A COLLECTION OF SEQUENCE SPACES

BY

J. R. CALDER AND J. B. HILL

Abstract. This paper concerns a collection of sequence spaces we shall refer to as $d_a$ spaces. Suppose $a = (a_1, a_2, \ldots)$ is a bounded number sequence and $a_i \neq 0$ for some $i$. Suppose $\mathcal{P}$ is the collection of permutations on the positive integers. Then $d_a$ denotes the set to which the number sequence $x = (x_1, x_2, \ldots)$ belongs if and only if there exists a number $k > 0$ such that

$$h_a(x) = \operatorname{lub}_{p \in \mathcal{P}} \sum_{i=1}^{m} |x_{\pi(1)}a_1| < k.$$ 

$h_a$ is a norm on $d_a$ and $(d_a, h_a)$ is complete.

We classify the $d_a$ spaces and compare them with $l_1$ and $m$. Some of the $d_a$ spaces are shown to have a semishrinking basis that is not shrinking. Further investigation of the bases in these spaces yields theorems concerning the conjugate space properties of $d_a$. We characterize the sequences $\beta$ such that, given $a$, $d_a = d_\beta$. A class of manifolds in the first conjugate space of $d_a$ is examined. We establish some properties of the collection of points in the first conjugate space of a normed linear space $S$ that attain their maximum on the unit ball in $S$. The effect of renorming $c_0$ and $l_1$ with $h_a$ and related norms is studied in terms of the change induced on this collection of functionals.

Introduction. The $d_a$ spaces were studied by W. L. C. Sargent [6] in 1960 and more recently by W. Ruckle [5] and D. J. H. Garling [2]. Some of the results in §§1 and III appear in one or more of the above papers, as will be indicated.

Throughout this paper if $S$ is a linear space and $g$ is a norm on $S$ then $(S, g)$ will denote $S$ with the norm $g$. The symbol $(S, g)^*$ denotes the first conjugate space of $(S, g)$ and $g^*$ denotes the conjugate norm on $(S, g)^*$ induced by $g$. If $H$ is a subset of $S$ then $L(H)$ denotes the linear span of $H$. The symbol $N(S)$ denotes the origin in $S$ and $U(S, g)$ denotes the unit ball in $(S, g)$. The term basis will refer to a Schauder basis.

I. $d_a$ spaces.

Definition 1.1. Suppose $n$ is a positive integer. Then $x^n$ denotes the number sequence $(x_1, x_2, \ldots)$ such that $x_i = 1$ if $i \leq n$ and $x_i = 0$ otherwise.

Definition 1.2. Suppose $a \in m$. Then $B(a)$ denotes the number sequence $(B_1(a), B_2(a), \ldots)$ defined as follows: for each $i$,

$$B_i(a) = h_a(x^n_i) \quad \text{if } i = 1,$$
$$= h_a(x^n_i) - h_a(x^{n-1}_i) \quad \text{if } i > 1.$$
Definition 1.3. \( Z \) denotes the number sequence collection to which the sequence \( a=(a_1, a_2, \ldots) \) belongs if and only if \( a \) is nonincreasing, \( a_i > 0 \) and for each \( i \), \( a_i \geq 0 \). \( Z_0 = c_0 \cap Z \) and \( Z_1 = l_1 \cap Z \).

The following lemma was found to be very useful in the investigation of the \( d_a \) spaces.

Lemma 1.1. Suppose \( a \in m \). Then \( B(a) \in Z \) and \( d_a = d_{B(a)} \). Moreover if \( x \in d_a \) then \( h_a(x) = h_{B(a)}(x) \).

Thus in the investigation of these spaces we need only consider the sequences in \( Z \).

Theorem 1.1 [2]. \( d_a = m \) if and only if \( a \in l_1 \).

Theorem 1.2 [2]. \( d_a = l_1 \) if and only if \( a \in m - c_0 \).

Thus the spaces fall naturally into three categories, (1) those that are \( l_1 \), (2) those that are \( m \) and (3) those that are "between" \( l_1 \) and \( m \).

Observation. If \( d_a = m \) then \( h_a \) is equivalent to the ordinary norm \( ||\cdot||_m \) on \( m \), and if \( d_a = l_1 \) then \( h_a \) is equivalent to the ordinary norm \( ||\cdot||_1 \) on \( l_1 \).

Definition 1.4. \( e = e_1, e_2, \ldots \) denotes the point sequence in \( m \) such that for each \( i \), \( e_i = (e'_1, e'_2, \ldots) \), \( e'_1 = 1 \) and \( e'_i = 0 \) if \( i \neq j \). If \( e \) is a basis for a normed linear space \( (S, g) \) then \( b = b_1, b_2, \ldots \) denotes the point sequence in \( (S, g)^* \) that is biorthogonal to \( e \). \( G_e \) denotes the closure of the linear span of \( b \). If \( e \) is a basis for \( (S, g) \) and \( f \in (S, g)^* \) then the number sequence \( (f_1, f_2, \ldots) \) is defined by \( f_i = f(e_i) \) for each \( i \).

Definition 1.5. \( T_1 \) denotes the linear transformation from \( (l_1, ||\cdot||_1)^* \) to \( m \) defined by \( T_1(f) = (f_1, f_2, \ldots) \) for each \( f \in (l_1, ||\cdot||_1)^* \).

It is well known that \( T_1 \) is a congruence (isometry) from \( [(l_1, ||\cdot||_1)^*, ||\cdot||_1] \) to \( (m, ||\cdot||_m) \). The following theorem shows that this relationship between \( l_1 \) and \( m \) does not necessarily exist between the \( d_a \) spaces that are \( l_1 \) and those that are \( m \).

Theorem 1.3. Suppose \( a \in Z_1 \). Then each two of the following statements are equivalent.

1. There exists a point \( \beta \in Z - Z_0 \) such that \( T_1 \) is a congruence from \( [(d_\beta, h)^*, h_\beta^*] \) to \( (d_a, h_a) \).
2. \( a_2 = 0 \).
3. There exists a number \( c \) such that if \( x \in d_a \) then \( h_a(x) = c \cdot ||x||_m \).

Proof. Suppose \( \beta \in Z - Z_0 \). \( d_a = m \) and \( d_\beta = l_1 \) and \( h_a \) is equivalent to \( ||\cdot||_m \) and \( h_\beta \) is equivalent to \( ||\cdot||_1 \). Hence \( (d_\beta, h_\beta)^* = (l_1, ||\cdot||_1)^* \) and \( T_1 \) is a reversible linear transformation from \( [(d_\beta, h_\beta)^*, h^*] \) onto \( (d_a, h_a) \). Now suppose \( T_1 \) is a congruence. Suppose further that for each positive integer \( n \), \( f^n = T_1^{-1}(\beta_1, \beta_2, \ldots, \beta_n, 0, 0, \ldots) \). Since \( T_1 \) is a congruence \( h_\alpha(T_1(f^n)) = h_\beta^*(f^n) \). But \( h_\beta^*(f^n) = 1 \) so \( h_\alpha(T_1(f^n)) = \beta_1 a_1 = 1 \). Thus \( \beta_1 = 1/\alpha_1 \) and \( h_\alpha(T_1(f^2)) = \beta_1 a_1 + \beta_2 a_2 = 1 \). Since \( \beta_2 \neq 0 \) then \( \alpha_2 = 0 \) and (1) implies (2).
Suppose now that $\alpha_2 = 0$ and $f = (f_1, f_2, \ldots) \in d_\alpha$. Then $h_\alpha(f) = \alpha_1 \cdot |f|_m$. So (2) implies (3).

Now suppose (3). $e_1 \in d_\alpha$ and $h_\alpha(e_1) = \alpha_1 = c \cdot |e|_m = c$. So $\alpha_1 = c$. Let $\beta = (\beta_1, \beta_2, \ldots)$ such that for each $i$, $\beta_i = 1/\alpha_i$. Then $\beta \in Z - Z_0$, $d_\beta = l_1$ and if $x \in d_\beta$ then $h_\beta(x) = (1/\alpha_1)|x|_1$. So $T_1$ is a congruence.

II. Bases in the $d_\alpha$ spaces. The following are some of the properties that a point sequence $p_1, p_2, \ldots$ may have and are listed here for easy reference.

**Definition 2.1.** Suppose $(S, g)$ is a normed linear space, $W$ is the set of positive integers and $Q$ is the collection of all finite subsets of $W$. Suppose further that $p = p_1, p_2, \ldots$ is a sequence each term of which is a point of $S$.

(i) $p$ is orthogonal means that if each of $H$ and $K$ is in $Q$, $H \subseteq K$ and $a_1, a_2, \ldots$ is a number sequence, then $g(\sum_{i \in H} a_i p_i) \leq g(\sum_{i \in K} a_i p_i)$.

(ii) $p$ is strictly orthogonal means that $p$ is orthogonal and if each of $H$ and $K$ is in $Q$, and $H \subseteq K$ and $a_1, a_2, \ldots$ is a number sequence, then the following two statements are equivalent.

1. $g(\sum_{i \in H} a_i p_i) = g(\sum_{i \in K} a_i p_i)$.
2. $H = K$ or $H \nsubseteq K$ and if $i \in K - H$ then $a_i = 0$.

(iii) $p$ is strictly coorthogonal means that if each of $H$ and $K$ is in $Q$, and $H \supseteq K$ and $a_1, a_2, \ldots$ is a number sequence then

$$g(\sum_{i \in W - K} a_i p_i) \leq g(\sum_{i \in W - H} a_i p_i)$$

and the following two statements are equivalent.

1. $g(\sum_{i \in W - K} a_i p_i) = g(\sum_{i \in W - H} a_i p_i)$.
2. $H = K$, or $H \nsubseteq K$ and if $i \in K - H$ then $a_i = 0$.

(iv) If $p$ is a basis, $p$ is unconditional means that if $x \in S$ and $x = \sum_{i=1}^{\infty} x_i p_i$ and if $r \in P$, then $x = \sum_{i=1}^{\infty} x(i)p_i$.

(v) If $p$ is a basis, $p$ is semishrinking means that there exists a number $c > 0$ such that

1. $0 < \text{glb}_1 (g(p_i)) \leq \text{lub}_1 (g(p_i)) < c$, and
2. if $f \in (S, g)^*$ then $\lim_{n \to \infty} f(p_n) = 0$.

(vi) If $p$ is a basis, $p$ is shrinking means that if $q = q_1, q_2, \ldots$ is the point sequence in $(S, g)^*$ that is biorthogonal to $p$ and $y_1, y_2, \ldots$ is a bounded point sequence in $S$ such that for each $j$, $\lim_{n \to \infty} q_j(y_n) = 0$, then if $f \in (S, g)^*$, $\lim_{n \to \infty} f(y_n) = 0$.

**Theorem 2.1.** Suppose $\alpha \in Z - Z_1$. Then the point sequence $e = e_1, e_2, \ldots$ in $(d_\alpha, h_\alpha)$ has the following properties:

1. $e$ is orthogonal;
2. $e$ is strictly orthogonal;
3. $e$ is strictly coorthogonal;
4. $e$ is a basis;
5. $e$ is unconditional;
6. $e$ is boundedly complete.
That $e$ is orthogonal, strictly orthogonal, and strictly coorthogonal is easily verified. Garling [2] has shown that the linear span of $e$, $L(e)$, is dense in $d_a$ and so it follows that since for each $i$, $e_i \neq N(d_a)$ and since $e$ is orthogonal, that $e$ is an unconditional basis for $d_a$. That $e$ is boundedly complete is obvious.

It may be noted that the collection of $d_a$ spaces can be enlarged as follows: if $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a bounded number sequence and $\alpha_i \neq 0$ for some $i$ and if $k \geq 1$, then $d_{a,k}$ denotes the set to which the number sequence $x = (x_1, x_2, \ldots)$ belongs only in the case that there exists a number $c$ such that

$$h_{a,k}(x) = \operatorname{lub}_{p \in \mathcal{P}} \left[ \sum_{i=1}^{\infty} |x_{p(i)}\alpha_i|^k \right]^{1/k} < c.$$ 

In this case, results similar to Lemma 1.1, Theorem 1.1, Theorem 1.2 and Theorem 2.1 may still be obtained. Theorem 1.1 becomes $d_{a,k} = m$ if and only if $\alpha \in l_k$. Theorem 1.2 becomes $d_{a,k} = l_k$ if and only if $\alpha \in m - c_0$. Again, if $d_{a,k} = m$ then $h_{a,k}$ is equivalent to $| \cdot |_m$ and if $d_{a,k} = l_k$ then $h_{a,k}$ is equivalent to the ordinary norm on $l_k$, $| \cdot |_k$.

Here then, we have spaces some of which are $l_k$, some $m$ and some "between" $l_k$ and $m$. The remainder of this paper deals with the $d_a$ (i.e. $d_{a,1}$) spaces.

A. Pelczyński and W. Szczenk [3], answering a question of I. Singer, constructed an example of a normed linear space with a basis that was semishrinking but not shrinking. J. R. Retherford [4] has shown that the space $(d)$, which is $d_a$ with $\alpha_i = 1/i$, also has a basis that is semishrinking but not shrinking.

**Theorem 2.2.** Suppose that $\alpha \in Z_0 - Z_1$. Then the basis $e$ for $(d_a, h_a)$ is semishrinking but not shrinking.

**Proof.** If $\alpha \in Z_0 - Z_1$ then there exists a point $x = (x_1, x_2, \ldots)$ in $d_a$ such that for each $i$, $x_i \geq x_{i+1} \geq 0$ and $x \not\in l_1$. Suppose $f \in (d_a, h_a)^*$ and that $\lim_{i \to \infty} f_i \neq 0$. Then there exists a number $c > 0$ and a subsequence $f_{n_j}, f_{n_2}, \ldots$ of $f_1, f_2, \ldots$ such that for each $i$, $|f_{n_i}| \geq c$. Let $y = (y_1, y_2, \ldots)$ be the point of $d_a$ such that $y_i = 0$ if $i \neq n_j$ for every $j$ and $y_i = x_i \cdot |f_{n_i}|/|f_{n_j}|$ if $i = n_j$ for some $j$. So if $N$ is a number there exists an integer $s$ such that

$$N < c \cdot \sum_{i=1}^{s} x_i \leq \sum_{i=1}^{s} |f_{n_i}| \cdot x_i = \sum_{i=1}^{n} f_i y_i.$$ 

So $f \not\in (d_a, h_a)^*$ and we have a contradiction. Hence $\lim_{i \to \infty} f_i = 0$. For each $i$, $h_a(e_i) = \alpha_i$ so $e$ is semishrinking.

For each positive integer $n$, let $S_n = \sum_{i=1}^{n} \alpha_i$ and $y_n = (1/S_n) \cdot \sum_{i=1}^{n} e_i$. Then $h_a(y_n) = 1$. Let $F$ denote the point of $(d_a, h_a)^*$ defined as follows: if $x \in d_a$ and $x = (x_1, x_2, \ldots)$, then $F(x) = \sum_{i=1}^{n} x_i \alpha_i$. For each $n$, $F(y_n) = 1$. But if $b = b_1, b_2, \ldots$ is the point sequence in $(d_a, h_a)^*$ biorthogonal to $e$ and if $j$ is a positive integer then $\lim_{n \to \infty} b_j(y_n) = 0$. So $e$ is not shrinking.

**Corollary 2.1.** Suppose $\alpha \in Z$. Then $[(d_a, h_a)^*, h_a^*]$ is not separable.
Proof. It is well known, for instance [1, p. 77], that if \( p=p_1, p_2, \ldots \) is an unconditional basis for a normed linear complete space \( (S, g) \) then \( p \) is shrinking if and only if \( [(S, g)^*, g^*] \) is separable. Thus if \( \alpha \in Z - Z_1 \), since \( e \) is an unconditional basis that is not shrinking we have that \( [(d_a, h_a)^*, h_a^*] \) is not separable. If \( \alpha \in Z_1 \), then \( (d_a, h_a) \) is isomorphic to \( (m, |m|) \) and thus \( [(d_a, h_a)^*, h_a^*] \) is not separable.

Definition 2.2. Suppose \( (S, g) \) is a normed linear space and that \( H \) is a linear manifold in \( (S, g)^* \). \( J^H \) denotes the transformation from \( S \) into \( (H, g^*)^* \) defined as follows: if \( x \in S \) and \( f \in H \) then \( J^H(x)(f) = f(x) \).

Theorem 2.3. Suppose \( \alpha \in Z - Z_1 \). Then \( J^{G_e} \) is a congruence.

Proof. \( (d_a, h_a) \) is complete and \( e \) is an unconditional basis for \( (d_a, h_a) \) that is boundedly complete. Thus it follows from a result of Singer [7] that \( J^{G_e} \) is a congruence.

Theorem 2.4. Suppose \( \alpha \in Z - Z_1 \) and \( b \) is the point sequence in \( (d_a, h_a)^* \) biorthogonal to \( e \). Then \( b \) is

1. orthogonal,
2. a basis for \( (G_e, h_a^*) \),
3. unconditional,
4. not boundedly complete,
5. not strictly orthogonal.

Proof. Since \( e \) is orthogonal \( b \) must be orthogonal and since \( b \) is orthogonal and \( L(b) \) is dense in \( G_e \) and since \( b_i \neq N(d_a, h_a)^* \) for each \( i \), it follows that \( b \) is an unconditional basis for \( G_e \). Since \( [(d_a, h_a)^*, h_a^*] \) is not separable there exists a point \( y \in (d_a, h_a)^* - G_e \). Suppose \( h_a^*(y) = c \). Suppose further that \( n \) is a positive integer and \( y^n = \sum_{i=1}^{n} a_i b_i \). Then \( h_a^*(y^n) \leq c \). But \( y \notin G_e \) so \( b \) is not boundedly complete. Suppose \( n \) is a positive integer and \( \alpha^n = \sum_{i=1}^{n} a_i b_i \). Let \( x = \frac{1}{\alpha} e_1 \). Then \( x \in U(d_a, h_a) \) and \( \alpha^*(x) = 1 \). Suppose \( y = (y_1, y_2, \ldots) \) is a point of \( U(d_a, h_a) \). Then \( |y| = \sum_{i=1}^{n} |y_i| b_i \leq h_a(y) = 1 \). So \( h_a^*(\alpha^n) = 1 \). Hence \( b \) is not strictly orthogonal.

Corollary 2.2. Suppose \( \alpha \in Z - Z_1 \) and \( (S, g) \) is a normed linear complete space. Then \( (G_e, h_a^*) \) is not isomorphic to \( [(S, g)^*, g^*] \).

Proof. Singer has shown [7] that a normed linear complete space \( (S, g) \) with an unconditional basis, \( p \), is isomorphic to the conjugate space of some normed linear space if and only if \( p \) is boundedly complete. Thus Corollary 2.2 follows.

Corollary 2.3. If \( \alpha \in Z \) then \( (d_a, h_a) \) is not reflexive.

In case \( \alpha \in Z_1 \) the question whether or not \( (d_a, h_a) \) is congruent to the conjugate space of some normed linear space is answered by the following theorem.

Theorem 2.5. Suppose \( \alpha \in Z_1 \) and \( g_\alpha \) is the norm on \( l_1 \) defined as follows: if \( x \in l_1 \) and \( x = (x_1, x_2, \ldots) \) then
\[
g_\alpha(x) = \text{lub} \left\{ \sum_{i=1}^{n} y_i x_i \bigm| y \in U(d_a, h_a), y = (y_1, y_2, \ldots) \right\}.
\]
Then each of the following statements is true.
(1) \((C_0, h_0^\ast, h_0^\ast)\) is congruent to \((l_1, g_a)\);
(2) \(g_a\) is equivalent to \(\cdot 1\);
(3) \((l_1, g_a^\ast, g_a^\ast)\) is congruent to \((d_a, h_a)\);
(4) \(((C_0, h_0^\ast, h_0^\ast), h_0^\ast)\) is congruent to \((d_a, h_a)\).

III. \(d_a = d_\beta\). W. J. Davis, in a private communication, has characterized the extreme points of \(U(d_a, h_a)\) in the case \(a \in Z_0 - Z_1\).

**Theorem 3.1 (Davis).** Suppose \(a \in Z_0 - Z_1, x \in d_a, x = (x_1, x_2, \ldots)\) and \(h_a(x) = 1\). Then (1) implies (2).

1. \(x\) is an extreme point.
2. There exists an integer \(n\) such that if \(i > n\) then \(x_i = 0\) and if \(x_i 
eq 0\) and \(x_k 
eq 0\) then \(|x_j| = |x_k|\).

This gives us the following result of Garling.

**Theorem 3.2 [2].** Suppose \(a \in Z_0 - Z_1, f \in (d_a, h_a^\ast)\) and \(r \in \mathcal{P}\). Suppose further that for each \(i\), \(|f_{r(i)}| \geq |f_{r(i+1)}|\). Then

\[
h_a^\ast(f) = \sup_{n=1}^{\infty} \frac{\sum_{i=1}^{n} |f_{r(i)}|}{\sum_{i=1}^{n} a_i}.
\]

Garling has also characterized \(G_e\).

**Theorem 3.3 [2].** Suppose \(a \in Z_0 - Z_1, f \in (d_a, h_a^\ast)\) and \(r \in \mathcal{P}\). Suppose further that for each \(i\), \(|f_{r(i)}| \geq |f_{r(i+1)}|\). Then \(f \in G_e\) if and only if

\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} |f_{r(i)}|}{\sum_{i=1}^{n} a_i} = 0.
\]

**Theorem 3.4 [5].** Suppose each of \(\alpha\) and \(\beta\) is in \(Z\) and for each positive integer \(S(\alpha, n) = \sum_{i=1}^{n} a_i\). Then \(d_a = d_\beta\) if and only if there exists a number \(k_1\) and a number \(k_2\) such that if \(n\) is a positive integer then \(S(\alpha, n) \leq k_1 S(\beta, n)\) and \(S(\beta, n) \leq k_2 S(\alpha, n)\).

**Observation.** Whenever \(d_a = d_\beta\) then \(h_a\) is equivalent to \(h_\beta\).

**Theorem 3.5.** Suppose \(\alpha \in Z - Z_1\) and \(f \in (d_a, h_a^\ast)\). Suppose further that \(\beta = (f_1, f_2, \ldots)\). Then each two of the following statements are equivalent.

1. \(d_a = d_\beta\);
2. \(f \in (d_a, h_a^\ast) - G_e\);
3. if \(F \in (d_a, h_a^\ast)\) such that for each \(i\), \(F_i = a_i\) then \(F \in (d_\beta, h_\beta^\ast)\).

IV. A collection of manifolds in \((d_a, h_a^\ast)\).

**Definition 4.1.** Suppose \((S, g)\) is a normed linear space and \(H\) is a linear manifold in \((S, g)^\ast\). The statement that \(H\) is absolutely total means that if \(x \in S\) then

\[
g(x) = \sup \{|f(x)| \mid f \in H \text{ and } g^\ast(f) = 1\}.
\]
Definition 4.2. The statement that a normed linear space \((S, g)\) has property \(t\) means that if \(H\) is an absolutely total linear manifold in \((S, g)^*\) then \(H\) is dense in \([S, g]^*, g^*\).

B. E. Wilder has shown [8] that each of \((c_0, \|\cdot\|_0)\) and \((c, \|\cdot\|_c)\), where \(\|\cdot\|_0\) and \(\|\cdot\|_c\) are the ordinary norms on \(c_0\) and \(c\) respectively, is a nonreflexive space with property \(t\). He has also shown that \((c_{0,1}, \|\cdot\|_{0,1})\), where \(\|\cdot\|_{0,1}\) is the ordinary max. norm on \(c_{0,1}\), does not have property \(t\). It has been conjectured that the only nonreflexive spaces that have property \(t\) are isomorphic to \((c_0, \|\cdot\|_0)\). The following theorem settles this conjecture in the negative.

Theorem 4.1. Suppose \(\alpha \in Z - Z_1\) and there exists a number \(M\) such that if \(i\) is a positive integer \(\alpha_i < (M+1)\alpha_{i+1}\). Then \((G_e, h_a^\*)\) has property \(t\).

Proof. Suppose for convenience that \(\alpha = 1\). \(J^G_e\) is a congruence from \((d_a, h_a)\) to \([(G_e, h_a^\*)^*, h_a^*\]). Let \(T\) denote the inverse of \(J^G_e\). Suppose that \(L\) is an absolutely total linear manifold in \((G_e, h_a^\*)^*\), and that \(n\) is a positive integer. Then \(b_n \in G_e\) and \(f > 0\) there exists a point \(f \in L\) such that \(h_a^\ast\ast(f) \leq 1\) and \(|h_a^\ast(b_n) - f(b_n)| < f(M+2)\).

Suppose \(T(f) = (f_1, f_2, \ldots)\). Then since \(h_a^\ast(b_n) = 1/\alpha_1 = 1\) and \(f(b_n) = b_n(T(f)) = f_n\), we have that \(|1 - f_n| < f(M+2)\).

\(h_a^\ast\ast(f) = h_a(\sum_{i=1}^\infty f_i b_i) \leq 1\), so \(f_n \leq 1\) and \(1 - f_n \leq 1 - f_n < f(M+2)\). Pick \(r \in \mathcal{P}\) such that \(r(1) = n\) and if \(i \geq 2\), \(|f_{i(0)}| \leq |f_{i(i+1)}|\). For each \(i\), let \(F_i = f_{i(0)}\). Then

\[
|F_1|\alpha_1 + \sum_{i=2}^\infty |F_i|\alpha_i = \sum_{i=1}^\infty |F_i|\alpha_i \leq h_a^\ast\ast(f) \leq 1.
\]

So

\[
\sum_{i=2}^\infty |F_i|\alpha_i \leq 1 - |f_n| < \frac{\varepsilon}{(M+2)}.
\]

Let \(x\) and \(y\) be points of \(d_a\) defined by \(x = -f_n \cdot e_n\) and \(y = f_n e_n - \sum_{i=1}^\infty f_i e_i\). Suppose \(p = p_1, p_2, \ldots\) is the point sequence in \((G_e, h_a^\ast)^*\) that is biorthogonal to \(b\). Then if \(i\) is a positive integer, \(T(p_i) = e_i\). So

\[
h_a^\ast\ast(p_n - f) = h_a(e_n - \sum_{i=1}^\infty f_i e_i) = h_a(x + y) \leq h_a(x) + h_a(y)
\]

\[
= 1 - f_n + h_a(y) < \frac{\varepsilon}{(M+2)} + h_a(y).
\]

Now

\[
h_a(y) = \sum_{i=1}^\infty |F_{i+1}|\alpha_i = \sum_{i=1}^\infty |F_{i+1}|\alpha_{i+1} + \sum_{i=1}^\infty |F_{i+1}| \cdot |\alpha_i - \alpha_{i+1}|
\]

\[
< \frac{\varepsilon}{(M+2)} + M \sum_{i=1}^\infty |F_{i+1}| \cdot |\alpha_{i+1} < \frac{\varepsilon}{(M+2)} + \frac{Me}{(M+2)} = \frac{\varepsilon(M+1)}{(M+2)}.
\]

So

\[
h_a^\ast\ast(p_n - f) = h_a(e_n - \sum_{i=1}^\infty f_i e_i) < \frac{\varepsilon}{(M+2)} + \frac{\varepsilon(M+1)}{(M+2)} = \varepsilon.
\]

Hence \(p_n\) is a point or a limit point of \(L\).
Suppose $s$ is a positive integer and each of $c_1, c_2, \ldots, c_s$ is a number. Suppose further that $x = c_1 p_1 + \cdots + c_s p_s$ and for each $j$, $1 \leq j \leq s$, let $y_1, y_2, \ldots$ be a point sequence in $L$ converging to $p_j$. If $\varepsilon > 0$ and if $j \leq s$ is a positive integer such that $c_j \neq 0$, then there exists a number $n_j$ such that if $i > n_j$ then $|y_i - p_j| < \varepsilon/|c_j|$. For each positive integer $j$, $j \leq s$, let $y_j = c_1 y_j^1 + \cdots + c_s y_j^s$. Then $y_j \in L$. Let $N = \max \{n_j\}$. Then if $i > N$,

$$h^s_{\ast \ast}(x - y_i) = h^s_{\ast \ast}(c_1(p_1 - y_i^1) + \cdots + c_s(p_s - y_i^s)) \leq |c_1| \cdot h^s_{\ast \ast}(p_1 - y_i^1) + \cdots + |c_s| \cdot h^s_{\ast \ast}(p_s - y_i^s) < \varepsilon.$$ 

So $x$ is a point or a limit point of $L$. Hence any point in $L(p)$ is a point or limit point of $L$ and thus the closure of $L$ contains $L(p)$. $L(e)$ is dense in $(d_\alpha, h_\alpha)$ and $J^{G_e}$ maps $L(e)$ onto $L(p)$ so $L(p)$ is dense in $[(G_e, h^s_\alpha)^s, h^{\ast \ast}_\alpha]$. Thus $L$ is dense in $[(G_e, h^s_\alpha)^s, h^{\ast \ast}_\alpha]$.

Every $\alpha$ in $Z - Z_0$ has the property that there is a number $M$ such that $\alpha_i < (M + 1)\alpha_{i+1}$ for each $i$. Some of the sequences in $Z_0 - Z_1$ have this property, for example $\alpha = (1, 1/2, \ldots, 1/i, \ldots)$, while some other sequences in $Z_0 - Z_1$ do not. If $\alpha \in Z_0 - Z_1$ then $(d_\alpha, h_\alpha)$ is not isomorphic to $(l_1, | |_1)$ and thus $(G_e, h^s_\alpha)$ is not isomorphic to $(c_0, | |_0)$. Thus we have the following corollary.

**Corollary 4.1.** There exists a nonreflexive normed linear space $(S, g)$ that has property $t$ but is not isomorphic to $(c_0, | |_0)$.

**Conjecture.** If $\alpha \in Z_0 - Z_1$ then $(G_e, h^s_\alpha)$ has property $t$.

**V. Regular functionals.**

**Definition 5.1.** Suppose $(S, g)$ is a normed linear space and $f \in (S, g)^*$. The statement that $f$ is regular on $(S, g)$ means that there exists a point $x \in S$ such that $g(x) = 1$ and $f(x) = g^*(f)$.

$R(S, g)$ denotes the subset of $(S, g)^*$ to which the point $f$ belongs only in the case that $f$ is regular on $(S, g)$.

**Theorem 5.1.** Suppose that $(S, g)$ is a normed linear space and that $p = p_1, p_2, \ldots$ is a monotone basis for $(S, g)$. Suppose further that $q = q_1, q_2, \ldots$ is the point sequence in $(S, g)^*$ that is biorthogonal to $p$. If $f \in L(q)$ then $f \in R(S, g)$.

**Proof.** Suppose $x \in S$ and $x = \sum_{i=1}^n x_ip_i$. If $n$ is a positive integer let $\bar{x}^n$ be the point of $E_n$ defined by $\bar{x}^n = (x_1, x_2, \ldots, x_n)$. Let $g_n$ denote the norm on $E_n$ defined by $g_n(\bar{x}^n) = g(\sum_{i=1}^n x_i p_i)$. Suppose $y \in L(q)$ and $y = y_1 q_1 + \cdots + y_n q_n$. Then if $x \in S$ and $x = \sum_{i=1}^n x_i p_i$, $y(x) = \sum_{i=1}^n y_i x_i$. Let $y'$ be the point of $[(E_n, g_n)^s, g^{\ast \ast}_s]$ defined as follows: if $x \in E_n$ and $x = (x_1, x_2, \ldots, x_n)$ then $y'(x) = \sum_{i=1}^n y_i x_i$. $y'$ is regular so there exists a point $z = (z_1, z_2, \ldots, z_n)$ in $E_n$ such that $g_n(z) = 1$ and $y'(z) = g^{\ast \ast}_s(y')$. Examine the point $x$ of $S$ defined by $x = \sum_{i=1}^n z_i p_i$. Let $g(x) = g_n(z) = 1$ and $y(x) = \sum_{i=1}^n y_i z_i = g^{\ast \ast}_s(y')$. Now suppose $r = \sum_{i=1}^n r_i p_i$ and $g(r) = 1$. Then $g_n(\bar{r}^n) \leq 1$ and

$$|y(r)| = \left| \sum_{i=1}^n y_i r_i \right| = |y'(\bar{r}^n)| \leq g^{\ast \ast}_s(y').$$

So $g^*(y) = g^{\ast \ast}_s(y')$ and $y \in R(S, g)$. 

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Corollary 5.1. Suppose \((S, g)\) is a normed linear complete space, \(p=p_1, p_2, \ldots\) is a basis for \((S, g)\) and \(q=q_1, q_2, \ldots\) is the point sequence in \((S, g)^*\) that is biorthogonal to \(p\). Then there exists a norm \(h\) on \(S\) such that \(h\) is equivalent to \(g\) and \(L(q) \subseteq R(S, h)\).

Proof. It is well known [1, p. 67] that there exists a norm \(h\) on \(S\) equivalent to \(g\) such that \(p\) is monotone in \((S, h)\). Hence, by Theorem 5.1, \(L(q) \subseteq R(S, h)\).

Theorem 5.2. Suppose \(e \in \mathbb{Z} - \mathbb{Z}_0\), \(e\) is the ordinary basis for \((d_0, h_0)\) and \(f \in (G_0, h_0^*)^*\). Suppose further that \(p=p_1, p_2, \ldots\) is the point sequence in \((G_0, h_0^*)^*\) that is biorthogonal to the basis \(b\) in \((G_0, h_0^*)\). Then \(L(p) = R(G_0, h_0^*)\).

Proof. Since \(b\) is an orthogonal basis for \((G_0, h_0^*)\) then, by Theorem 5.1, \(L(p) \subseteq R(G_0, h_0^*)\). Suppose \(f \in (G_0, h_0^*)^* - L(p)\). \(J_{G_0}\) is a congruence from \((d_0, h_0)\) to \([(G_0, h_0^*)^*, h_0^{**}]\) and for each \(i\), \(J_{G_0}(e_i) = p_i\). Thus if \(T\) denotes the inverse of \(J_{G_0}\) and \(T(f) = (f_1, f_2, \ldots)\), then \(T(f) \in d_0 - L(e)\). Pick \(r \in \mathcal{P}\) such that for each \(i\), \(|f_{ri_0}| \geq |f_{ri_{i+1}}|\) and let \(F_i = |f_{ri_0}|\). Suppose \(n\) is a positive integer and \(a_n = \sum_{i=1}^{n} a_i b_i\). Then if \(T(F) = (F_1, F_2, \ldots)\),

\[
F(a_n) = \sum_{i=1}^{n} F_i a_i < \sum_{i=1}^{n+1} F_i a_i = F(a_{n+1}).
\]

Suppose \(y \in U(G_0, h_0^*)\). Then \(\lim_{n \to \infty} |y_i| = 0\). Pick \(s \in \mathcal{P}\) such that for each \(i\), \(|y_{si_0}| \geq |y_{si_{i+1}}|\) and let \(Y_i = |y_{si_0}|\). Let \(Y = \sum_{i=1}^{\infty} Y_i b_i\) and \(c_1 = \text{glb} a_i\). Since \(\lim_{i \to \infty} y_i = 0\) there exists a number \(n_1\) such that if \(i > n_1\) then \(Y_i < c_1\). Let \(c_2 = \sum_{i=1}^{n_1} F_i (a_i - Y_i)\). Suppose \(c_2 < 0\). Then \(\sum_{i=1}^{n_1} F_i a_i < \sum_{i=1}^{n_2} F_i Y_i\). There exists a number \(t > 0\) such that the point \(z\) of \((d_0, h_0)\) defined by \(z = \sum_{i=1}^{n_1} t' F_i b_i\) has norm 1. So

\[
t \sum_{i=1}^{n_1} F_i a_i = 1 < t \sum_{i=1}^{n_1} F_i Y_i = Y(z).
\]

So \(h_0^*(Y) > 1\). But \(h_0^*(Y) = h_0^*(y) = 1\) so \(c_2 \geq 0\), and

\[
\sum_{i=1}^{n_1+1} F_i (a_i - Y_i) = c_3 > c_2 \geq 0.
\]

Let \(n_2\) be a positive integer such that \(\sum_{i=n_1+1}^{\infty} F_i Y_i < c_3/2\). Suppose \(N = \max\{n_1 + 1, n_2\}\). Then if \(n > N\),

\[
F(a_n) = F(Y) = \sum_{i=1}^{n} F_i (a_i - Y_i) - \sum_{i=n+1}^{\infty} F_i Y_i > c_3 - \frac{c_3}{2} = \frac{c_3}{2} > 0.
\]

So \(F(a_n) > F(Y)\). \(F(Y) \geq F(y)\) and so \(F\) is not regular and \(f\) is not regular. Hence \(R(G_0, h_0^*) = L(e)\) and the theorem is proved.

It may be noted that if we define the norm \(h_{a,0}\) on \(c_0\) by \(h_{a,0}(x) = h_{a}^*(T_{a}^{-1}(x))\) for each \(x \in c_0\), then \(T_{a}\) restricted to \(G_0\) is a congruence from \((G_0, h_0^*)\) to \((c_0, h_{a,0})\) that maps the basis \(b\) in \((G_0, h_0^*)\) onto the basis \(e\) in \((c_0, h_{a,0})\). Thus Theorem 4.2 gives us, in case \(a_1 = 1\) and \(a_2 = 0\), the usual characterization of \(R(c_0, |\cdot|_0)\).
Theorem 5.3. Suppose that \( \alpha \in Z_1 \) and \( e \) is the ordinary basis for \((c_0, h_a)\). Then \( L(b) = R(c_0, h_a) \) if and only if \( \alpha_2 = 0 \).

Proof. If \( \alpha_2 = 0 \) then \((c_0, h_a)\) is congruent to \((c_0, | \cdot |_0)\) and \( h_a = \alpha_1 | \cdot |_0 \). So \( R(c_0, h_a) = R(c_0, | \cdot |_0) \). But \( R(c_0, | \cdot |_0) = L(b) \) so \( R(c_0, h_a) = L(b) \). Suppose \( \alpha_2 \neq 0 \). If \( \alpha' = \sum_{i=1}^{\infty} \alpha_i b_i \), then \( h_a^*(\alpha') = 1 \). Suppose \( \alpha' \notin L(b) \) and \( y = (y_1, y_2, \ldots) \) is the point of \( U(c_0, h_a) \) defined as follows: \( y_1 = 1/\alpha_1 \) and \( y_i = 0 \) if \( i > 1 \). Then \( \alpha'(y) = 1 \) so \( \alpha' \in R(c_0, h_a) \) and \( L(b) \neq R(c_0, h_a) \). Suppose now that \( \alpha' \in L(b) \). Then there exists an integer \( n \) such that \( \alpha_n \neq 0 \) and \( \alpha_{n+1} = 0 \). Let \( f \) be the point of \((c_0, h_a)\)* defined as follows:

\[
f_i = \alpha_i \quad \text{if } 1 \leq i \leq n-1 \quad \text{and} \quad f_i = \frac{\alpha_n}{2^{i-n+1}} \quad \text{if } i \geq n.
\]

Then \( f \notin L(b) \) and it can be shown that \( f \) is regular on \((c_0, h_a)\). Hence \( L(b) \neq R(c_0, h_a) \), and the theorem is proved.

Definition 5.2. Suppose \( g \) is a norm on \( l_1 \). The statement that \( g \) has property \( r \) means that

1. \( g \) is equivalent to \( | \cdot |_1 \); and
2. if \( x = (x_1, x_2, \ldots) \) is a point in \( l_1 \) and \( s \in \mathcal{P} \) and if \( y = (y_1, y_2, \ldots) \) is the point in \( l_1 \) such that for each \( i, y_i = |x_{s_i}| \), then \( g(y) = g(x) \).

Theorem 5.4. Suppose that \( g \) is a norm on \( l_1 \) and \( g \) has property \( r \). Suppose that \( f \in (l_1, g)^* \) and that if \( j \) is a positive integer then \( |f_j| < \text{lub}_j |f_i| \). Then \( f \notin R(l_1, g) \).

This result is well known in case \( g = | \cdot |_1 \) and a proof may be constructed similar to the proof of that case.

Theorem 5.5. Suppose that \( \alpha \in Z - Z_0 \). Then only one of the following statements is true.

1. For each positive integer \( i, \alpha_i = \alpha_1 \).
2. \( R(d_a, h_a) \) is a proper subset of \( R(l_1, | \cdot |_1) \).

Proof. Suppose (1) is true. Then the transformation \( T \) from \((d_a, h_a)\) to \((l_1, | \cdot |_1)\) defined by \( T(x) = \alpha_1 x \), for each \( x \in d_a \), is a congruence and \( R(d_a, h_a) = R(l_1, | \cdot |_1) \). So (2) is not true. It is well known that \( f \in (l_1, | \cdot |_1)^* \) then \( f \notin R(l_1, | \cdot |_1) \) if and only if for each positive integer \( j, |f_j| < \text{lub}_j |f_i| \). Therefore, since \( h_a \) has property \( r \), \( R(d_a, h_a) \subseteq R(l_1, | \cdot |_1) \). Suppose (1) is not true. Let \( n \) be the least integer such that \( \alpha_n > \alpha_{n+1} \). Let \( f \) be the point of \((d_a, h_a)\)* defined as follows:

\[
f_i = 1 \quad \text{if } 1 \leq i \leq n \quad \text{and} \quad f_i = \frac{i-n}{i-n+1} \quad \text{if } i > n; \quad f \in R(l_1, | \cdot |_1).
\]

However it can be shown that \( f \) is not in \( R(d_a, h_a) \), so (2) is true.

Thus it is seen that, in case \( \alpha \in Z - Z_0, R(d_a, h_a) \) is largest when \((d_a, h_a)\) is congruent to \((l_1, | \cdot |_1)\).
Definition 5.3. Suppose \((S, g)\) is a normed linear space and \(H\) is a linear manifold in \((S, g)^*\). The statement that \(H\) is maximal regular in \((S, g)^*\) means that

1. \(H \subseteq R(S, g)\).
2. If \(L\) is a linear manifold in \((S, g)^*\) and \(H\) is a proper subset of \(L\), then there exists a point \(f \in L - H\) such that \(f\) is not in \(R(S, g)\).

Definition 5.4. Suppose that \((S, g)\) is a normed linear space. Then \(Q\) denotes the transformation from \((S, g)\) to \([[(S, g)^*, g^*], g^{**}]\) defined as follows: if \(x \in S\) and \(f \in (S, g)^*\) then \(Q_x(f) = f(x)\). \(Q(S)\) denotes the image of \(Q\).

Theorem 5.6. Suppose \(g\) is a norm on \(l_1\) and \(g\) has property \(r\). Suppose further that the ordinary basis \(e\) for \((l_1, g)\) is orthogonal. Then \(G_e\) is maximal regular.

Proof. By Theorem 5.1, \(G_e \subseteq R(l_1, g)\). Suppose \(L\) is a linear manifold in \((l_1, g)^*\) and \(G\) is a proper subset of \(L\). Suppose further that \(f \in L - G_e\) and \(T_1(f) = (f_1, f_2, \ldots)\). Consider the following two cases.

I. Suppose there exists a positive integer \(n\) such that \(F_n = T_1^{-1}(f_{n+1}, f_{n+2}, \ldots)\) is not regular on \((l_1, g)\). In this case let \(y\) be the point of \(G_e\) such that \(y_i = -f_i\) if \(1 \leq i \leq n\) and \(y_i = 0\) if \(i > n\). Then \(y + f \in L\) and \(y + f\) is not regular on \((l_1, g)\).

II. Suppose that for each positive integer \(n\), \(F_n = T_1^{-1}(f_{n+1}, f_{n+2}, \ldots)\) is regular on \((l_1, g)\). Let \(n_1\) denote the least integer such that \(|f_{n_1}| = |T_1(f)|_m\). Let \(n_2\) denote the least integer such that \(n_2 > n_1\) and \(|f_{n_2}| = |T_1(F_{n_1})|_m\). If \(j\) is a positive integer, \(j \geq 2\), let \(n_j\) denote the least integer such that \(n_j > n_{j-1}\) and \(|f_{n_j}| = |T_1(F_{n_{j-1}})|_m\). Then \(|f_{n_1}|, |f_{n_2}|, \ldots\) is a nonincreasing sequence of \(|f_1|, |f_2|, \ldots\) converging to a number \(k > 0\). Let \(s_1, s_2, \ldots\) be the subsequence of \(f_1, f_2, \ldots\) to which the number \(f_j\) belongs only in the case that \(|f_j| \geq k\). For each positive integer \(i\), let \(d_i = |f_{s_i}| - k + k/2s_i\). Define \(y = (y_1, y_2, \ldots)\) as follows:

\[
y_i = \begin{cases} 0 & \text{if } i \neq s_j \text{ for every } j, \\ -d_i & \text{if } i = s_j \text{ for some } j \text{ and } s_j \geq 0, \\ d_i & \text{if } i = s_j \text{ for some } j \text{ and } s_j < 0.
\end{cases}
\]

It can be shown that \(Y = T_1^{-1}(y)\) is in \(G_e\) so \(Y + f \in L\) and \(f + Y\) is not regular on \((l_1, g)\).

Corollary 5.2. Suppose \(a \in Z_1\). Then \(Q(c_0)\) is maximal regular in \([[(c_0, h_a)^*, h_a^*], h_a^{**}]\).

Corollary 5.3. Suppose \(a \in Z - Z_0\) and \(e\) is the ordinary basis for \((d_a, h_a)\). Then \(Q(G_e)\) is maximal regular in \([[(G_e, h_a)^*, h_a^{**}], h_a^{***}]\).

Conjecture. Suppose \((S, g)\) is a normed linear space. Then \(Q(S)\) is maximal regular in \([[(S, g)^*, g^*], g^{**}]\).

References


