ON GENERALIZED COMMUTING ORDER OF AUTOMORPHISMS WITH QUASI-DISCRETE SPECTRUM

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0.0. Introduction. Abramov [1] has defined the notions of an automorphism of a finite measure space with quasi-discrete spectrum, using the concepts of quasi-proper function and quasi-proper value introduced by Halmos and von Neumann [12]. For this class of quasi-proper functions Abramov defines an ascending sequence of abelian groups, which turns out to be a complete set of invariants for the classification of automorphisms with quasi-discrete spectrum. In addition he proves a representation theorem. In [11] an analogous theorem was proved for a homeomorphism of a compact space. Adler [2] has introduced the generalized commuting order of an automorphism on a finite measure space. The generalized commuting order is conjugacy invariant for automorphisms. [2] proves that the generalized commuting order of a totally ergodic translation of the measure space consisting of a compact metric abelian group is two. Furthermore, [5] gives conditions that every member of the generalized commuting order 2 have quasi-discrete spectrum.

In this paper we discuss the result first obtained by Abramov [1] in §1. In §2 we show a result stronger than the representation theorem of Abramov, Hahn and Parry. Our main result is to know an answer to the following question raised by Adler [2]. Let T be an automorphism of a finite measure space, then are there automorphisms T for \( CN(T) = n \) for an integer \( n \) including \( CN(T) = \infty \)? In §3, we mention a few examples concerning this question. Furthermore, for a totally ergodic automorphism of a finite measure space with quasi-discrete spectrum, we generalize the obtained examples.

I benefited from reading the papers by Adler [2], Hahn [10] and Hoare and Parry [14].

0. Preliminaries. By a dynamical system we mean a pair \((X, T)\) where X is a compact Hausdorff space and T is a homeomorphism of X onto itself. We say that \((X, T)\) is minimal if \(X\) contains no nonempty closed \(T\)-invariant set, and totally minimal if \((X, T^m)\) is minimal for any integer \(m \neq 0\). Throughout, a homeomorphism \(T\) is bicontinuous of \(X\) onto itself. Let \(C(X)\) be the Banach algebra of continuous complex valued functions on a compact Hausdorff space \(X\). A homeomorphism \(T\) induces an isometric isomorphism \(V_T\) of the Banach algebra \(C(X)\), \(V_Tf(x) = f(Tx)\).
Let \((X, T)\) be totally minimal. We recall the following definition of quasi-proper function [10]. Let \(G(T)_0\) be a group \(\{a \in K : V_x f(x) = af(x), |f(x)| = 1 \text{ for } f \in C(X)\}\) where \(K\) is the unit circle in the complex plane. For \(i > 0\) let \(G(T)_i \subset C(X)\) be the group of all functions \(f\) such that \(V_x f = gf, |f(x)| = 1 \text{ for } g \in G(T)_{i-1}\). We put \(G(T) = \bigcup_{i \geq 0} (T)_i\). \((X, T)\) is said to have quasi-discrete spectrum if \(G(T)\) spans \(C(X)\), and have discrete spectrum if \(G(T)_{\leq 1}\) spans \(C(X)\). If it ever happens that \(G(T)_n = G(T)_{n+1}\), then \(G(T)_n = G(T)_{n+k}\) for all \(k\) and in this case we define \(GN(T) = \min \{n : G(T)_n = G(T)_{n+1}\}\) and otherwise \(GN(T) = \infty\). It follows that \(G(T) = K \times O(T)\) where \(O(T)\) is a subgroup of \(G(T)\) isomorphic to the factor group \(G(T)/K\) and the elements of \(O(T)\) are linearly independent. If \((X, T)\) is totally minimal and has quasi-discrete spectrum, then there exists a unique \(T\)-invariant finite Borel measure [10]. Two compact Hausdorff spaces \(X\) and \(Y\) are homeomorphic if and only if their corresponding Banach algebras \(C(X)\) and \(C(Y)\) are isomorphic [9]. Whenever a compact Hausdorff space \(X\) is metrizable, Halmos and von Neumann [12] have proved that if \((X, T)\) is minimal and \(T\) an isometric homeomorphism on \(X\) then \(X\) isometric to an interval into \(X\) a multiplication so that \(X\) becomes (with the original topology of \(X\)) a compact metric abelian group and \(T\) becomes a translation. But a homeomorphism \(T\) of the circle such that no power of \(T\) has a fixed point is homeomorphic to a translation [8]. Hahn and Parry [11] have proved that if \((X, T)\) is totally minimal and has quasi-discrete spectrum then there exist a compact abelian group with the normalized Haar measure and a totally ergodic affine transformation \(A\) on the space, and \(T\) is homeomorphic to \(A\). Let \(X_n\) be the \(n\)-dimensional torus, i.e., \(X_n = \mathbb{R}^n/\sim\) where \(\mathbb{R}^n\) is the Euclidean plane and \(\sim\) is the equivalence relation which identifies \(n\)-points in the plane if their corresponding coordinates differ by integers. A metric on \(X_n\) can be defined in terms of the metric on \(\mathbb{R}^n\) by taking the distance between \(n\)-points of \(X_n\) to be the minimal distance between any representatives of these points in \(\mathbb{R}^n\). The set of functions \([\psi_{p_1, p_2, \ldots, p_n}]\):

\[
\psi_{p_1, p_2, \ldots, p_n}(x_1, x_2, \ldots, x_n) = \exp \left[2\pi i(p_1 x_1 + p_2 x_2 + \cdots + p_{n-1} x_{n-1} + p_n x_n)\right]
\]

where \(p_i = 0, \pm 1, \pm 2, \ldots\) and \(i = 1, 2, \ldots, n\), forms a complete system of \(C(X_n)\). The set of generators of a compact metric connected abelian group has a positive measure with respect to its Haar measure [13]. It is known that a translation \(T_r : x \mapsto x + r\) on a compact abelian group \(X\) is ergodic if and only if \(r\) is a generator of \(X\). Let \((\Omega, \Sigma, \mu)\) be a finite measure space where \(\Omega\) is a set of elements, \(\Sigma\) a \(\sigma\)-field of measurable subsets of \(X\), and \(\mu\) a finite measure on \(\Sigma\). We denote by \(\Sigma(\mu)\) the Boolean \(\sigma\)-algebra by identifying sets in whose symmetric difference has zero measure, and \(\mu\) is induced on the elements of \(\Sigma(\mu)\) in the natural way. Let \(L^2(\Sigma)\) be the Hilbert space of complex-valued square integrable functions defined on \((\Omega, \Sigma, \mu)\), but sometimes we use two symbols \(L^2(\Omega)\) and \(L^2(\Sigma(\mu))\) instead of \(L^2(\Sigma)\). Let \(T\) be an automorphism of \((\Omega, \Sigma, \mu)\) and we denote by \(V_T : f(x) \mapsto f(Tx)\) \((f \in L^2(\Sigma))\) the linear isometry induced by \(T\). An automorphism of the measure
algebra is called a metric automorphism. An automorphism \( T \) of \((\Omega, \Sigma, \mu)\) induces a metric automorphism in the natural way and sometimes we denote by \( T' \) an induced metric automorphism. \( T \) is said to be totally ergodic if \( T^n \) is ergodic for every integer \( n \neq 0 \). We recall the following definition of quasi-proper function for a totally ergodic automorphism of \((\Omega, \Sigma, \mu)\) \[1\]. This definition is an analogue to that of a totally minimal dynamical system. Let \( G_\Phi(T)_0 = \{ \alpha \in K : V_\tau f = \alpha f \text{ a.e.,} \| f \|_2 = 1 \text{ for } f \in L^2(\Sigma) \} \), and for \( i > 0 \) let \( G_\Phi(T)_i \subset L^2(\Sigma) \) be the set of all normalized functions \( f \) such that \( V_\tau f = gf \text{ a.e.} \) where \( g \in G_\Phi(T)_{i-1} \). Then \( G_\Phi(T)_i \) is the set of quasi-proper functions of order \( i \). \( T \) is said to have quasi-discrete spectrum if \( G_\Phi(T) = \bigcup_{i>0} G_\Phi(T)_i \) spans \( L^2(\Sigma) \). Since \( G_\Phi(T) \) is a group, we follow that \( G_\Phi(T)_n = K \times G_\Phi(T)_{n-1} \) where \( G_\Phi(T) \) is a subgroup of \( G_\Phi(T)_n \). We denote by \( G_\Phi(T)_n \) the least positive integer \( n \) for which \( G_\Phi(T)_n = G_\Phi(T)_{n+1} \) does happen and otherwise \( G_\Phi(T)_n = \infty \). Halmos and von Neumann \[12\] shows that a linear isometry \( V \) on \( L^2(\Sigma) \) onto itself is induced by an automorphism of the measure algebra if and only if both \( V \) and \( V^{-1} \) send every bounded function onto a bounded function and \( V(fg) = Vf \cdot Vg \) whenever \( f \) and \( g \) are bounded functions. A necessary and sufficient condition that a closed subspace \( H \) of \( L^2(\Sigma) \) be of the form \( H = L^2(\mu) \) where \( \Phi(\mu) \) is the smallest \( \sigma \)-algebra of \( \Sigma(\mu) \) with respect to which all functions in \( H \) are measurable is that \( H \) contains a dense subalgebra consisting of bounded functions, constant functions and their complex conjugations \[6\]. If \( G \) is any group, and \( a \) any element of \( G \), then we define subsets \( C_n(a) \) \((n = 0, 1, 2, \ldots)\) of \( G \) in the following way:

\[
C_0(a) = \{ e \},
\]

\[
C_n(a) = \{ b \in G : bab^{-1}a^{-1} \in C_{n-1}(a) \} \quad (n = 1, 2, \ldots).
\]

It is clear that \( C_n(a) \subset C_{n+1}(a) \), \( n = 0, 1, 2, \ldots \). The least \( n \) for which \( C_n(a) = C_{n+1}(a) \) is called the generalized commuting order of \( a \) in \( G \), and we denote by \( CN(a) \) such an integer \( n \). If \( bab^{-1}a^{-1} = a' \) where \( b, a, a' \in G \) then it is clear that \( CN(a) = CN(a') \). But the converse does not hold. Adler \[2\] has shown the following results: let \( T_a \) be the translation by \( a \) in a compact separable abelian group. If \( T_a \) is totally ergodic, then \( CN(T_a) = 2 \) and \( C_1(T_a) \) is the group of translations, and \( C_2(T_a) \) is the group consisting of translations composed with continuous group automorphisms; let \( r \) be an irrational number of the 1-dimensional torus \( X_1 \), and let \( n \) be an integer, then for \( T_{r,n}(x_1, x_2) = (x_1 + r, x_2 + nx_1) \) \((\text{additions mod 1})\), \( CN(T_{r,n}) = 3 \), and \( C_1(T_{r,n}), C_2(T_{r,n}) \) and \( C_3(T_{r,n}) \) are groups. By Adler's ideas, \[4\] has proved, without the representation theorem due to Halmos and von Neumann, that the generalized commuting order of a totally ergodic metric automorphism with discrete spectrum on the measure algebra associated with a finite measure space is two.

1. Properties of automorphisms with quasi-discrete spectrum. If \( X \) is a compact abelian group, \( r \in X \) and \( \beta \) is a continuous group automorphism of \( X \), then \( T(x) = T_r \beta(x) \) is called an affine transformation of \( X \) onto itself. For a totally ergodic, (with respect to the Haar measure) affine transformation on \( X \), both definitions of

\[
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\]
the word "quasi-discrete spectrum," introduced by Abramov [1] and Hahn and Parry [11], coincide.

The next result was first obtained by Hahn and Parry [11].

**Lemma 1.1.** Let $X$ be a compact connected abelian group with Haar measure on $X$. If a totally ergodic affine transformation $T(x) = Tr \beta(x)$, $x \in X$, has quasi-discrete spectrum, then $(X, T)$ is a totally minimal dynamical system.

**Proof.** Since the totally ergodic affine transformation $T$ has quasi-discrete spectrum, we see that $O_{\mu}(T)$ ($\mu$ is Haar measure on $X$) is equal to the character group of $X$. Let $C_n$ ($n = 1, 2, \ldots$) be a set

$$\{g : B^ng = 1, g \text{ a character of } X\}$$

where $B$ is a homomorphism on the character group of $X$ defined by $B(g) = g^{-1}V_{\beta}g$. Then $\bigcup_{k=1}^{\infty} C_k$ is equal to the character group of $X$ if and only if $T$ has quasi-discrete spectrum [14]. Suppose that $x, y, z$ are in $X$. Suppose that $\{n_j : j \in \Delta\}$ is a net of integer such that

$$\lim T^{n_j}x = \lim T^{n_j}y = z.$$ 

Then $g(T^{n_j}x) = \lim g(T^{n_j}y) = g(z)$ for every character $g$ of $X$. Here we prove by induction that if $g$ is a quasi-proper function belonging to $C_n$ for any integer $n$ then $g(x) = g(y)$. If $n = 1$, then $g(x) = g(y)$ since $B(g) = 1$ and

$$\lim g(\beta^{n_j}(x \beta^{-1})) = 1.$$ 

Suppose now that all characters which are quasi-proper functions belonging to $C_n$ annihilate $xy^{-1}$. Let $g$ be a quasi-proper function of $C_{n+1}$. Then $B^{n+1}g = 1$. Thus $B^n(Bg) = 1$ and $Bg \in C_n$. Therefore $Bg(x) = Bg(y)$ and $g(\beta(xy^{-1})) = g(xy^{-1})$ which gives $g(x) = g(y)$. We have shown that the character group is equal to $\bigcup_{k=1}^{\infty} C_k$ and every character $g$ satisfies $g(x) = g(y)$. By the duality theorem, we have $x = y$. But this is a definition of distal. Let $N$ be the smallest $\beta$-invariant subgroup of the character group containing characters $f_{i_1}, f_{i_2}, \ldots, f_{i_n}$ and let $\text{ann}(N)$ be the annihilator of $N$. Then $X/\text{ann}(N)$ is metrizable. If $T'$ is the affine transformation on $X/\text{ann}(N)$ induced by $T$, we see that $T'$ is totally ergodic and distal. From ergodicity of $T'$, there is an element $x' \in X/\text{ann}(N)$ such that $\{T'^{n}x' : n = 0, \pm 1, \pm 2, \ldots\}$ is dense in $X/\text{ann}(N)$. Moreover, since $T'$ is distal, $(X/\text{ann}(N), T')$ is minimal [7].

This fact and connectedness of $X$ guarantee that $(X, T)$ is totally minimal.

The idea of the following theorem is essentially contained in Abramov [1].

**Theorem 1.2.** Let $(\Omega, \Sigma, \mu)$ be a normalized measure space, and let $Q$ be a totally ergodic automorphism of $(\Omega, \Sigma, \mu)$ with quasi-discrete spectrum. Then there exist a compact connected abelian group $X$ with the normalized Haar measure and affine transformation $T(x) = T_\alpha \beta(x)$, $x \in X$, where $\alpha \in X$ and $\beta$ is a continuous group automorphism of $X$, and $Q$ is conjugate to $T$. Furthermore, the dynamical system $(X, T)$ is totally minimal. If $(\Omega, \Sigma, \mu)$ is separable, then $X$ is metrizable.
Proof. We denote by $X$ the character group of $O_u(Q)$ imposed by the discrete topology. If $(\Omega, \Sigma, \mu)$ is separable, $O_u(Q)$ is countable so that $X$ is metrizable. $X$ is a compact abelian group with the normalized Haar measure. Let $\langle \cdot, \cdot \rangle$ denote the pairing between $O_u(Q)$ and its dual. To define the linear isometry, we put

$$V \left( \sum_{k=1}^{n} r_k f_k \right) = \sum_{k=1}^{n} r_k \langle \cdot, f_k \rangle, \quad f_k \in O_u(Q).$$

Then $V$ is an isometry which can be extended uniquely to an isometry of $L^2(\Sigma)$ onto $L^2(X)$. We suppose that $V$ is an extended linear isometry. Since $V$ satisfies the conditions of the multiplication theorem, there exists a metric isomorphism $\varphi$ such that $V = V_\varphi$. Now define $V'$ on $L^2(X)$ by $V' = V_\varphi V_\varphi V_\varphi^{-1}$ and put $O(Q) = \{ \langle \cdot, f \rangle : f \in O_u(Q) \}$. Then $V'$ has quasi-discrete spectrum and $K \times O(Q)$ is invariant under $V'$. Here we show that $V'$ is an operator induced by an affine transformation on $X$. $V'$ is an automorphism of $K \times O(Q)$ onto itself and a subgroup $K \times 1$ is mapped identically onto itself. We define maps

$$P: O(Q) \to O(Q), \quad r: O(Q) \to K$$

by $V'g = r(g)P(g)$, $g \in O(Q)$. We have $r(fg) = r(f)r(g)$ and $P(fg) = P(f)P(g)$ for $f, g \in O(Q)$. Therefore $r(\cdot)$ and $P(\cdot)$ are homomorphisms of $O(Q)$. To show that $P(\cdot)$ is one-to-one, let us put $P(f) = P(g)$ for $f, g \in O(Q)$, then we have $V'(fg^{-1}) = r(fg^{-1})$ and $fg^{-1} = r(fg^{-1}) \in O(Q)$, i.e., $f(x) = g(x)$ for all $x \in X$. It is clear that $P(\cdot)$ is onto. We have shown that $P(\cdot)$ is an automorphism of $O(Q)$. $P(\cdot)$ therefore induces a continuous group automorphism $\beta$ of $X$. Since $r$ is a homomorphism of $O(Q)$ into $K$, $r$ is an element of $X$. Therefore

$$V'g(x) = r(g)P(g) = g(r(\beta x)) = g(T_\beta x)$$

for all $x \in X$ and all $g$. We have proved that $V'$ is an operator induced by $T_\beta$, and $Q$ is conjugate to $T_\beta$. Since $T_\beta$ is totally ergodic and has quasi-discrete spectrum, it follows that $X$ is connected. It is clear from Lemma 1.1 that $(X, T)$ is totally minimal.

The next corollary is the result of Hahn and Parry [11] and Hoare and Parry [14].

Corollary 1.3. Let $X$ be a compact connected abelian group with Haar measure on $X$. An ergodic affine transformation $T$ has quasi-discrete spectrum if and only if $(X, T)$ is totally minimal.

2. Behavior of affine transformations with quasi-discrete spectrum. We see that the continuous group automorphisms of $X_\alpha$ are in correspondence with the invertible linear transformations of $R^n$ which preserve subset $Z_\alpha$ of $R^n$ consisting of points with integer coordinates. Therefore if a fixed base is chosen in $X_\alpha$, the automorphisms of $X_\alpha$ are in one-to-one correspondence with $n \times n$ unimodular matrices. Let $\beta$ be a continuous group automorphism of $X_\alpha$ and let $[\beta]$ denote the
corresponding matrix. If \([\beta]=[a_{ij}: i,j=1,2,\ldots, n]\) then the automorphism \(\beta\) is given by

\[
\beta((x_1, x_2, \ldots, x_n)+Z_n) = \left(\left(\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \ldots, \sum_{j=1}^{n} a_{nj}x_j\right) + Z_n\right).
\]

This equation is denoted by

\[
\beta(x_1, x_2, \ldots, x_n) = \left(\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \ldots, \sum_{j=1}^{n} a_{nj}x_j\right) \pmod{1}.\]

**Theorem 2.1.** Let \(T\) be a homeomorphism of \(X_n\) onto itself. If a dynamical system \((X_n, T)\) is totally minimal and has quasi-discrete spectrum, then there is an affine transformation \(T, \beta\) of \(X_n\) homeomorphic to \(T\). Furthermore, \(T, \beta\) is homeomorphic some affine transformation given by some matrix

\[
\begin{bmatrix}
1 & 0 & & 0 \\
0 & 1 & & \\
& & \ddots & & 0 \\
0 & \cdots & & 1
\end{bmatrix}
\quad \text{and} \quad
r' = \begin{bmatrix}
r'_1 \\
r'_2 \\
\vdots \\
r'_n
\end{bmatrix}.
\]

In particular, if \(a_{ij}=0\) for \(i \neq j\) such that \(2 \leq i \leq l\) and \(1 \leq j \leq n\), then the numbers \(r'_1, r'_2, \ldots, r'_n\) are integrally independent.

**Proof.** Since \((X_n, T)\) is a totally minimal dynamical system with quasi-discrete spectrum, elements of \(O(T)\) are linearly independent and \(O(T)\) spans \(C(X_n)\). We denote by \(X\) the character group of \(O(T)\) imposed by the discrete topology. \(X\) is homeomorphic to \(X_n\). Thus the rank of the character group \(O(T)\) of \(X\) is equal to the number \(n\) since \(X\) is the \(n\)-dimensional topological space. Since \(X\) is connected and locally connected, \(O(T)\) is the direct product of the free cyclic groups \(C_j\), \(j=1, 2, \ldots, n\). Therefore \(X\) is isomorphic to the \(n\)-dimensional torus \(X_n\). If \(T'\) is the homeomorphism on \(X\) induced by \(T\), \((X, T')\) is a totally minimal dynamical system with quasi-discrete spectrum. Since \(X\) is the \(n\)-dimensional torus, we may suppose that \(X=X_n\). Then

\[
V_T(K \times [\psi_{p_1, p_2, \ldots, p_n}]) = (K \times [\psi_{p_1, p_2, \ldots, p_n}]).
\]

As we did in Theorem 1.2, we follow that \(T'\) is an affine transformation \(T, \beta\) such that \(T\) is a translation of \(X_n\) and \(\beta\) a continuous group automorphism of \(X_n\). Thus \(G(T, \beta) = K \times [\psi_{p_1, p_2, \ldots, p_n}]\) and \(GN(T, \beta)\) is finite. If \(GN(T, \beta) = m\) and

\[
G(T, \beta)_i = [\psi_{p_1, p_2, \ldots, p_i}], \quad i = 1, 2, \ldots, m,
\]

we choose a base dependent on \((1)\) in \(X_n\). For the base in \(X_n\), there exists some unimodular matrix

\[
[\beta'] = [a_{ij}: i,j = 1, 2, \ldots, n]
\]
so that the continuous group automorphism given by the matrix \([\beta']\) is isomorphic to \(\beta\). Thus \(T_r \beta\) is homeomorphic to the affine transformation \(T_r \beta'\) where

\[
r' = \begin{bmatrix}
    r_1' \\
    \vdots \\
    r_n'
\end{bmatrix}.
\]

Since the operator \(V_{\beta'}\) is identical on \([\psi_{p_1, p_2, \ldots, p_{l'}]}\), it follows that

\[
[\beta'] = \begin{bmatrix}
    E_1 & 0 \\
    \vdots & \ddots \\
    \star & \star & 0
\end{bmatrix}
\]

where \(E_1\) is the identity matrix of order \(l_1 \times l_1\). For every \(g \in [\psi_{p_2, p_3, \ldots, p_{l_2}}]\),

\[
V_{\beta'} g(x_1, x_2, \ldots, x_{l_2}) = g'(x_1, x_2, \ldots, x_{l_2}) g(x_1, x_2, \ldots, x_{l_2})
\]

where

\[
g'(x_1, x_2, \ldots, x_{l_2}) = \exp \left[ 2\pi i \left( \sum_{k=1}^{l_1} p_k a_{k1} x_1 + \sum_{k=1}^{l_2} p_k a_{k2} x_2 + \cdots + \sum_{k=1}^{l_m} p_k a_{kn} x_n \right) \right]
\cdot \exp \left[ -2\pi i (p_1 x_1 + p_2 x_2 + \cdots + p_{l_1} x_{l_1}) \right],
\]

and \(g'\) is an element of \(G(T_r \beta') = [\psi_{p_1, p_2, \ldots, p_{l_1}}]\). From this fact, the form of \([\beta']\) is the following matrix

\[
[\beta'] = \begin{bmatrix}
    E_1 & 0 \\
    \vdots & \ddots \\
    \star & \star & 0
\end{bmatrix}
\]

where \(E_1\) is the identity matrix of order \(l_1 \times l_1\) and \(E_2\) the identity matrix of order \((l_2 - l_1) \times (l_2 - l_1)\). From such an argument we see that the form of \([\beta']\) is the following triangular matrix

\[
[\beta'] = \begin{bmatrix}
    E_1 & 0 \\
    \vdots & \ddots \\
    \star & \star & 0
\end{bmatrix}
\]
where \( E_j \) is the identity matrix of order \((l_j-l_{j-1}) \times (l_j-l_{j-1})\) (but \( l_0 = 0 \)) for \( j=1, 2, \ldots, m \). The fact that \( r'_1, r'_2, \ldots, r'_m \) are integrally independent follows immediately from the fact that \((X, T, \beta^T)\) is minimal.

It is well known that on the 1-dimensional torus \( X_1 \) there exist only two continuous group automorphisms, the identical automorphism and another automorphism \( \beta \) for which \( \beta x = -x, \, x \in X_1 \).

We have the next corollary here.

**Corollary 2.2.** Let \( T \) be a homeomorphism of \( X_1 \) onto itself and if a dynamical system \((X_1, T)\) is totally minimal and has quasi-discrete spectrum, then there is a translation of \( X_1 \) homeomorphic to \( T \).

Let \( T \) be a homeomorphism of the 2-dimensional torus \( X_2 \), and defined by

\[
T: (x_1, x_2) \rightarrow (x_1 + r, x_2 + nx_1)
\]

where \( r \) is a real number and \( n \) an integer. Such a transformation is called a skew product transformation of \( X_2 \) [3].

**Corollary 2.3.** If \((X_2, T)\) is totally minimal and has quasi-discrete spectrum, then there is a skew product transformation of \( X_2 \) homeomorphic to \( T \).

This is direct from Theorem 2.1.

**Corollary 2.4.** Let \((X_n, T)\) be a totally minimal dynamical system with quasi-discrete spectrum and let \( CN(T) = 2 \). Then there is a following affine transformation \( T, \beta \) homeomorphic to \( T \),

\[
T_1, \beta: (x_1, x_2, \ldots, x_n)
\]

\[
\rightarrow \left( x_1 + r_1, \ldots, x_i + r_i, x_{i+1} + \sum_{j=1}^{i} a_{i+1,j}x_j + r_{i+1}, \ldots, x_n + \sum_{j=i}^{n} a_{n,j}x_j + r_n \right)
\]

(additions mod 1)

where each \( a_{ij} \) is some integer and each \( r_j \) some real number, and moreover the numbers \( r_1, r_2, \ldots, r_i \) are integrally independent.

**Corollary 2.5.** Let \((X_n, T)\) be a totally minimal dynamical system with quasi-discrete spectrum and let \( GN(T) = n \). Then there is a following affine transformation \( T, \beta \) homeomorphic to \( T \),

\[
T_1, \beta: (x_1, x_2, \ldots, x_n)
\]

\[
\rightarrow \left( x_1 + r_1, x_2 + a_{21}x_1 + r_2, \ldots, x_{n-1} + \sum_{j=1}^{n-2} a_{n-1,j}x_j + r_{n-1}, x_n + \sum_{j=1}^{n-1} a_{n,j}x_j + r_n \right)
\]

(additions mod 1)

where each \( a_{ij} \) is some integer, but \( a_{ij} \) is nonzero for \( j=2, \ldots, n \), and each \( r_j \) some real number, but \( r_1 \) irrational.
Proof. From Theorem 2.1, \( T \) is homeomorphic to some affine transformation \( T, \beta \) such that the matrix \([\beta]\) is of the following form
\[
[\beta] = \begin{bmatrix}
1 & a_{21} & 1 & 0 \\
0 & a_{31} & a_{32} & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_{n1} & \cdots & a_{n,n-1} & 1
\end{bmatrix}
\]
and
\[
r = \begin{bmatrix}
r_1 \\
r_2 \\
\vdots \\
r_n
\end{bmatrix}
\]
where each number \( a_{ij} \) is some integer and each number \( r_j \) some real. But \( a_{i,j-1} \) is nonzero for \( j=2, \ldots, n \) since \( GN(T) = n \) and from (totally) minimality of \((X_n, T, \beta)\), \( r_1 \) is irrational.

Corollary 2.6. Let \( T \) be a totally ergodic automorphism of a finite measure space \((\Omega, \Sigma, \mu)\) with quasi-discrete spectrum. Then there exist metric automorphisms \( W \) and \( S \) such that \( W \) has each function of \( O_n(T) \) as a proper function and \( V_S \) maps \( O_n(T) \) onto itself, and the metric automorphism of \( T \) is equal to \( SW \).

The proof of Corollary 2.6 is similar to [5].

3. Generalized commuting order of transformations. As pointed out in Adler [2], it is interesting to know an answer to the following question: are there examples for \( CN(T) = n \) for an integer \( n \) including \( CN(T) = \infty \)? The next example shows that this question has a positive answer.

Theorem 3.1. Let \( T \) be an affine transformation of \( X_n \) and let \((X_n, T)\) be a totally minimal dynamical system (with quasi-discrete spectrum). If \( GN(T) = n \), then \( CN(T) = n + 1 \).

Proof. From Theorem 2.1 we may suppose that the affine transformation \( T \) is written as follows
\[
T: (x_1, x_2, \ldots, x_n) 
\rightarrow \left(x_1 + r_1, x_2 + a_{21}x_1 + r_2, \ldots, x_n + \sum_{j=1}^{n-2} a_{n-1,j}x_j + r_{n-1}, x_n + \sum_{j=1}^{n-1} a_{n,j}x_j + r_n \right)
\]
(additions mod 1)
where each number \( a_{ij} \) is an integer and \( r_j, j=1, 2, \ldots, n \), are real. Since \( GN(T) = n \), from Corollary 2.5, the integer \( a_{j,j-1} \) is nonzero for \( j=2, \ldots, n \) and \( r_1 \) irrational. We show by induction that members of \( C_n(T) \), \( n=1, 2, \ldots \), are affine transformations. If \( n=1 \), then for \( S_1 \in C_1(T) \) we have \( S_1 T = TS_1 \) and therefore
\[
V_{S_1}(K \times [\psi_{p_1, p_2, \ldots, p_n}] = K \times [\psi_{p_1, p_2, \ldots, p_n}].
\]
From this relation and the proof of Theorem 1.2, we see that \( S_1 \) is an affine
transformation. Let members of $C_n(T)$ be affine transformations and let $S_{n+1} \in C_{n+1}(T)$. Then $S_{n+1}TS_{n+1}^{-1} = S_nT$ where $S_n \in C_n(T)$. $T$ and $S_nT$ are affine transformations, and $(X_n, T)$ and $(X_n, S_nT)$ are totally minimal dynamical systems (with quasi-discrete spectrum). Therefore it follows that

$$V_{S_{n+1}}(K \times [\psi_{p_1, p_2, \ldots, p_n}]) = K \times [\psi_{p_1, p_2, \ldots, p_n}].$$

We see easily that $S_{n+1}$ is an affine transformation. We have shown that the members of $\bigcup_{n=0}^{\infty} C_n(T)$ are affine transformations. If $S_{n+2} \in C_{n+2}(T)$ then we can write $S_{n+2}TS_{n+2}^{-1} = S_{n+1}T$ for some $S_{n+1} \in C_{n+1}(T)$. Since $S_{n+1}T$ is homeomorphic to $T$, $(X_n, S_{n+1}T)$ is totally minimal and $GN(S_{n+1}T) = n$. The affine transformation $S_{n+1}$ has a representation as follows:

$$S_{n+1} = T_{\tau_{n+1}} \beta_{n+1}$$

where $T_{\tau_{n+1}}$ is a translation of $X_n$ and $\beta_{n+1}$ is a continuous group automorphism of $X_n$. We put $T = T_{\tau_{n}}\beta$ for convenience. Since $S_{n+1}T$ is homeomorphic to $T$ and since

$$S_{n+1}T = T_{\tau_{n+1} + \beta_{n+1}(\tau)} \beta_{n+1} \beta$$

has quasi-discrete spectrum and $GN(S_{n+1}T) = n$, we see by induction that the matrix $[\beta_{n+2}]$ is lower triangular. $[\beta_{n+1}\beta]$ is a lower triangular matrix such that the numbers 1 appear throughout the diagonal, because the spectrum of $S_{n+1}T$ is quasi-discrete. Thus we follow that $[\beta_{n+1}]$ is a matrix such that

$$\begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & \ddots \\
1 & 1 & 1
\end{bmatrix}.$$ 

We show now that $C_{n+1}(T) = C_{n+2}(T)$. Let $S_{n+2} \in C_{n+2}(T)$. Then $S_{n+2}TS_{n+2}^{-1} = S_{n+1}$ where $S_{n+1} \in C_{n+1}(T)$. Furthermore, for the affine transformation $S_{n+1} = T_{\tau_{n+1}} \beta_{n+1}$,

$$S_{n+1}TS_{n+1}^{-1}T^{-1} = S_n$$

where $S_n \in C_n(T)$ and $S_n = T_{\tau_n} \beta_n$. Then from (1),

$$\beta_{n+1} \beta = \beta_n \beta \beta_{n+1},$$

$$r_{n+1} + \beta_{n+1}(r) = r_n + \beta_n(r) + \beta_n \beta(r_{n+1}).$$

From the fact that the spectrum of $S_nT$ is quasi-discrete, the matrix $[\beta_n\beta]$ is a lower triangular matrix such that the numbers 1 appear throughout the diagonal.
Furthermore, since the numbers 1 appear throughout the diagonal of \([\beta_{n+1}]\) and from the relation (2), we see that

\[
(4) \quad [\beta_n] = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & b_{31} & 0 & 1 \\
& & \ddots & \ddots \\
& & & b_{n1} \ldots b_{n,n-2} & 0 & 1
\end{bmatrix}
\]

where each \(b_{ij}\) is some integer. Therefore, from the relations (3) and (4), it is easily to see that

\[
r_n = \begin{bmatrix}
0 \\
\vdots \\
r_{n-1} \\
r_n
\end{bmatrix}
\]

Next, for the affine transformation \(S_n = T_{r_n} \beta_n\), we have \(S_nT S_n^{-1} = S_{n-1} T\) where \(S_{n-1} \in C_{n-1}(T)\), and therefore \(S_{n-1}\) is equal to an affine transformation \(T_{r_{n-1}} \beta_{n-1}\) such that \(\beta_{n-1}\) has a unimodular matrix

\[
[\beta_{n-1}] = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & c_{41} & 0 & 0 \\
& & \ddots & \ddots \\
& & & c_{n1} \ldots c_{n,n-2} & 0 & 0 & 1
\end{bmatrix}
\]

\[
\text{and} \quad r_{n-1} = \begin{bmatrix}
0 \\
0 \\
0 \\
r_{n-1,1} \\
r_{n-1,2} \\
r_{n-1,3}
\end{bmatrix}
\]

Following this argument step-by-step, we get that \(S_3 T S_3^{-1} = S_2 T\) where \(S_2 \in C_2(T)\), and the behavior of \(S_2\) is the following

\[S_2(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{n-1}, x_n + r_{2n}) \quad \text{(addition mod 1)}\]

where \(r_{2n}\) is some real number. Therefore it follows that \(S_2\) commutes with \(T\), i.e., \(S_2 \in C_1(T)\). From this fact,

\[S_3 \in C_2(T), \ldots, S_{n-1} \in C_{n-2}(T), S_n \in C_{n-1}(T)\]

and, from \(S_{n+1} T S_{n+1}^{-1} T^{-1} = S_n\),

\[S_{n+1} \in C_n(T) \quad \text{and} \quad S_{n+2} \in C_{n+1}(T)\]

Thus we have shown that \(C_{n+1}(T) = C_{n+2}(T)\).

We give a translation \(T_d:\)

\[(x_1, x_2, \ldots, x_n) \rightarrow (x_1 + d_1, x_2 + d_2, \ldots, x_n + d_n) \quad \text{(additions mod 1)}\]
where \(d_1, d_2, \ldots, d_{n-1}\) and \(d_n\) are nonzero real numbers, but not integers. Then we have the following equation

\[
T_d TT_d^{-1} T^{-1}(x_1, x_2, \ldots, x_n) = (x_1, x_2 + d_2^{(2)}, x_3 + d_3^{(2)}, \ldots, x_n + d_n^{(2)}) \quad \text{(additions mod 1)}
\]

where

\[
d_k^{(2)} = -\sum_{j=1}^{k-1} a_k d_j, \quad k = 2, 3, \ldots, n.
\]

Putting

\[
T_d^{(a)}(x_1, x_2, \ldots, x_n) = (x_1, x_2 + d_2^{(2)}, x_3 + d_3^{(2)}, \ldots, x_n + d_n^{(2)}) \quad \text{(additions mod 1)},
\]

we have that

\[
T_d^{(a)} TT_d^{-1} T^{-1}(x_1, x_2, \ldots, x_n) = (x_1, x_2, x_3 + d_3^{(3)}, x_4 + d_4^{(3)}, \ldots, x_n + d_n^{(3)}) \quad \text{(additions mod 1)}
\]

where

\[
d_k^{(3)} = -\sum_{j=2}^{k-1} a_k d_j^{(2)}, \quad k = 3, 4, \ldots, n.
\]

We obtain from the argument that

\[
T_d^{(a)} T(x_1, x_2, \ldots, x_n) = TT_d^{(a)}(x_1, x_2, \ldots, x_n)
\]

where \(d_n^{(n)} = -a_n a_{n-1} d_{n-1}^{(n-1)}\) and

\[
T_d^{(a)}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n + d_n^{(n)}) \quad \text{(addition mod 1)}.
\]

Since \(a_{n-1}\) is a nonzero integer for \(j=2, \ldots, n\), we can choose nonzero real numbers (but not integers) \(d_1, d_2, \ldots, d_{n-1}\) and \(d_n\) such that

\[
d_k^{(2)} \neq 0 \quad \text{for} \quad k = 2, 3, 4, \ldots, n,
\]

\[
d_k^{(3)} \neq 0 \quad \text{for} \quad k = 3, 4, 5, \ldots, n,
\]

\[
d_k^{(4)} \neq 0 \quad \text{for} \quad k = 4, 5, 6, \ldots, n,
\]

\[
\vdots
\]

\[
d_k^{(n-1)} \neq 0 \quad \text{for} \quad k = n-1, n,
\]

\[
d_n^{(n)} \neq 0.
\]

In particular, we can choose \(d_1\) such that the number \(-\frac{1}{2} \cdot d_1\) is the first coordinate \(r_1\) of \(r = (r_1, r_2, \ldots, r_n)\). For such real numbers \(d_j, j = 1, 2, \ldots, n\), and the translation \(T_d\) where \(d = (d_1, d_2, \ldots, d_n)\), it follows that \(T_d^{(a)} T = TT_d^{(a)}\), and since \(d_n^{(n)} \neq 0\) (mod 1), \(T_d^{(a)}\) is not the identical map. Thus we have that \(T_d \in C_n(T) - C_{n-1}(T)\).

Here we put \(S = T_b \beta'\) where \(\beta'\) is a group automorphism of \(X_n\) such that \(\beta' x = -x, x \in X_n\), and \(T_b\) is a translation determined by the element \(b\) satisfying the equation \(b + [\beta'][r] = d + r + [\beta][b]\). Then \(STS^{-1} T^{-1} = T_d\) and therefore \(S \in C_{n+1}(T) - C_n(T)\).

We have shown that \(CN(T) = n+1\).
Corollary 3.2. Let $T$ be an affine transformation and let $(X_1, T)$ be a totally minimal dynamical system (with quasi-discrete spectrum). If $GN(T) = 1$, then $CN(T) = 2$.

The corollary was shown by Adler [2].

Corollary 3.3. Let $T$ be a skew product transformation of $X_2$ and let $(X_2, T)$ be a totally minimal dynamical system (with quasi-discrete spectrum) and let $GN(T) = 2$. Then $CN(T) = 3$.

The statement is direct from Theorem 3.1.

The following theorem is a result better than Theorem 3.1.

Theorem 3.4. Let $T$ be an affine transformation of $X_n$ and let $(X_n, T)$ be a totally minimal dynamical system (with quasi-discrete spectrum) with $GN(T) = m$. Then $CN(T) = m + 1$.

Proof. We put $T = T_r \beta$ where $T_r$ is a translation of $X_n$ and $\beta$ a continuous group automorphism of $X_n$. From Theorem 2.1 and the analogous argument of Corollary 2.5, we may suppose that since $GN(T) = m$, the affine transformation $T$ is written as follows

$$T_r \beta: (x_1, x_2, \ldots, x_n)$$

$$\rightarrow \left( x_1 + r_1, \ldots, x_{i_1} + r_{i_1}, x_{i_1+1} + \sum_{j=1}^{i_1} a_{i_1+j-1} x_j + r_{i_1+j-1}, \ldots, \right)$$

$$x_{i_2} + \sum_{j=1}^{i_2} a_{i_2+j-1} x_j + r_{i_2+j-1}, \ldots, x_{i_{m-1}+1} + \sum_{j=1}^{i_{m-1}} a_{i_{m-1}+1+j-1} x_{i_{m-1}+j-1}, \ldots, x_{i_m} + \sum_{j=1}^{i_m} a_{i_m+j-1} x_j + r_{i_m} \right)$$

(aditions mod 1).

Here each $a_{ij}$ is an integer and each $l_i$ an integer such that $l_{n} = n$ and there is at the least one nonzero integer in

$$\{a_{i-1+1}, a_{i-1+1}, a_{i-1+1}, a_{i-1+1}, \ldots, a_{i-1+1}\} \quad (l_0 = 0)$$

for $i=2, 3, \ldots, m$. Since $T$ is totally ergodic, the real numbers $r_k$, $k = 1, 2, \ldots, l_1$, are integrally independent. Thus the matrix $[\beta]$ is of the form

$$\begin{bmatrix}
E_{11} & & & & \\
A_{21} & E_{22} & & & \\
A_{31} & A_{32} & E_{33} & & \\
& & & & \\
A_{m1} & & & A_{m,m-1} & E_{mm}
\end{bmatrix}$$

(2)

where $E_{ii}$, $i=1, 2, \ldots, m$, are the identity matrices of order $(l_i - l_{i-1}) \times (l_i - l_{i-1})$. 
and $A_{ij}$, $i \neq j$, $i = 2, 3, \ldots, m$, $j = 1, 2, \ldots, m$, are matrices of order $(l_i - l_{i-1}) \times (l_j - l_{j-1})$ (but $l_0 = 0$). Since $T$ has quasi-discrete spectrum and $GN(T) = m$, the blocks of $[\beta]$, $A_{i+1}$, $i = 2, 3, \ldots, m$, are nonzero matrices and there is at the least one nonzero integer in the first row

$$(a_{i-1} + 1, a_{i-1} + 2, \ldots, a_{i-1} + m)$$

of the matrix $A_{i+1}$. For each positive integer $k$ and each $S_k \in C_k(T)$ with $S_k = Tr \beta_k$, it follows by induction that the matrix $[\beta_k]$ is of the form

$$
\begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1m} \\
B_{21} & B_{22} & \cdots & B_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
B_{m1} & B_{m2} & \cdots & B_{mm}
\end{bmatrix}
$$

(3)

where $B_{ij}$, $i, j = 1, 2, \ldots, m$, are matrices of order $(l_i - l_{i-1}) \times (l_j - l_{j-1})$. Because, if $k = 1$, then $\beta_1 \beta = \beta \beta_1$. Since the spectrum of $T_\beta$ is quasi-discrete, the matrix $[\beta_1]$ is of the form (3). Let $[\beta_2]$ be of the form (3). Then the rank of the group $O(T) \cap G(T)$ is equal to the rank of the group $O(S_kT) \cap G(S_kT)$ since $V_{s_k\beta_1}G(T) = G(S_kT)$ for $i = 1, 2, \ldots, m$. Thus we see that the matrix $[\beta_2, \beta]_i$ is of the form (2). Since $\beta_{k+2} = \beta_k \beta_{k+1}$, the matrix $[\beta_{k+1}]$ is of the form (3). We show that $C_{m+1}(T) = C_{m+2}(T)$. If $S_{m+2} \in C_{m+2}(T)$ with $S_{m+2} = Tr_{m+2} \beta_{m+2}$, then we have $S_{m+2}TS_{m+2}^{-1} = S_{m+1}T$ where $S_{m+1} \in C_{m+1}(T)$ with $S_{m+1} = Tr_{m+1} \beta_{m+1}$. From the equation above, we have $\beta_{m+2} = \beta_m \beta_{m+2}$. Since the matrix $[\beta_{m+1}]$ is of the form (2), the matrix $[\beta_{m+1}, \beta]$ is also of the form (2). For the affine transformation $S_{m+1} = Tr_{m+1} \beta_{m+1}$, $S_{m+1}TS_{m+1}^{-1} = S_mT$ where $S_m \in C_m(T)$ with $S_m = Tr_m \beta_m$. Therefore $\beta_{m+1} = \beta_m \beta_{m+1}$ and

$$r_{m+1} + \beta_{m+1}(r) = r_m + \beta_m(r) + \beta_m \beta(r_{m+1}).$$

(4)

Since $[\beta_m]$ and $[\beta_{m+1}]$ are of the form (2) and $[\beta_{m+1}] = [\beta_m \beta_{m+1}]$, it follows that the blocks of $[\beta_m]$, $A_{i+1}$, $i = 2, 3, \ldots, m$, are zero matrices. From (4) and the form of the matrix $[\beta_m]$ obtained above, we see that $T_m$ is of the form

$$r_m = \begin{bmatrix}
0 \\
\vdots \\
0 \\
r_{m1} + 1 \\
\vdots \\
r_{mn}
\end{bmatrix}.$$
but it follows that $S_1$ is the identity map since the continuous group automorphism $\beta_2$ is the identity map and $r_2 = (0, \ldots, 0, r_{2m-1}, \ldots, r_{2n})$. Therefore $S_1 \in C_0(T)$ and $S_{m+2} \in C_{m+1}(T)$. Consequently, $C_{m+1}(T) = C_{m+2}(T)$. Since the blocks $A_{i_{i-1}}$, $i=2, 3, \ldots, m$, of the matrix $[\beta]$ are nonzero and moreover the first row in $A_{i_{i-1}}$, $(a_{i_{i-1}+1} l_{i-2} + 1, a_{i_{i-1}+1} l_{i-2} + 2, \ldots, a_{i_{i-1}+1} l_{i-1})$, is a nonzero vector. By the same manner used in Theorem 3.1, we can construct an affine transformation $S = T_m \beta'$ such that $S \in C_{m+1}(T)$, but $S \notin C_m(T)$. Therefore $CN(T) = m+1$.

**Theorem 3.5.** Let $T$ be an affine transformation of $X_n$ and let $(X_n, T)$ be a totally minimal dynamical system (with quasi-discrete spectrum) with $GN(T) = m$. Then $C_{m+1}(T)$ is a subgroup of the group consisting of all homeomorphisms of $X_n$.

**Proof.** As in Theorems 3.1 and 3.4, we may suppose that the affine transformation $T = T_{r} \beta$ is written as follows

$$T: (x_1, x_2, \ldots, x_n) \rightarrow \left( x_1 + r_1, \ldots, x_{l_1} + r_{l_1}, x_{l_1+1} + \sum_{j=1}^{l_1} a_{l_1+1} x_j + r_{l_1+1}, \ldots, x_{l_2} + \sum_{j=1}^{l_2} a_{l_2} x_j + r_{l_2}, \ldots, x_{l_{m-1}} + \sum_{j=1}^{l_{m-1}} a_{l_{m-1}+1} x_j + r_{l_{m-1}+1}, \ldots, x_{l_m} + \sum_{j=1}^{l_m} a_{l_m} x_j + r_{l_m} \right)$$

(additions mod 1),

where each $a_{ij}$ is an integer and indices $l_j, j=1, 2, \ldots, m$, are integers such that $l_m = n$ and there is at the least one nonzero integer in

$$\{a_{lj_{j-1}+1} l_j + 1, a_{lj_{j-1}+1} l_j + 2, \ldots, a_{lj_{j-1}+1} l_j + 1 \} \quad \text{(but } l_0 = 0).$$

Since $(X_n, T)$ is (totally) minimal, real numbers $r_k, k=1, 2, \ldots, l_1$ are integrally independent. We consider affine transformations $Q_k = T_{r_k} \beta_k, k=0, 1, 2, \ldots, m$, where $T_{r_k}$ are translations of $X_n$ and $\beta_k$ a continuous group automorphism of $X_n$.

Suppose now that the matrix $[\beta_m]$ is of the form

$$\begin{bmatrix}
A_{11} & & & \\
A_{21} & A_{22} & & \\
A_{31} & A_{32} & A_{33} & \\
\vdots & \vdots & \ddots & \ddots \\
A_{m1} & \cdots & & A_{m, m-1} & A_{mm}
\end{bmatrix}$$

where $A_{ij}, i,j=1, 2, \ldots, m$, are matrices of order $(l_i - l_{i-1}) \times (l_j - l_{j-1})$ (but $l_0 = 0$).

The form of $[\beta]$ has the form (1) and in particular $A_{ii}, i=1, 2, \ldots, m$, are the identity matrices. We put

$$Q_{m-j} T Q_{m-j}^{-1} T^{-1} = Q_{m-1-j}, \quad j = 0, 1, 2, \ldots, m-1.$$
From the relation (2), the matrices \([\beta_{m-j}], j = 1, 2, \ldots, m-1\), are of the form (1). Moreover, from \([\beta_{m-j}] = [\beta_{m-1-j} \beta_{m-j}], [\beta_{m-j}], j = 1, 2, \ldots, m\), are the matrices such that

\[
\begin{bmatrix}
E_{11} & & & \\
& \ddots & & 0 \\
& & \ddots & 0 \\
B_{ij} & 0 & \cdots & 0 \\
& & \cdots & \cdots \\
& & & \ddots \\
B_{m1} & \cdots & B_{m-1-j} & 0 & \cdots & 0 & E_{mm}
\end{bmatrix}
\]

where each \(E_{ij}\) is the identity matrix of order \((l_i - l_{i-1}) \times (l_i - l_{i-1})\) and each \(B_{ij}\) is the matrix of order \((l_i - l_{i-1}) \times (l_j - l_{j-1})\) (but \(l_0 = 0\)). From this fact and the relation (2), the translation \(T_{r_{m-j}}\) is of the form

\[
r_{m-j} = \begin{bmatrix}
0 \\
\vdots \\
0 \\
r_{m-j} l_{j-1} + 1 \\
\vdots \\
r_{m-j} n
\end{bmatrix}, \quad j = 2, 3, \ldots, m-1.
\]

Thus it follows that if an affine transformation \(S\) is a transformation consisting of a translation composed with continuous group automorphisms \(\beta'\) of \(X_n\) such that the matrix \([\beta']\) is of the form (1), then \(S \in C_{m+1}(T)\). Here we show that \(C_{m+1}(T)\) is a subgroup. For \(S, S'' \in C_{m+1}(T)\) with \(S = T_r \beta'\) and \(S'' = T_r \beta''\). Let us put

\[
D = (S''S^{-1})T(S''S^{-1})^{-1}T^{-1},
\]

then \(\beta^{(3)} = (\beta' \beta'^{-1})\beta(\beta' \beta'^{-1})^{-1}\beta^{-1}\) is a continuous group automorphism of \(X_n\) and \(T_{(3)}\) a translation of \(X_n\) where

\[
r^{(3)} = r'' + \beta' r' - \beta\beta' r'' - \beta' r - \beta' r' - \beta' \beta' r'' - \beta' \beta' r' - \beta' \beta' r'' - \beta^{-1} r.
\]

The affine transformation \(D = T_{(3)} \beta^{(3)}\) belongs to \(C_{m+2}(T)\), because the matrices \([\beta']\) and \([\beta'']\) are of the form (1). Consequently \(S''S^{-1} \in C_{m+1}(T)\) since

\[
D = (S''S^{-1})T(S''S^{-1})^{-1}T^{-1} \quad \text{and} \quad CN(T) = m+1.
\]

We have shown that \(C_{m+1}(T)\) is a subgroup.

As before, \((\Omega, \Sigma, \mu)\) is a finite measure space and \(T\) is a totally ergodic automorphism of \((\Omega, \Sigma, \mu)\) with quasi-discrete spectrum. We consider a normalized measure space \((\Omega, \Sigma, \mu)\).

In order to prove the following lemma, we invoke properties of entropy.
Lemma 3.6. Let $S$ and $W$ be as in Corollary 2.6 and let $T'$ be a metric automorphism induced by $T$ such that $T' = SW$. If for any $f \in O_\mu(T)$, $Y$ is a subgroup generated by an orbit of $f$ under $V_s$, then $Y$ is finitely generated.

Proof. We denote by $G$ the subgroup of $Y$ generated by the set

$$\{ V_s^i f : j = 1, 2, \ldots \}. $$

If $V_s G \neq G$. Then it is well known that $T' = SW$ has positive entropy. On the other hand, since $T$ is totally ergodic and has quasi-discrete spectrum, the entropy of $T$ is zero. Thus $V_s G = G$ and

$$f = V_s^{n_1} f^{q_1} \cdot V_s^{n_2} f^{q_2} \cdot \ldots \cdot V_s^{n_k} f^{q_k}$$

where each $n_j$ is an integer and each $q_j$ an integer. If $G'$ is a subgroup generated by $\{ V_s^i f : j = 1, 2, \ldots, k \}$ where $k = \max \{ n_1, n_2, \ldots, n_k \}$, it follows that $V_s G' = G'$. Thus $Y = G'$ and $Y$ is finitely generated.

We denote by $T'$ the metric automorphism induced by an automorphism $T$.

Theorem 3.7. If $G_\mu N(T) = m$, then $CN(T') = m + 1$. Furthermore, if $G_\mu N(T) = +\infty$ then $CN(T') = +\infty$.

Proof. Since $T$ is totally ergodic and has quasi-discrete spectrum, $O_\mu(T)$ is an orthonormal base of $L^2(\Sigma(\mu))$, and, by Corollary 2.6, the metric automorphism $T'$ has a representation $T' = SW$ on $\Sigma(\mu)$ for metric automorphisms $S$ and $W$ such that $V_s O_\mu(T) = O_\mu(T)$ and $W$ has each function in $O_\mu(T)$ as a proper function. For any $f \in O_\mu(T)$, it follows from Lemma 3.6 that $Y(f)$, the smallest subgroup generated by an orbit of $f$ under $V_s$, is finitely generated. Since $T$ is totally ergodic $Y(f)$ is torsion free. Here we suppose that the number of generators of $Y(f)$ is $n$. Since there exists a nontrivial $T$-invariant sub $\sigma$-algebra $\Phi(\mu)$ such that

$$L^2(\Phi(\mu)) = \text{span} \ Y(f)$$

and $T'$ has quasi-discrete spectrum on $L^2(\Phi(\mu))$, it follows from the proof of Theorem 1.2 that there exists the $n$-dimensional torus $X_n$ such that the dynamical system $(X_n, A_f)$ is totally minimal, and that $T'$ restricted to $\Phi(\mu)$ is isomorphic to the metric automorphism $A_f$ induced by $A_f$; in other words, $\varphi_f T' = A_f \varphi_f$, where $\varphi_f$ is a metric isomorphism from the measure algebra $\Phi(\mu)$ to the measure algebra associated with the measure space consisting of the Borel field of $X_n$ and the normalized Haar measure. We show first that $CN(T') = m + 1$ if $G_\mu N(T) = m$ and $m$ is an integer. Suppose now that $G_\mu N(A_f) \leq G_\mu N(T) - 1 (= m - 1)$ for each $f \in O_\mu(T)$.

Then we have

$$G_\mu(T) \cap Y(f) = G_\mu(T)_{m-1} \cap Y(f),$$

and

$$G_\mu(T) = \bigcup \{ G_\mu(T) \cap (K \times Y(f)) : f \in O_\mu(T) \}$$

$$= \bigcup \{ G_\mu(T)_{m-1} \cap (K \times Y(f)) : f \in O_\mu(T) \}$$

$$= G_\mu(T)_{m-1}.$$
This contradicts $G_a N(T) = m$. Therefore there exists a function $f \in O_a(T)$ such that $GN(A_f) = m$. Let us put for each $f \in O_a(T)$

$$C_0(A_f) = \{ A : A' = \varphi_f S' \varphi_f^{-1} \text{ for } S' \in C_0(T') \}$$

and

$$C_n(A_f) = \{ A : A' = \varphi_f S' \varphi_f^{-1} \text{ for } S' \in C_n(T') \}, \quad n = 1, 2, \ldots .$$

$C_n(A_f)$, $n = 0, 1, 2, \ldots$, are the generalized commuting classes of affine transformations with respect to the affine transformation $A_f$. It follows from Theorem 3.4 that for $f \in O_a(T)$ with $GN(A_f) = m$, $CN(A_f) = m+1$. It is clear that $CN(A_g) \leq m$ for $g \in O_a(T)$ with $GN(A_g) \leq m-1$. Consequently we see that

$$m+1 = \max \{ n : C_n(A_f) = C_{n+1}(A_f) \text{ for } f \in O_a(T) \}.$$ 

Therefore we have $CN(T') = m+1$. It remains to show that if $G_a N(T) = \infty$ then $CN(T') = \infty$. For the finitely generated group $Y(f)$ of a member $f$ of $O_a(T')$, we denote by $g_0$ a function $g$ such that $g \in O_a(T)$ and $g \notin Y(f)$, and by $g_1$ a function $g$ such that $g \in O_a(T)$ and $g \notin Y(f) \cup Y(g_0)$ and so on. Then we can choose infinitely many set functions $\{ g_j : j = 0, 1, 2, \ldots \}$ such that $g_j$ belongs to a group of distinct order for $j = 0, 1, 2, \ldots$. Here we put

$$G_{i_m} = \prod_{j=0}^m Y(g_j), \quad m = 0, 1, 2, \ldots ,$$

and let the index $l_m$ be the least integer such that $(K \times \prod_{j=0}^m Y(g_j)) \cap G_a(T) \subseteq G_a(T_{i_m})$. Then $V(0) \subseteq G_{i_m}$ and $l_k \uparrow +\infty$ as $k \to +\infty$. Let $X$ be the dual space of a discrete group $G_{i_m}$ and $Q$ a transformation on $X$ induced by $T$. Then $Q$ is totally ergodic with respect to Haar measure $\mu'$ on $X$ and has quasi-discrete spectrum and $G_a N(Q) = l_{i_m}$. We see from Theorem 3.4 that the generalized commuting order of $Q$ is $l_{i_m}+1$. From this, we follow that $CN(T') \geq l_{i_m}+1$. Since $m$ is an arbitrary positive integer, we have $CN(T') = \infty$.

**Theorem 3.8.** If $CN(T') = m$, then $G_a N(T) = m - 1$. Furthermore, if $CN(T') = +\infty$ then $G_a N(T) = +\infty$.

The proof is an application of Theorems 3.1, 3.4 and 3.7.

**Corollary 3.9.** If $G_a N(T) = m$, then $C_{m+1}(T')$ is a subgroup of the group consisting of all metric automorphisms of $\Sigma(\mu)$ onto itself. Furthermore, if $G_a N(T) = +\infty$ then $C_\infty(T') = \bigcup_{n>0} C_n(T')$ is a subgroup.

This corollary is proved by Theorems 3.5 and 3.7.

**References**


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