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APPLICATIONS OF THE TUMURA-CLUNIE THEOREM

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Abstract. Some applications of the Tumura-Clunie theorem are given. Most of these concerned fixed points of compositions of entire functions.

I. **Introduction.** The purpose of this note is to develop some applications of the Tumura-Clunie theory (Hayman [2, p. 69]). It is assumed that the reader is familiar with the basic quantities of Nevanlinna Theory: $T(r,f)$, $m(r,f)$, $N(r,f),\dots$ and their elementary properties.

II. **Notation, terminology.** 1. f will always denote a nonconstant meromorphic function.

2. We shall denote by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$, as $r\to+\infty$, possibly outside a set of r of finite measure.

3. We shall denote by $a(z)(a_1(z),\dots)$ meromorphic functions satisfying $T(r, a(z))=S(r,f)$.

4. By a differential polynomial $P(f)$ we understand a polynomial in $f(z)$ and its derivatives whose coefficients are of the form $a(z)$.

III. **Statement and applications of the Tumura-Clunie theorem.**

Theorem 1 (Tumura-Clunie [2]). *Suppose $g(z)=f^n(z)+P_{n-1}(f)$, and that*

$$N(r,f)+N(r,1/g) = S(r,f). \tag{1}$$

Then $g(z)=h^n(z)$, $h(z)=f(z)+a(z)/n$ and $h^{n-1}(z)a(z)$ is obtained by substituting $h(z)$ for $f(z)$, $h'(z)$ for $f'(z)$, etc. in the terms of degree $n-1$ in $P_{n-1}(f)$.

Remark. The conclusion still holds good if the condition (1) is replaced by:

$$N(r,f)+N(r,1/g)=S_0(r,f),$$

where $S_0(r,f)$ denotes any quantity which satisfies $S_0(r,f)=o(T(r,f))$ as $r\to+\infty$ through a set of r of infinite measure.

We have

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COROLLARY (1-1). *Let $A(z)$, $B(z)$ be entire functions, $f(z)=e^{A(z)}$, $T(r,f)=O(T(r, e^{B(z)}))$. Then the identity*

$$a_1(z)e^{A(z)}+a_2(z)e^{B(z)} = a_3(z) \tag{1-1}$$

cannot hold for an a_3 which is not identically equal to 0.

Proof. The result follows immediately if $a_1(z)\equiv 0$ or $a_2(z)\equiv 0$. Therefore we may assume that both $a_1(z)$ and $a_2(z)$ are not identically equal to zero.

Multiplying both sides of (1-1) by $a_1(z)$, we have

$$a_1^2(z)e^{A(z)}-a_1(z)a_3(z) = -a_1(z)a_2(z)e^{B(z)}. \tag{1-2}$$

Setting $a_1(z)\exp[A(z)/2]=H(z)$, $-a_1(z)a_2(z)e^{B(z)}=g(z)$, we have

$$H^2(z)-a_1(z)a_3(z) = g(z). \tag{1-3}$$

We note that

$$\begin{aligned} N(r, H)+N(r, 1/g) &= N(r, a_1(z))+N(r, 1/a_1(z)a_2(z)) \\ &= S(r, e^{A(z)}) = S(r, H). \end{aligned} \tag{1-4}$$

Therefore, the Tumura-Clunie theorem is applicable to the identity (1-3), and leads to

$$H^2(z) = g(z). \tag{1-5}$$

This forces $a_1(z)a_3(z)\equiv 0$, therefore $a_3(z)\equiv 0$. In order to state Theorem 2, we need

DEFINITION 1. To each function $\lambda(r)$, positive, continuous and nondecreasing on $0<r<+\infty$, we associate the class F_λ of meromorphic functions f satisfying $T(r,f)=O(\lambda(r))$, as $r\to\infty$ for all r except a set of r of finite measure. It is easily verified that F_λ is a field.

THEOREM 2. *Let f be a meromorphic function with*

$$N(r,f) = S(r,f) \tag{2-1}$$

satisfying

$$f^n(z)+P_{n-1}(f) = b(z), \tag{2-2}$$

where $P_{n-1}(f)$ is a differential polynomial in f of degree at most $n-1$, and all the coefficients in $P_{n-1}(f)$ belong to F_λ and $b(z)\in F_\lambda$, $b(z)\not\equiv 0$. Then $f\in F_\lambda$.

Proof. Assume the statement is false, i.e., $f\notin F_\lambda$. This implies that $T(r, b(z))=S_0(r,f)$.

Combining this with the hypothesis on $N(r,f)$, we have

$$N(r,f)+N(r, 1/b(z)) = S_0(r,f). \tag{2-3}$$

Therefore, Theorem 1 is applicable by the remark made after Theorem 1. Thus we obtain $(f(z)+a(z))^n = b(z)$. Combined with $T(r, a(z)) = S(r, f)$, $T(r, b(z)) = S_0(r, f)$ this leads to the contradiction $T(r, f) = S_0(r, f)$. Thus the theorem is proved.

A consequence of Theorem 2 is

COROLLARY (2-1) (HELLERSTEIN AND RUBEL [4]). *F_λ is algebraically closed in the field of all meromorphic functions.*

The following theorem generalizes a result of Rosenbloom [3].

THEOREM 3. *Let f be a transcendental meromorphic function with $N(r, f) = S(r, f)$. Then $R(f)$ has infinitely many fixed points, where $R(z) = P(z)/Q(z)$, a rational function, and $P(z)$, $Q(z)$ are two relative prime polynomials with degree of $P(z) \geqq 2$.*

Proof. Assume $R(z) - z$ has finitely many fixed points. That is

$$\frac{P(f) - zQ(f)}{Q(f)} \tag{3-1}$$

has finitely many zeros.

Since $P(z)$, $Q(z)$ are relatively prime, the zeros of $R(z)-z$ are just the zeros of $P(f) - zQ(f)$. Thus

$$P(f) - zQ(f) = A(z)e^{B(z)}, \tag{3-2}$$

where $A(z)$ is a polynomial and $B(z)$ is an entire function. Consequently,

$$(\alpha_0 z + \beta_0) f^n(z) + (\alpha_1 z + \beta_1) f^{n-1}(z) + \cdots + (\alpha_n z + \beta_n) f(z) + \alpha z + \beta = A(z)e^{B(z)}, \tag{3-3}$$

where n is an integer $\geqq 2$, α_i $(i = 0, 1, 2, \ldots, n)$, β_j $(j = 0, 1, 2, \ldots)$, α, β are constants and it is assumed that at least one of $\alpha_0, \beta_0 \neq 0$.

According to the Tumura-Clunie theorem, we obtain that

$$\left[f(z) + \frac{1}{n}\frac{\alpha_1 z + \beta_1}{\alpha_0 z + \beta_0}\right]^n = \frac{A(z)e^{B(z)}}{\alpha_0 z + \beta_0}. \tag{3-4}$$

Combining (3-3) and (3-4) we obtain

$$R_1(z) f^{n-1}(z) + R_2(z) f^{n-2}(z) + \cdots + R_n(z) f(z) \equiv R_0(z), \tag{3-5}$$

where $R_i(z)$, $i = 1, 2, \ldots, n$, are rational functions and

$$R_0(z) = \frac{\alpha z + \beta}{\alpha_0 z + \beta_0} - \left(\frac{\alpha_1 z + \beta_1}{\alpha_0 z + \beta_0}\right)^n \not\equiv 0.$$

We conclude as before that

$$(f(z) + R(z))^{n-1} = T(z), \tag{3-6}$$

where $R(z)$, $T(z)$ are rational functions. This is impossible because f is transcendental and our proof is completed.

Along the same lines we have

THEOREM 4. *Let $f(z)$, $g(z)$ be two transcendental entire functions and let $f(z)$ have finitely many fixed points. Let $A(z)$ be entire, and suppose that $A(z) \not\equiv g(z)-z$, and that $A(z)$ satisfies the condition*

$$T(r, A(z)) = S(r, f). \tag{4-1}$$

(*In the case $A(z) \equiv 0$ we also formally recognize that the condition* (4-1) *is fulfilled.*) *Then $f(g)(z)-A(z)$ has infinitley many fixed points.*

Proof. Assume $f(g)(z)-A(z)$ has finitely many fixed points, that is

$$f(g)(z)-A(z)-z = P_1(z) \exp [Q_1(z)], \tag{4-2}$$

where $P_2(z)$ is a polynomial and $Q_2(z)$ is an entire function.

On the other hand, according to the hypothesis, we have

$$f(z)-z = P_2(z) \exp [Q_2(z)], \tag{4-3}$$

where $P_2(z)$ is a polynomial and $Q_2(z)$ is an entire function. Consequently,

$$f(g)(z)-g(z) = P_2(g) \exp [Q_2(g)]. \tag{4-4}$$

Combining (4-2) and (4-4) we obtain

$$g(z)-z-A(z) = P_1(z) \exp [Q_1(z)]-P_2(g) \exp [Q_2(g(z))]. \tag{4-5}$$

Since for any two entire transcendental functions f and g

$$T(r, f(g))/\{T(r, f)+T(r, g)\} \to \infty$$

(Hayman [2, pp. 50, 54]), we can apply Corollary (1-1) and conclude

$$g(z)-z-A(z) \equiv 0. \tag{4-6}$$

This contradicts the hypothesis, so our proof is completed.

REMARK. ($f(z)=e^z+z$ shows that the assumption $A(z) \not\equiv g(z)-z$ is crucial.) In the special case $A(z) \equiv 0$, we obtain another result of Rosenbloom [3].

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