{R_2 - r_2} \log G(R_1, R_2).$$

Taking  $R_1 = \alpha r_1, R_2 = \alpha r_2$  ( $\alpha > 1$ ), we have from (2.5)

$$\log M(r_1, r_2) \leq \{(\alpha + 1)/(\alpha - 1)\}^2 \log G(\alpha r_1, \alpha r_2).$$

Therefore

$$\begin{aligned} \log g_k(\alpha r_1, \alpha r_2) &\geq \{(\alpha - 1)/(\alpha + 1)\}^2 \{1 - 1/\alpha^{k+1}\}^2 \log M(r_1/\alpha, r_2/\alpha) \\ &= H\{\log M(r_1/\alpha, r_2/\alpha)\}, \end{aligned}$$

where  $H > 0$  and is independent of  $r_1$  and  $r_2$ . Hence

$$(2.6) \quad \lim_{r_1, r_2 \rightarrow \infty} \left\{ \frac{\sup \log \log g_k(r_1, r_2)}{\inf \log(r_1 r_2)} \right\} \geq \lim_{r_1, r_2 \rightarrow \infty} \left\{ \frac{\sup \log \log M(r_1, r_2)}{\inf \log(r_1 r_2)} \right\} = \begin{cases} \rho \\ \lambda \end{cases}$$

Combining (2.3) and (2.6), we get (2.1).

3. Let  $\phi(r_1, r_2)$  be a "slowly changing" function; that is,  $\phi(r_1, r_2) > 0$  and is continuous for  $r_1 > r_1^0, r_2 > r_2^0$  and for every constant  $l, m > 0, \phi(lr_1, mr_2) \sim \phi(r_1, r_2)$  as  $r_1$  or  $r_2$ , or  $r_1$  and  $r_2$ , tend to infinity.

Also let

$$(3.1) \quad \lim_{r_1, r_2 \rightarrow \infty} \left\{ \frac{\sup \log g_k(r_1, r_2)}{\inf (r_1 r_2)^\rho \phi(r_1, r_2)} \right\} = \begin{cases} p \\ q \end{cases} \quad (0 < q \leq p < \infty),$$

and

$$(3.2) \quad \lim_{r_1, r_2 \rightarrow \infty} \left\{ \frac{\sup \log G(r_1, r_2)}{\inf (r_1 r_2)^\rho \phi(r_1, r_2)} \right\} = \begin{cases} c \\ d \end{cases} \quad (0 < d \leq c < \infty).$$

**THEOREM 2.** *If  $f(z_1, z_2)$  be an entire function of finite nonzero order  $\rho$ , then*

$$(3.3) \quad \{(k + 1)/(k + \rho + 1)\}^2 d \leq q \leq p \leq \{(k + 1)/(k + \rho + 1)\}^2 c;$$

$$(3.4) \quad c \leq p \{1 + \rho/(k + 1)\}^2 \{1 + (k + 1)/\rho\}^{2\rho/(k + 1)};$$

$$(3.5) \quad c + \{\rho(2k + 3\rho + 2)/(k + \rho + 1)^2\} d \leq p \{1 + \rho/(k + 1)\}^2 \{1 + (k + 1)/\rho\}^{2\rho/(k + 1)}.$$

**Proof.** For  $0 < \eta < 1$ , from (1.2) we have

$$\begin{aligned} (3.6) \quad &\log g_k(r_1 + \eta r_1, r_2 + \eta r_2) \\ &= \frac{(k + 1)^2}{(r_1 r_2)^{k+1} (1 + \eta)^{2(k+1)}} \int_0^{r_1 + \eta r_1} \int_0^{r_2 + \eta r_2} (x_1 x_2)^k \log G(x_1, x_2) dx_1 dx_2 \\ (3.7) \quad &< \frac{A}{(r_1 r_2)^{k+1}} + \left(\frac{r_1^0}{r_1}\right)^{k+1} \left\{ \frac{\log G(r_1^0, r_2)}{(1 + \eta)^{2(k+1)}} + \frac{((1 + \eta)^{k+1} - 1) \log G(r_1^0, r_2 + \eta r_2)}{(1 + \eta)^{2(k+1)}} \right\} \\ &+ \left(\frac{r_2^0}{r_2}\right)^{k+1} \left\{ \frac{\log G(r_1, r_2^0)}{(1 + \eta)^{2(k+1)}} + \frac{((1 + \eta)^{k+1} - 1) \log G(r_1 + \eta r_1, r_2^0)}{(1 + \eta)^{2(k+1)}} \right\} \\ &+ \frac{(k + 1)((1 + \eta)^{k+1} - 1)}{(1 + \eta)^{2(k+1)}} \left\{ \frac{1}{r_1^{k+1}} \int_{r_1^0}^{r_1} x_1^k \log G(x_1, r_2 + \eta r_2) dx_1 \right. \\ &\quad \left. + \frac{1}{r_2^{k+1}} \int_{r_2^0}^{r_2} x_2^k \log G(r_1 + \eta r_1, x_2) dx_2 \right\} \end{aligned}$$

$$\begin{aligned}
 & + \frac{(c + \varepsilon)(k + 1)^2}{(r_1 r_2)^{k+1}(1 + \eta)^{2(k+1)}} \int_{r_1^0}^{r_1} \int_{r_2^0}^{r_2} (x_1 x_2)^{k+\rho} \phi(x_1, x_2) dx_1 dx_2 \\
 & + \frac{((1 + \eta)^{k+1} - 1)^2}{(1 + \eta)^{2(k+1)}} \log G(r_1 + \eta r_1, r_2 + \eta r_2),
 \end{aligned}$$

where  $A$  is a constant,

$$\begin{aligned}
 (3.8) \quad & \sim \frac{A}{(r_1 r_2)^{k+1}} + \left(\frac{r_1^0}{r_1}\right)^{k+1} \left\{ \frac{\log G(r_1^0, r_2)}{(1 + \eta)^{2(k+1)}} + \frac{((1 + \eta)^{k+1} - 1) \log G(r_1^0, r_2 + \eta r_2)}{(1 + \eta)^{2(k+1)}} \right\} \\
 & + \left(\frac{r_2^0}{r_2}\right)^{k+1} \left\{ \frac{\log G(r_1, r_2^0)}{(1 + \eta)^{2(k+1)}} + \frac{((1 + \eta)^{k+1} - 1) \log G(r_1 + \eta r_1, r_2^0)}{(1 + \eta)^{2(k+1)}} \right\} \\
 & + \left\{ \frac{k + 1}{k + \rho + 1} \right\} \frac{(c + \varepsilon)(r_1 r_2)^\rho ((1 + \eta)^{k+1} - 1)}{(1 + \eta)^{2(k+1)}} \\
 & \qquad \qquad \qquad \cdot \{ \phi(r_1, r_2 + \eta r_2) + \phi(r_1 + \eta r_1, r_2) \} \\
 & + \left\{ \frac{k + 1}{k + \rho + 1} \right\}^2 \frac{(c + \varepsilon)(r_1 r_2)^\rho \phi(r_1, r_2)}{(1 + \eta)^{2(k+1)}} \\
 & + \frac{((1 + \eta)^{k+1} - 1)^2}{(1 + \eta)^{2(k+1)}} \log G(r_1 + \eta r_1, r_2 + \eta r_2)
 \end{aligned}$$

from [3, Lemma 5].

Hence, dividing both the sides by  $\{(r_1 + \eta r_1)^\rho (r_2 + \eta r_2)^\rho \phi(r_1 + \eta r_1, r_2 + \eta r_2)\}$  and taking limit, we get

$$\begin{aligned}
 (3.9) \quad \limsup_{r_1, r_2 \rightarrow \infty} \left\{ \frac{\log g_k(r_1, r_2)}{(r_1 r_2)^\rho \phi(r_1, r_2)} \right\} & \leq c \left[ \left\{ \frac{k + 1}{k + \rho + 1} \right\}^2 \frac{1}{(1 + \eta)^{2(k + \rho + 1)}} \right. \\
 & \left. + 2 \left\{ \frac{k + 1}{k + \rho + 1} \right\} \frac{((1 + \eta)^{k+1} - 1)}{(1 + \eta)^{2(k + \rho + 1)}} + \frac{((1 + \eta)^{k+1} - 1)^2}{(1 + \eta)^{2(k+1)}} \right].
 \end{aligned}$$

Since  $\eta$  is arbitrary, we get

$$(3.10) \quad p \leq \{(k + 1)/(k + \rho + 1)\}^2 c.$$

Next, from (3.6), we have

$$\begin{aligned}
 (3.11) \quad & \log g_k(r_1 + \eta r_1, r_2 + \eta r_2) \\
 & > \frac{(d - \varepsilon)(k + 1)^2}{(r_1 r_2)^{k+1}(1 + \eta)^{2(k+1)}} \int_{r_1^0}^{r_1} \int_{r_2^0}^{r_2} (x_1 x_2)^{k+\rho} \phi(x_1, x_2) dx_1 dx_2 \\
 & + \frac{(d - \varepsilon)(k + 1)((1 + \eta)^{k+1} - 1)}{(1 + \eta)^{2(k+1)}} \left\{ \frac{r_2^0}{r_1^{k+1}} \int_{r_1^0}^{r_1} x_1^{k+\rho} \phi(x_1, r_2) dx_1 \right. \\
 & \qquad \qquad \qquad \left. + \frac{r_1^0}{r_2^{k+1}} \int_{r_2^0}^{r_2} x_2^{k+\rho} \phi(r_1, x_2) dx_2 \right\} \\
 & + \frac{((1 + \eta)^{k+1} - 1)^2}{(1 + \eta)^{2(k+1)}} \log G(r_1, r_2)
 \end{aligned}$$

$$\sim \frac{(d-\epsilon)(r_1 r_2)^\rho \phi(r_1, r_2)}{(1+\eta)^{2(k+1)}} \left[ \left\{ \frac{k+1}{k+\rho+1} \right\}^2 + 2 \left\{ \frac{k+1}{k+\rho+1} \right\} \{(1+\eta)^{k+1} - 1\} \right] + \left[ \frac{\{(1+\eta)^{k+1} - 1\}}{(1+\eta)^{2(k+1)}} \right] \log G(r_1, r_2),$$

using [3, Lemma 5].

Hence

$$(3.12) \quad q \geq \frac{d}{(1+\eta)^{2(k+\rho+1)}} \left[ \left\{ \frac{k+1}{k+\rho+1} \right\} + \{(1+\eta)^{k+1} - 1\} \right]^2.$$

Since  $\eta$  is arbitrary, we get

$$(3.13) \quad q \geq \{(k+1)/(k+\rho+1)\}^2 d.$$

Combining (3.10) and (3.13), the inequality (3.3) follows. Also, from (3.11)

$$\{(1+\eta)^{k+1} - 1\}^2 c \leq p(1+\eta)^{2(k+\rho+1)} - d \left[ \left\{ \frac{k+1}{k+\rho+1} \right\}^2 + 2 \left\{ \frac{k+1}{k+\rho+1} \right\} \{(1+\eta)^{k+1} - 1\} \right],$$

and so for all  $\eta > 0$

$$(3.14) \quad c \leq p(1+\eta)^{2(k+\rho+1)} / \{(1+\eta)^{k+1} - 1\}^2.$$

The right-hand side of this inequality has the least value when  $(1+\eta)^{k+1} = \{1 + (k+1)/\rho\}$ , therefore

$$c \leq p \{1 + \rho/(k+1)\}^2 \{1 + (k+1)/\rho\}^{2\rho/(k+1)}.$$

Also, from (3.14) we get

$$c + \frac{\rho(2k+3\rho+2)}{(k+\rho+1)^2} d \leq p \{1 + \rho/(k+1)\}^2 \{1 + (k+1)/\rho\}^{2\rho/(k+1)},$$

which completes the proof of the theorem.

4. THEOREM 3. Let  $f(z_1, z_2)$  be an entire function of finite order  $\rho$  and if

$$(4.1) \quad \lim_{r_1, r_2 \rightarrow \infty} \left\{ \begin{array}{l} \sup \log g_k(r_1, r_2) \\ \inf (r_1 r_2)^\rho \phi(r_1, r_2) \end{array} \right\} = \left\{ \begin{array}{l} p \\ q \end{array} \right\},$$

and

$$(4.2) \quad \log G(r_1, r_2) \sim \beta(r_1 r_2)^\rho \phi(r_1, r_2),$$

for large values of  $r_1$  and  $r_2$ , where  $\beta$  is a constant and  $0 < q \leq p < \infty$  and  $\phi(r_1, r_2)$  as defined in §3. Then

- (i)  $f(z_1, z_2)$  is of regular growth;
- (ii)  $(k+\rho+1)^2 p = (k+1)^2 \beta = (k+\rho+1)^2 q$ ; and
- (iii)  $\lim_{r_1, r_2 \rightarrow \infty} \log g_k(r_1, r_2) / \log G(r_1, r_2) = \{(k+1)/(k+\rho+1)\}^2$ .

**Proof.** (i) Taking the logarithm of (4.2), the result follows.

(ii) From (1.2), we have

$$(4.3) \quad \log g_k(r_1, r_2) < \frac{A}{(r_1 r_2)^{k+1}} + \left\{ \frac{r_1^0}{r_1} \right\}^{k+1} \log G(r_1^0, r_2) + \left\{ \frac{r_2^0}{r_2} \right\}^{k+1} \log G(r_1, r_2^0) \\ + \frac{\beta(k+1)^2}{(r_1 r_2)^{k+1}} \int_{r_1^0}^{r_1} \int_{r_2^0}^{r_2} (x_1 x_2)^{k+\rho} \phi(x_1, x_2) dx_1 dx_2 \\ \text{from (4.2) and } A \text{ is a constant,} \\ \sim \frac{A}{(r_1 r_2)^{k+1}} + \left\{ \frac{r_1^0}{r_1} \right\}^{k+1} \log G(r_1^0, r_2) + \left\{ \frac{r_2^0}{r_2} \right\}^{k+1} \log G(r_1, r_2^0) \\ + \beta \left\{ \frac{k+1}{k+\rho+1} \right\}^2 (r_1 r_2)^\rho \phi(r_1, r_2) \\ \text{by repeated application of [3, Lemma 5].}$$

Taking the limit, we get

$$(4.4) \quad \lim_{r_1, r_2 \rightarrow \infty} \frac{\log g_k(r_1, r_2)}{(r_1 r_2)^\rho \phi(r_1, r_2)} \leq \beta \left\{ \frac{k+1}{k+\rho+1} \right\}^2.$$

Also, from (1.2), we have

$$(4.5) \quad \log g_k(r_1, r_2) > \frac{(k+1)^2}{(r_1 r_2)^{k+1}} \int_{r_1^0}^{r_1} \int_{r_2^0}^{r_2} (x_1 x_2)^k \log G(x_1, x_2) dx_1 dx_2 \\ \sim \beta \left\{ \frac{k+1}{k+\rho+1} \right\}^2 (r_1 r_2)^\rho \phi(r_1, r_2) \text{ from (4.2).}$$

Hence

$$(4.6) \quad \lim_{r_1, r_2 \rightarrow \infty} \frac{\log g_k(r_1, r_2)}{(r_1 r_2)^\rho \phi(r_1, r_2)} \geq \beta \left\{ \frac{k+1}{k+\rho+1} \right\}^2.$$

Combining (4.4) and (4.6), the result follows.

(iii) Further, if we divide by  $\log G(r_1, r_2)$ , a positive increasing function, the inequalities (4.3) and (4.5) respectively, we get

$$(4.7) \quad \frac{\log g_k(r_1, r_2)}{\log G(r_1, r_2)} < \frac{1}{\log G(r_1, r_2)} \left[ \frac{A}{(r_1 r_2)^{k+1}} + \left\{ \frac{r_1^0}{r_1} \right\}^{k+1} \log G(r_1^0, r_2) \right. \\ \left. + \left\{ \frac{r_2^0}{r_2} \right\}^{k+1} \log G(r_1, r_2^0) \right] \\ + \beta \left\{ \frac{k+1}{k+\rho+1} \right\}^2 \frac{(r_1 r_2)^\rho \phi(r_1, r_2)}{\log G(r_1, r_2)}$$

and

$$(4.8) \quad \frac{\log g_k(r_1, r_2)}{\log G(r_1, r_2)} > \beta \left\{ \frac{k+1}{k+\rho+1} \right\}^2 \frac{(r_1 r_2)^\rho \phi(r_1, r_2)}{\log G(r_1, r_2)}.$$

Since (4.2) holds, on proceeding to limits, the inequalities (4.7) and (4.8) lead to the result.

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