TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 152, November 1970

EXISTENCE AND UNIQUENESS OF FIXED-POINTS FOR SEMIGROUPS OF AFFINE MAPS(1)

BY
ROBERT E. HUFF

Abstract. The Day fixed-point theorem is extended to include both existence and uniqueness. For uniqueness of fixed-points, continuity for pointwise limits of a semigroup of continuous affine maps is needed; necessary and sufficient conditions for this are obtained and compared with the stronger condition of equicontinuity. The comparison is between, on the one hand, the above condition, separate continuity, and weak compactness, and, on the other hand, equicontinuity, joint continuity, and strong compactness. An extension of the Kakutani fixed-point theorem results. Also as a corollary, known necessary and sufficient conditions for continuity of the convolution operation are obtained.

1. **Introduction.** M. M. Day [3] proved that a left-amenable semigroup S acting as continuous affine maps on a compact convex set K leaves some point in K fixed. In [10] it was shown that these hypotheses imply that there is an affine map on K which sends each point in K into a fixed-point in its closed convex orbit; also if S is equicontinuous and right amenable, then there is at most one fixed-point in every such orbit. In §3 below we derive extensions of these results by different methods. The assumption of equicontinuity for uniqueness can be replaced by the assumption that every pointwise limit of convex combinations of members of S is continuous; we obtain necessary and sufficient conditions for this latter assumption to hold, together with analogous necessary and sufficient conditions for equicontinuity. The entire development is an exploitation of properties of the affine space of a compact convex set, developed in §2. In §4 we apply our results to obtain new proofs of known necessary and sufficient conditions for continuity of convolutions.

If S is a nonvoid set, then $m(S)$ denotes the space of all bounded, real-valued functions on S, with pointwise linear operations, the uniform norm, and pointwise ordering ($f \geqq 0$ if and only if $f(s) \geqq 0$ for all s in S).

Presented to the Society, January 25, 1969 under the title *On uniqueness of fixed-points*; received by the editors October 24, 1969.

AMS 1969 *subject classifications*. Primary 4785, 5485; Secondary 4601, 5428.

Key words and phrases. Fixed-points, equicontinuity, separate continuity, joint continuity, weak compactness, almost periodic function, weakly almost periodic functions, convolutions, amenable semigroups, affine space, functions spaces, invariant means.

(1) The research, supported in part by a National Science Foundation Fellowship, formed part of the author's doctoral dissertation prepared at the University of North Carolina under the direction of Professor B. J. Pettis, whose guidance is gratefully acknowledged.

By a *function space* on a set S we mean a linear subspace X of $m(S)$ such that X contains the constant functions; X is *separating* if for every two distinct points x, y in S there exists an f in X such that $f(x) \neq f(y)$. $M(X)$ denotes the set of positive linear functionals on X of norm one; the elements of $M(X)$ will be called *X-means*. $M(X)$ is a weak* compact convex subset of X^*; we shall always assume that $M(X)$ is given the weak* topology. The *natural map* $q_X: S \to M(X)$ is given by $[q_X(s)](f) = f(s)$ $(f \in X, s \in S)$. The separation theorem implies that $M(X) = \mathrm{cl}\,(\mathrm{co}\,(q_X(S)))$ (co (A) denotes the convex hull of a set A; cl (co (A)), its closure). For these and other properties of means, see [2].

Let S be a set which is a semigroup under a binary operation denoted by juxtaposition. A function space X on S is said to be *left* [respectively, *right*] *invariant* provided $f \circ L_s$ $[f \circ R_s]$ is in X whenever f is in X and s is in S, where $L_s: S \to S$ $[R_s: S \to S]$ is defined by $L_s(t) = st$ $[R_s(t) = ts]$. An X-mean is then said to be *left* [*right*] *invariant* if $\mu(f \circ L_s) = \mu(f)$ $[\mu(f \circ R_s) = \mu(f)]$ for all f in X and s in S. S is said to be *left* [*right*] *amenable* if there exists a left [right] invariant $m(S)$-mean [2].

A function f on a semigroup S is said to be *strongly* [*weakly*] *almost periodic* if the set $\{f \circ L_s : s \in S\}$ is relatively compact [relatively weakly compact] in $m(S)$; equivalently, if $\{f \circ R_s : s \in S\}$ is relatively compact [relatively weakly compact] in $m(S)$ (see [4, p. 80] for references). The function space AP (S) [WAP (S)] of strongly [weakly] almost periodic functions is complete and two-sided invariant [6]. Von Neumann proved that there exists a two-sided invariant AP (S)-mean if S is a group (see [7, §18]); more generally, Ryll-Nardzewski has shown that there exists a two-sided invariant WAP (S)-mean if S is a group [18].

Throughout this paper, for any space of functions the letter p will denote the topology of pointwise convergence, and the letter u will denote the topology of uniform convergence.

The symbol $\|\cdot\|$ will be used for all norm topologies, and w will denote the weak topology for any normed linear space.

2. **Spaces of affine functions.** In this section we develop some needed properties of certain spaces of functions on a compact convex subset K of a locally convex (Hausdorff) space (following [1] we shall call such a set a *convex compactum*).

If K is a convex compactum, the function space $A(K)$, consisting of all real-valued, continuous, affine maps on K, is separating. The natural map $q_{A(K)}: K \to M(A(K))$ is continuous, one-to-one, and affine; thus $q_{A(K)}(K)$ is compact and convex. But then $M(A(K)) = \mathrm{cl}\,(\mathrm{co}\,(q_{A(K)}(K))) = q_{A(K)}(K)$; thus, $q_{A(K)}$ is an affine homeomorphism of K onto $M(A(K))$. From this and the fact that $A(K)^*$ is positively generated (which follows from [13, Theorem 23.6]), the weak topology for $A(K)$ coincides with the topology of pointwise convergence on K.

Let X be any function space on a set S, and let ψ be a map on S to a convex compactum K such that $f \circ \psi$ is in X for all f in $A(K)$. Let $\psi^*: A(K) \to X$ be given by $\psi^*(f) = f \circ \psi$ $(f \in A(K))$, and let $\psi^{**}: M(X) \to M(A(K))$ be given by $\psi^{**}(\mu) = \mu \circ \psi^*$ $(\mu \in M(X))$. Then the function $\rho = (q_{A(K)})^{-1} \circ \psi^{**}$ is a continuous affine

map on $M(X)$ to K such that $\rho \circ q_X = \psi$; furthermore, since $M(X) = \mathrm{cl}\,(\mathrm{co}\,(q_X(S)))$, ρ is the only continuous affine map with this property. For every μ in $M(X)$, we have $f(\rho(\mu)) = \mu(f \circ \psi)$ for all f in $A(K)$. (These remarks are the content of Lemma 1.4 of [17].)

We remark that everything in the previous two paragraphs holds if one replaces $A(K)$ by any separating subfunction space of $A(K)$. The facts above will be used heavily in the next two sections.

The notation $A(K, K)$ will be used for the semigroup (under composition) of all continuous affine maps on K to K.

The product space $A(K)^{A(K)}$ with its *strong* [respectively, *weak*] *operator topology* is the space $((A(K), \|\cdot\|)^{A(K)}, p)$ $[((A(K), w)^{A(K)}, p)]$. $\mathscr{L}(A(K))$ denotes the subspace of $A(K)^{A(K)}$ consisting of all bounded linear operators on $A(K)$.

Suppose φ is in $A(K, K)$. Then φ has an "adjoint" φ^* in $\mathscr{L}(A(K))$ given by $\varphi^*(f) = f \circ \varphi\,(f \in A(K))$. The essential properties of φ^* are (i) $\|\varphi^*\| = 1$, (ii) $\varphi^*(C) = C$ (where C denotes any constant function), and (iii) φ^* is order preserving. Let $\mathscr{A}(K)$ denote the subset of $\mathscr{L}(A(K))$ consisting of all members of $\mathscr{L}(A(K))$ with properties (i), (ii), and (iii); and let β: $A(K, K) \to \mathscr{A}(K)$ be defined by $\beta(\varphi) = \varphi^*$ $(\varphi \in A(K, K))$. $\mathscr{A}(K)$ is closed in $A(K)^{A(K)}$ with respect to either the weak or the strong operator topology. $\mathscr{A}_s(K)$ $[\mathscr{A}_w(K)]$ denotes $\mathscr{A}(K)$ with the induced strong [weak] operator topology.

ASSERTION. *β is an affine homeomorphism of $(A(K, K), p)$ [respectively, $(A(K, K), u)$] onto $\mathscr{A}_w(K)$ [respectively, $\mathscr{A}_s(K)$].*

Proof. β is affine and one-to-one. To see that β is onto, let ψ be an element of $\mathscr{A}(K)$. Then ψ has an adjoint ψ^*: $M(A(K)) \to M(A(K))$; $(q_{A(K)})^{-1} \circ \psi^* \circ q_{A(K)}$ is in $A(K, K)$, and $\psi = \beta((q_{A(K)})^{-1} \circ \psi^* \circ q_{A(K)})$.

Let $\{\varphi_\alpha\}$ be a net in $A(K, K)$, and let φ_0 be a point in $A(K, K)$.

We have $\varphi_\alpha \to \varphi_0$ in $(A(K, K), p) \Leftrightarrow \varphi_\alpha(k) \to \varphi_0(k)$ in K for all k in $K \Leftrightarrow f \circ \varphi_\alpha(k) \to f \circ \varphi_0(k)$ for all f in $A(K)$ and all k in $K \Leftrightarrow f \circ \varphi_\alpha \to f \circ \varphi_0$ in $(A(K), p) = (A(K), w)$ for all f in $A(K, K) \Leftrightarrow \beta(\varphi_\alpha) \to \beta(\varphi_0)$ in $\mathscr{A}_w(K)$. Hence β is a homeomorphism of $(A(K, K), p)$ onto $\mathscr{A}_w(K)$.

Now suppose $\varphi_\alpha \to \varphi_0$ in $(A(K, K), u)$. Let f be in $A(K)$. Let $\varepsilon > 0$ be given and consider the member U of the uniformity for K given by

$$U = \{(x, y) \in K \times K : |f(x) - f(y)| < \varepsilon\}.$$

There exists an index α_0 such that $\alpha \geqq \alpha_0$ implies $(\varphi_\alpha(k), \varphi_0(k))$ is in U for all k in K, i.e. $|f \circ \varphi_\alpha(k) - f \circ \varphi_0(k)| < \varepsilon$ for all k in K, or $\|[\beta(\varphi_\alpha)](f) - [\beta(\varphi_0)](f)\| \leqq \varepsilon$. Thus for every f in $A(K)$, $[\beta(\varphi_\alpha)](f) \to [\beta(\varphi_0)](f)$ in $(A(K), \|\cdot\|)$, i.e. $\beta(\varphi_\alpha) \to \beta(\varphi_0)$ in $\mathscr{A}_s(K)$.

Conversely, suppose $\beta(\varphi_\alpha) \to \beta(\varphi_0)$ in $\mathscr{A}_s(K)$. Let U be a member of the uniformity of K. There exist $f_1, \ldots, f_n$ in $A(K)$ such that

$$\{(x, y) \in K \times K : |f_i(x) - f_i(y)| < 1;\, i = 1, \ldots, n\} \subset U.$$

There is an index α_0 such that $\alpha \geqq \alpha_0$ implies

$$\|[\beta(\varphi_\alpha)](f_i) - [\beta(\varphi_0)](f_i)\| = \|f_i \circ \varphi_\alpha - f_i \circ \varphi_0\| < 1$$

for all $i=1,\ldots,n$, i.e. $|f_i(\varphi_\alpha(k))-f_i(\varphi_0(k))|<1$ for all $i=1,\ldots,n$, and all k in K. Hence $(\varphi_\alpha(k), \varphi_0(k))$ is in U for all k in K, and so $\varphi_\alpha \to \varphi_0$ in $(A(K, K), u)$. We have now proved that β is a homeomorphism of $(A(K, K), u)$ onto $\mathscr{A}_s(K)$.

THEOREM 1. *Let K be a convex compactum, and let S be a semigroup of continuous affine maps on K to K. Let T denote the closure of S in (K^K, p). The following three statements are equivalent*:

(1) *T consists of continuous (affine) functions.*

(2) *The mapping $(T, p)\times K \to K$ given by $(t, k) \to t(k)$ is separately continuous.*

(3) *For every f in $A(K)$ the set $f\circ S=\{f\circ s : s\in S\}$ is relatively compact in $(A(K), w)$.*

THEOREM 2. *Let K, S, and T be as in Theorem 1. The following three statements are equivalent*:

(1) *S is equicontinuous on K.*

(2) *The mapping $(T, p)\times K \to K$ given by $(t, k) \to t(k)$ is jointly continuous.*

(3) *For every f in $A(K)$, the set $f\circ S$ is relatively compact in $(A(K), \|\cdot\|)$.*

Proofs. The equivalence of (1) and (2) in Theorem 1 is clear; for Theorem 2 it follows from [12, Theorems 14, 15, 16, pp. 232–233].

Suppose (1) holds in Theorem 1 [respectively, Theorem 2]. Then T is a compact subset of $(A(K, K), p)$ [$(A(K, K), u)$ by [12, Theorems 14, 15, p. 232]], so that $\beta(T)$ is compact in $\mathscr{A}_w(K)\subset((A(K), w)^{A(K)}, p)$ [$\mathscr{A}_s(K)\subset((A(K), \|\cdot\|)^{A(K)}, p)$]. Therefore, if f is in $A(K)$, $(\beta(T))(f)$ is compact in $(A(K), w)$ [$(A(K), \|\cdot\|)$] and contains $f\circ S$; hence (3).

Conversely, suppose (3) holds in Theorem 1 [respectively, Theorem 2]. Let $P=(\prod\{\mathrm{cl}\,(f\circ S) : f\in A(K)\}, p)$, where $\mathrm{cl}\,(f\circ S)$ denotes the closure of $f\circ S$ in $(A(K), w)$ [$(A(K), \|\cdot\|)$]. Then P is contained in $((A(K), w)^{A(K)}, p)$ [$((A(K), \|\cdot\|)^{A(K)}, p)$] and contains $\beta(S)$. P is compact, so that the closure $\mathrm{cl}\,(\beta(S))$ of $\beta(S)$ in

$$((A(K), w)^{A(K)}, p)$$

[$((A(K), \|\cdot\|)^{A(K)}, p)$] is compact; $\mathrm{cl}\,(\beta(S))$ is contained in $\mathscr{A}(K)$ since $\mathscr{A}(K)$ is closed. Thus $\beta^{-1}(\mathrm{cl}\,(\beta(S)))$ is compact in $(A(K, K), p)$ [$(A(K, K), u)$] and contains S. Hence (1) holds in Theorem 1 [respectively, Theorem 2 by [12, Theorem 5, p. 223, and Theorem 16, p. 233]].

3. **Fixed-point theorems.** Throughout this section let K be a convex compactum, and let S be a semigroup of continuous affine maps on K to K. If k is in K, then $O(k)$ denotes the closed convex hull of $\{s(k) : s\in S\}$.

The function space K^K is a convex compactum under the topology of pointwise convergence. For each k in K, let $k^\wedge: K^K \to K$ be given by $k^\wedge(\varphi)=\varphi(k)$ $(\varphi\in K^K)$. Let Z be a separating subfunction space of $A(K)$, and let X be a function space on S such that $f\circ k^\wedge|S$ is in X for every k in K and every f in Z. Let Y be the linear subspace of $A(K^K)$ generated by $\{f\circ k^\wedge : k\in K, f\in Z\}$. Let $\psi: S\to K^K$ be the

injection map. Then $g \circ \psi$ is in X for all g in Y. From §2, we know that there exists a unique continuous affine map $\rho\colon M(X) \to K^K$ such that $\rho \circ q_X = \psi$; we have

$$f \circ k^\wedge(\rho(\mu)) = \mu(f \circ k^\wedge \circ \psi) = \mu(f \circ k^\wedge)$$

for all f in Z, k in K, and μ in $M(X)$.

Since ρ is continuous and affine and since $M(X) = \mathrm{cl}\,(\mathrm{co}(q_X(S)))$, $\rho(M(X))$ is the closed convex hull of S in (K^K, p); hence for each k in K, $O(k) = [\rho(M(X))](k)$.

The following is a strengthening of Day's fixed-point theorem [3] which asserted only existence of some fixed point in K under the same hypotheses. The result here was essentially proven in [10] by different methods.

THEOREM 3. *With the above notation, if X is left invariant and if μ_l is a left-invariant X-mean, then $\rho(\mu) \circ \rho(\mu_l) = \rho(\mu_l)$ for all μ in $M(X)$. In particular, $[\rho(\mu_l)](k)$ is a fixed point in $O(k)$ for every k in K.*

Proof. Choose s in S, k in K, and f in Z. We have

$$\begin{aligned} f\{[\rho(q_X(s)) \circ \rho(\mu_l)](k)\} &= f\{[s \circ \rho(\mu_l)](k)\} \\ &= f \circ s \circ k^\wedge[\rho(\mu_l)] = \mu_l(f \circ s \circ k^\wedge) = \mu_l(f \circ k^\wedge \circ L_s) \\ &= \mu_l(f \circ k^\wedge) = f\{[\rho(\mu_l)](k)\}. \end{aligned}$$

Hence $\rho(q_X(s)) \circ \rho(\mu_l) = \rho(\mu_l)$ for all s in S. Since the map which sends μ into $\rho(\mu) \circ \rho(\mu_l)$ is continuous and affine, and since $M(X) = \mathrm{cl}\,(\mathrm{co}\,(q_X(S)))$, we have $\rho(\mu) \circ \rho(\mu_l) = \rho(\mu_l)$ for all μ in $M(X)$.

The above theorem is a result on *existence* of fixed points; it asserts that under correct hypotheses on S and X, the existence of a left-invariant X-mean μ_l implies the existence of an affine map $\rho(\mu_l)$, which is the pointwise limit of convex combinations of members of S, such that $[\rho(\mu_l)](k)$ is a fixed point in $O(k)$ for every k in K.

Heyneman [9] first showed, under quite different circumstances, that the existence of a right-invariant mean for the action semigroup is connected with *uniqueness* of fixed points (see also [4, §7]). We modify the argument in the above proof to obtain such a uniqueness result.

With the notation above, if X is right invariant and if μ_r is a right-invariant X-mean, then one obtains, by computations similar to those in the above proof, that $\rho(\mu_r) \circ \rho(q_X(s)) = \rho(\mu_r)$ for all s in S. If, furthermore, $\rho(\mu_r)$ is continuous on K, then the map which sends μ into $\rho(\mu_r) \circ \rho(\mu)$ is continuous and affine, implying that $\rho(\mu_r) \circ \rho(\mu) = \rho(\mu_r)$ for all μ in $M(X)$. Under these conditions, suppose k is in K and x is a fixed point in $O(k)$. Then x is fixed under every member of $\rho(M(X))$ $(= \mathrm{cl}\,(\mathrm{co}\,(S))$ in $(K^K, p))$. Also $x = [\rho(\mu)](k)$ for some μ in $M(X)$; so that $x = [\rho(\mu_r)](x) = [\rho(\mu_r)]([\rho(\mu)](k)) = [\rho(\mu_r)](k)$. Thus $[\rho(\mu_r)](k)$ is the only possible fixed point in $O(k)$. We have proven the following theorem.

THEOREM 4. *With the above notation, if X is right invariant, if μ_r is a right-invariant X-mean, and if $\rho(\mu_r)$ is continuous on K, then $\rho(\mu_r) \circ \rho(\mu) = \rho(\mu_r)$ for all μ*

in $M(X)$, and for every k in K the only possible fixed point in $O(k)$ is the point $[\rho(\mu_r)](k)$.

Direct computation shows that the conclusion on uniqueness holds if we assume only that $\rho(\mu_r)$ is continuous at the fixed points in K.

For uniqueness of fixed points, we are therefore interested in conditions which insure continuity of the members of $\rho(M(X))$. In this direction, we have the following two corollaries to the development in §2.

COROLLARY 1 (TO THEOREM 1). *With the above notation, the following statements are equivalent:*

(1) *Every member of $\rho(M(X))$ is continuous on K to K.*

(2) *The map $(\mu, k) \to [\rho(\mu)](k)$ is separately continuous on $M(X) \times K$ to K.*

(3) *For every f in $A(K)$ the set $f \circ S$ is relatively compact in $(A(K), w)$.*

COROLLARY 2 (TO THEOREM 2). *With the above notation, the following statements are equivalent:*

(1) *S is equicontinuous on K.*

(2) *The map $(\mu, k) \to [\rho(\mu)](k)$ is jointly continuous on $M(X) \times K$ to K.*

(3) *For every f in $A(K)$, the set $f \circ S$ is relatively compact in $(A(K), \|\cdot\|)$.*

Kakutani [11] proved that if S is an equicontinuous group, then there exists a fixed point in K (see [5, p. 457]). The development above gives the following extension of Kakutani's result.

THEOREM 5. *Let K be a convex compactum and let G be a group of continuous affine maps on K to K such that the closure of G in (K^K, p) consists of continuous functions. Then there is a continuous affine map $\sigma: K \to K$ such that for each k in K, $\sigma(k)$ is a fixed point in $O(k)$; furthermore, $\sigma(k)$ is the only fixed point in $O(k)$.*

Proof. By Theorem 1, $f \circ G$ is relatively compact in $(A(K), w)$ for every f in $A(K)$. Since for each k in K the adjoint of $(k^\wedge|G): G \to K$ is continuous on $(A(K), w)$ to $(m(G), w)$, it follows that $f \circ k^\wedge|G$ is a weakly almost periodic function on G for every f in $A(K)$ and k in K. The theorem now follows from Ryll-Nardzewski's result that there exists a two-sided invariant mean for WAP (G) [18, Theorems 3 and 4, and Corollary 1].

COROLLARY 3. *Let K be a convex compactum and let G be an equicontinuous group of affine maps on K to K. Then the conclusions of Theorem 5 hold.*

4. **Continuity of convolutions.** As additional application of the above development, we give here proofs of results of Pym [16] on continuity of convolutions.

Let X be a complete, two-sided invariant function space on a semigroup S, and assume that X is introverted [2], i.e. for every μ in $M(X)$ and every f in X, the two functions $L(\mu, f)$ and $R(\mu, f)$ on S, given by $[L(\mu, f)](s) = \mu(f \circ L_s)$ and $[R(\mu, f)](x) = \mu(f \circ R_s)$ $(s \in S)$, are again in X.

Let $\psi: S \to M(X)^{M(X)}$ be defined by $[\psi(x)](\mu) = \mu \circ L_s^*$ $(s \in S, \mu \in M(X))$, where $L_s^*: X \to X$ is given by $L_s^*(f) = f \circ L_s$ $(f \in X)$.

Define a mapping $(\hat{\ }): X \to A(M(X))$ $[(\hat{\ }): f \to f^\wedge]$ by $f^\wedge(\mu) = \mu(f)$ $(f \in X, \mu \in M(X))$. $(\hat{\ })$ is a linear isometry and maps constants into constants; hence it maps X onto a closed subfunction space $X^\wedge$ of $A(M(X))$. That $X^\wedge = A(M(X))$ follows from [14, p. 31].

For each μ in $M(X)$, $\mu^\wedge: M(X)^{M(X)} \to M(X)$ is given by

$$\mu^\wedge(\varphi) = \varphi(\mu) \qquad (\varphi \in M(X)^{M(X)}).$$

For each $f^\wedge$ in $X^\wedge$ (i.e., each f in X), we have $(f^\wedge \circ \mu^\wedge) \circ \psi = L(\mu, f)$, which is in X by supposition. Hence $g \circ \psi$ is in X for all g in the linear span of

$$\{f^\wedge \circ \mu^\wedge : f^\wedge \in A(M(X)), \mu \in M(X)\}$$

in $A(M(X)^{M(X)})$. This is precisely the set-up for the development in §3, where K has been replaced by $M(X)$.

There is a unique continuous affine map $\rho: M(X) \to M(X)^{M(X)}$ such that $\rho \circ q_X = \psi$.

Define a binary operation $*$ on $M(X)$ by $\mu * \nu = [\rho(\mu)](\nu)$. Since ρ is continuous, $\mu * \nu$ is continuous in μ for each fixed ν in $M(X)$.

By direct computation it can be shown that $*$ is the convolution product as defined in [15] (originally in [8]; see also [2, §6]).

THEOREM (PYM [16]). *Of the following statements,* (1) *is equivalent to* (2), *and* (3) *is equivalent to* (4).

(1) $\mu * \nu$ *is separately continuous.*

(2) $X \subset \mathrm{WAP}(S)$.

(3) $\mu * \nu$ *is jointly continuous.*

(4) $X \subset \mathrm{AP}(S)$.

Proof. Statement (1) is equivalent to asserting that every $\rho(\mu)$ is continuous, which, by Corollary 1, is equivalent to asserting that $f \circ L_s$ is relatively compact in (X, w) for all f in X, i.e. $X \subset \mathrm{WAP}(S)$. Similarly, the equivalence of (3) and (4) follows from Corollary 2.

Note that if we had started by defining ψ by $[\psi(x)](\mu) = \mu \circ R_s^*$, and proceeded to define a binary operation "$\circ$" by $\mu \circ \nu = [\rho(\nu)](\mu)$, then we would have obtained the "evolution product" as defined in [15].

REFERERCES

1. L. N. Argabright, *Invariant means and fixed points; A sequel to Mitchell's paper*, Trans. Amer. Math. Soc. **130** (1968), 127–130. MR **36** #667.

2. M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544. MR **19**, 1067.

3. ———, *Fixed-point theorems for compact convex sets*, Illinois J. Math. **5** (1961), 585–590. MR **25** #1547.

4. K. deLeeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. **105** (1961), 63–97. MR **24** #A1632.

5. N. Dunford and J. T. Schwartz, *Linear operators*, I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

6. W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67** (1949), 217–240. MR **12**, 112.

7. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. I: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York, and Springer-Verlag, Berlin, 1963. MR **28** #158.

8. E. Hewitt and H. S. Zuckerman, *Finite dimensional convolution algebras*, Acta Math. **93** (1955), 67–119. MR **17**, 1048.

9. R. G. Heyneman, *Duality in general ergodic theory*, Pacific J. Math. **12** (1962), 1329–1341. MR **26** #6803.

10. R. E. Huff, *Some applications of a general lemma on invariant means*, Illinois J. Math. **14** (1970), 216–221.

11. S. Kakutani, *Two fixed-point theorems concerning bicompact convex sets*, Proc. Imp. Acad. Japan **14** (1938), 242–245.

12. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955. MR **16**, 1136.

13. J. L. Kelley and I. Namioka, *Linear topological spaces*, The University Series in Higher Math., Van Nostrand, Princeton, N. J., 1963. MR **29** #3851.

14. R. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N. J., 1966. MR **33** #1690.

15. J. S. Pym, *The convolution of linear functionals*, Proc. London Math. Soc. (3) **14** (1964), 431–444. MR **31** #617.

16. ———, *The convolution of functionals on spaces of bounded functions*, Proc. London, Math. Soc. (3) **15** (1965), 84–104. MR **30** #3367.

17. N. W. Rickert, *Amenable groups and groups with the fixed point property*, Trans. Amer. Math. Soc. **127** (1967), 221–232. MR **36** #5260.

18. C. Ryll-Nardzewski, *On fixed points of semigroups of endomorphisms of linear spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), vol. II, part I, Univ. of California Press, Berkeley, Calif., 1967, pp. 55–61. MR **35** #5977.

PENNSYLVANIA STATE UNIVERSITY,
UNIVERSITY PARK, PENNSYLVANIA 16802