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THE SPECTRUM OF PARTIAL DIFFERENTIAL OPERATORS ON $L^p(R^n)$(1)

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Abstract. The purpose of this paper is to prove that if the polynomial $P(\xi)$ associated with a partial differential operator P on $L^p(R^n)$, with constant coefficients, has the growth property, $|P(\xi)|^{-1}=O(|\xi|^{-r})$, $|\xi|\to\infty$ for some $r>0$, then the spectrum of P is either the whole complex plane or it is the numerical range of $P(\xi)$; and if $P(\xi)$ has some additional property (all the coefficients of $P(\xi)$ being real, for example), then the spectrum of P is the numerical range for those p sufficiently close to 2. It is further shown that the growth property alone is not sufficient to ensure that the spectrum of P is the numerical range of $P(\xi)$.

1. **Introduction.** It is well known that the spectrum of a partial differential operator on $L^2(R^n)$ with constant coefficients associated with $P(D)=\sum_{|\alpha|\leqq m} a_\alpha D^\alpha$ is precisely the closure of the numerical range $\mathcal{N}(P(\xi))=\{P(\xi)\mid \xi\in R^n\}$ of $P(\xi)$. Here a_α's are complex constants, $\alpha=(\alpha_1,\ldots,\alpha_n)$, α_j's nonnegative integers, $|\alpha|=\sum_{j=1}^n \alpha_j$, $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, and $D_j=(1/i)\partial/\partial x_j$, $1\leqq j\leqq n$. By a differential operator P on $L^p(R^n)$ we understand the closure with respect to the L^p norm of the operator: $u\to P(D)u$, $u\in\mathcal{S}$, where $\mathcal{S}$ is the class of all complex-valued functions $u\in C^\infty(R^n)$ such that $\sup_{x\in R^n}|x^\alpha D^\beta u(x)|<\infty$ for all multi-indices α and β. Further we will be concerned only with differential operators with constant coefficients and $P(\xi)$ always stands for a polynomial in n variables $\xi=(\xi_1,\ldots,\xi_n)$ with complex coefficients. If $P(D)$ is an elliptic operator on $L^p(R^n)$, $1\leqq p<\infty$, with constant coefficients, then also $\sigma(P)=\mathcal{N}(P(\xi))$ (Balslev [1]). However, the nonmultiplier example given by Littman, McCarthy and Rivière [5] shows that such a characterization cannot be made for a general differential operator. Thus, there is a limit to the possibility of characterizing the spectrum of a partial differential operator on $L^p(R^n)$ with constant coefficients by the numerical range of the associated polynomial. Consequently attention may now be directed toward finding the largest class of differential operators for which such a characterization can be made or alternatively toward finding some other simple means of characterizing the spectrum. Recently M. Schechter [6] announced a characterization for somewhat

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larger class of differential operators than the class of hypoelliptic operators with some restriction on p (for the precise statement see [6]). The main purpose of this paper is to establish the following theorems:

THEOREM 1. *Let P be a differential operator on $L^p(R^n)$, $1 \leqq p < \infty$ (with constant coefficients), such that*

$$|P(\xi)|^{-1} = O(|\xi|^{-r}), \qquad |\xi| \to \infty,$$

for some $r > 0$. Then, either

(i) $\sigma(P) = \boldsymbol{C}$, *or*

(ii) $\sigma(P) = \mathcal{N}(P(\xi))$.

THEOREM 2. *Let P be a differential operator on $L^p(R^n)$ with the following properties:*

(1) $|P(\xi)|^{-1} = O(|\xi|^{-r})$, $|\xi| \to \infty$, *for some $r > 0$;*

(2) *there exists a $\mu_0 \in \boldsymbol{C}$ such that*

$$\mathcal{N}(P(\xi) - \mu_0) \subset \{z \in \boldsymbol{C} \,|\, |\operatorname{Im} z| \leqq K|\operatorname{Re} z|^{\alpha}\}$$

for some constants α, $0 < \alpha < 1$, and K. Then $\rho(P) \neq \varnothing$ for those p satisfying

$$|1/p - 1/2| < r/(2\chi(m - r - 1) + n),$$

where χ is the smallest integer such that $2\chi > n$ and m is the degree of $P(\xi)$.

The §§2 and 3 are devoted to the proofs of these theorems. In §4 we will show that the growth property of $P(\xi)$ alone does not assure that $\sigma(P) = \mathcal{N}(P(\xi))$, so that the case (i) does occur.

Throughout this paper we assume the results established in Hörmander [3]. The definitions of some undefined terms and symbols in this paper can be found in Hörmander [3] and [4].

2. **Proof of Theorem 1.** To prove Theorem 1 we need four lemmas. The first three are either well known or their proofs are straightforward, and the proofs will not be given.

LEMMA 1. *If P is a differential operator on $L^p(R^n)$, $1 \leqq p < \infty$, with constant coefficients, then*

$$\|Pu\|_p \geqq K\|u\|_p \quad \textit{for all } u \in \mathcal{S} \tag{3}$$

if and only if

$$\left\| \mathcal{F}^{-1}\left(\frac{1}{P(\xi)}\,\hat{u}(\xi)\right)\right\|_p \leqq \frac{1}{K}\,\|u\|_p \quad \textit{for all } u \in \mathcal{S}, \tag{4}$$

where K is some positive constant.

LEMMA 2. *If λ is not in the closure of $\mathcal{N}(P(\xi))$, then there exists a $\delta_\lambda > 0$ such that*

$$|P(\xi) - \lambda| \geqq \delta_\lambda |P(\xi)| \quad \textit{for all } \xi \in R^n. \tag{5}$$

LEMMA 3. *If* $|P(\xi)|^{-1}=O(|\xi|^{-r})$, $|\xi|\to\infty$, *for some* $r>0$, *then* $\mathcal{N}(P(\xi))$ *is closed.*

LEMMA 4. *If* $|P(\xi)|^{-1}=O(|\xi|^{-r})$, $|\xi|\to\infty$, *for some* $r>0$, *and if* $0\notin\mathcal{N}(P(\xi))$, *then there exists a positive integer* N *such that* $1/P(\xi)^N\in M_p^p$, $1\leqq p\leqq\infty$.

Proof. Since $|P(\xi)|^{-1}=O(|\xi|^{-r})$, $|\xi|\to\infty$, and $0\notin\mathcal{N}(P(\xi))$, we have

$$|P(\xi)| \geqq C_1(1+|\xi|)^r \quad \text{for all } \xi\in R^n \tag{6}$$

for some positive constant C_1. Let N be a positive integer such that

$$2Nr-2|\beta|(m-r-1) > n \quad \text{for all } \beta \tag{7}$$

with $|\beta|\leqq\chi$, where χ is the smallest integer such that $2\chi>n$ and m is the degree of $P(\xi)$. Let $S(\xi)=P(\xi)^N$. Then $S^{(\alpha)}(\xi)=\partial^{|\alpha|}S(\xi)/\partial\xi^\alpha$ can be written as

$$S^{(\alpha)}(\xi) = \sum_{j=1}^{\min(|\alpha|,N)} P(\xi)^{N-j}F_j^\alpha(\xi)$$

where $F_j^\alpha(\xi)$ is a polynomial of degree at most $jm-|\alpha|$. Estimating $F_j^\alpha(\xi)$ by $(1+|\xi|)^{jm-|\alpha|}$ and using (6), we obtain

$$|S^{(\alpha)}(\xi)/S(\xi)| \leqq C_\alpha(1+|\xi|)^{|\alpha|(m-r-1)}$$

for all $\xi\in R^n$, for some constant C_α.

Noting that the ratio of the number of factors in the numerator and the denominator in the second factor of the expression

$$\frac{1}{S(\xi)}\cdot\frac{S^{(\mu_1)}(\xi)S^{(\mu_2)}(\xi)\cdots S^{(\mu_k)}(\xi)}{S(\xi)^k} \tag{8}$$

remains invariant upon differentiation and that $D^\beta(1/S(\xi))$ consists of a linear combination of the terms of the form (8) in which $\sum_{j=1}^k|\mu_j|=|\beta|$, we obtain

$$\left|D^\beta\left(\frac{1}{S(\xi)}\right)\right| \leqq A_\beta\frac{(1+|\xi|)^{|\beta|(m-r-1)}}{(1+|\xi|)^{rN}} \tag{9}$$

for all $\xi\in R^n$ for some constant A_β. We have

$$\begin{aligned}\int_{R^n}&\left|\mathscr{F}^{-1}\left(\frac{1}{S(\xi)}\right)(x)\right|dx\\ &\leqq\left[\int_{R^n}\frac{1}{(1+|x|^2)^\chi}\,dx\right]^{1/2}\left[\int_{R^n}(1+|x|^2)^\chi\left|\mathscr{F}^{-1}\left(\frac{1}{S(\xi)}\right)(x)\right|^2dx\right]^{1/2}\\ &\leqq C\left[\sum_{|\beta|\leqq\chi}b_\beta\int_{R^n}\left|D^\beta\left(\frac{1}{S(\xi)}\right)\right|^2d\xi\right]^{1/2}\end{aligned} \tag{10}$$

where b_β's are the multinomial coefficients. Each integral in the sum is finite by (9) and (7). Hence $\mathscr{F}^{-1}(1/S(\xi))\in L^1(R^n)$, and so

$$\frac{1}{S(\xi)} = 1/P(\xi)^N\in M_p^p, \qquad 1\leqq p\leqq\infty.$$

Proof of Theorem 1. Suppose $\sigma(P)\neq \boldsymbol{C}$. Let $\lambda\in \boldsymbol{C}$ such that $\lambda\notin \mathscr{N}(P(\xi))$. Then by Lemmas 3 and 2 there exists a $\delta_\lambda>0$ such that $|P(\xi)-\lambda|\geqq \delta_\lambda|P(\xi)|$ for all $\xi\in R^n$. Hence $|P(\xi)-\lambda|\geqq \delta_\lambda C_0|\xi|^r$ whenever $|\xi|\geqq R_0$. $0\notin \mathscr{N}(P(\xi)-\lambda)$, so that by Lemma 4 there exists a positive integer N such that $(1/[P(\xi)-\lambda]^N)\in M_p^p$, $1\leqq p\leqq\infty$. Hence by Lemma 1, there exists a constant $K>0$ such that

$$\|[P(D)-\lambda]^N u\|_p \geqq K\|u\|_p \quad \text{for all } u\in\mathscr{S}. \tag{11}$$

Let P_λ be the closed operator associated with $P(\Gamma)-\lambda$. For each $u\in\mathscr{S}$ we have by definition

$$P_\lambda u = [P(D)-\lambda]u.$$

Moreover, $[P(D)-\lambda]u\in\mathscr{S}$ whenever $u\in\mathscr{S}$. Hence by induction $P_\lambda^N u=[P(D)-\lambda]^N u$ for each $u\in\mathscr{S}$. Thus,

$$\|P_\lambda^N u\|_p \geqq K\|u\|_p \quad \text{for all } u\in\mathscr{S}.$$

Since $\mathscr{S}$ is dense in $L^p(R^n)$, $1\leqq p<\infty$, and $\mathscr{S}$ is contained in the range of P_λ^N, this shows $0\in\rho(P_\lambda^N)$ or $0\notin\sigma(P_\lambda^N)$. We assert $0\notin\sigma(P_\lambda)$. Suppose the contrary. By assumption $\rho(P)\neq\varnothing$, so that $\rho(P_\lambda)\neq\varnothing$. Thus the spectral mapping theorem for closed operators (Dunford and Schwartz [2, Theorem 10, p. 604]) applies: $[\sigma(P_\lambda)]^N=\sigma(P_\lambda^N)$, so that if $0\in\sigma(P_\lambda)$, then $0\in\sigma(P_\lambda^N)$, contradicting our conclusion reached above. Hence $0\notin\sigma(P_\lambda)$ or $\lambda\notin\sigma(P)$. Therefore, $\sigma(P)\subset\mathscr{N}(P(\xi))$, which together with the well-known fact that the closure of $\mathscr{N}(P(\xi))$ is contained in $\sigma(P)$ proves that (ii) holds. Thus, we have the dichotomy.

3. **Proof of Theorem 2.** The idea of the proof is rather simple. From Lemma 4 we know that if (1) is satisfied and $0\notin\mathscr{N}(P(\xi))$, then there exists a positive integer N such that the operator:

$$u\to\mathscr{F}^{-1}\left(\frac{1}{P(\xi)^N}\,\hat{u}(\xi)\right)$$

is a bounded operator on $L^p(R^n)$, $1\leqq p\leqq\infty$. The idea is to extract the Nth root of the above operator and show that it is bounded and coincides with the operator: $u\to\mathscr{F}^{-1}((1/P(\xi))\hat{u}(\xi))$ on $\mathscr{S}$.

Proof of Theorem 2. Before we begin the proof we remark that if $m-r-1<0$, where m is the degree of $P(\xi)$, $P(D)$ is elliptic and Balslev's result together with the condition $\mathscr{N}(P(\xi))\neq\boldsymbol{C}$ (condition (2)) shows that the assertion is true. Thus we will be concerned with the case $m-r-1\geqq 0$.

We first bring $\mathscr{N}(P(\xi))$ around the real axis and push it so far away from the imaginary axis that it is contained in a very narrow sector centered around the real axis. This statement is made precise below. Let $P(\xi)=R(\xi)-\mu_1+i[Q(\xi)-\mu_2]$ where $\mu_0=\mu_1+iu_2$, $\mu_1,\mu_2\in R$, and $R(\xi)$ and $Q(\xi)$ are polynomials with real coefficients. Using (2) and (1), we obtain

$$[(R(\xi)-\mu_1)^2+K^2(R(\xi)-\mu_1)^{2\alpha}]^{1/2} \geqq C_0|\xi|^r \tag{12}$$

whenever $|\xi|\geqq R_0$ for some constants $C_0>0$ and R_0. This shows that $|R(\xi)-\mu_1|\to\infty$

as $|\xi| \to \infty$. Hence the first term in the brackets in (12) predominates as $|\xi| \to \infty$, and we have $|R(\xi)-\mu_1| \geqq C_1|\xi|^r$ whenever $|\xi| \geqq R_1$ for some constants $C_1>0$ and R_1. Since $R(\xi)$ and μ_1 are real, we can thus assume (if necessary by multiplying $P(\xi)$ by -1) that

$$R(\xi)-\mu_1 \geqq C_1|\xi|^r \quad \text{whenever } |\xi| \geqq R_1. \tag{13}$$

Let N be a fixed positive integer such that $2Nr-2\chi(m-r-1)>n$. Let $\alpha_0>0$ be so small that $N\alpha_0 \leqq \pi/4$. Then choose $\lambda_0>0$ so large that

$$K/[\lambda_0+R(\xi)-\mu_1]^{1-\alpha} \leqq \tan \alpha_0, \tag{14}$$

$$\lambda_0+R(\xi)-\mu_1 \geqq |R(\xi)-\mu_1|, \tag{15}$$

and

$$\lambda_0+R(\xi)-\mu_1 \geqq 4 \tag{16}$$

for all $\xi \in R^n$.

Let $P_0(\xi)=\lambda_0+R(\xi)-\mu_1+i[Q(\xi)-\mu_2]$ and set

$$S_\lambda(\xi) = 1-|\lambda|+i\lambda-P_0(\xi)^N. \tag{17}$$

From our construction we obtain

$$\begin{aligned} |S_\lambda(\xi)| &\geqq |\operatorname{Re} S_\lambda(\xi)| \geqq |\lambda|+|P_0(\xi)|^N \cos(\pi/4)-1 \\ &\geqq |\lambda|+\tfrac{1}{2}|P_0(\xi)|^N \quad \text{for all } \xi \in R^n. \end{aligned} \tag{18}$$

In particular,

$$|S_\lambda(\xi)| \geqq \tfrac{1}{2}|P_0(\xi)|^N \quad \text{for all } \xi \in R^n. \tag{19}$$

Using this, just as in the proof of Lemma 4 we obtain

$$|S_\lambda^{(\alpha)}(\xi)/S_\lambda(\xi)| \leqq C_\alpha(1+|\xi|)^{|\alpha|(m-r-1)} \quad \text{for all } \xi \in R^n, \tag{20}$$

and

$$\left| D^\alpha\left(\frac{1}{S_\lambda(\xi)}\right)\right| \leqq A_\alpha \frac{(1+|\xi|)^{|\alpha|(m-r-1)}}{|\lambda|+C_2(1+|\xi|)^{Nr}} \tag{21}$$

for all $\xi \in R^n$, for some constants C_α and A_α. Here $C_2>0$ is a constant such that

$$|P_0(\xi)| \geqq C_2^{1/N}(1+|\xi|)^r \quad \text{for all } \xi \in R^n. \tag{22}$$

For $|\alpha|=\chi$, we obtain, using (21),

$$\begin{aligned} &\int_{R^n} \left| D^\alpha\left(\frac{1}{S_\lambda(\xi)}\right)\right|^2 d\xi \\ &\qquad \leqq C[|\lambda|+C_2]^{-2}+C'|\lambda|^{-2+(2\chi(m-r-1)+n)/Nr} \int_{R^n} \frac{|\xi|^{2\chi(m-r-1)}}{[1+C_2|\xi|^{Nr}]^2}\, d\xi, \end{aligned}$$

where C and C' are some constants (independent of λ). The integral on the right is finite since $2Nr-2\chi(m-r-1)>n$. The first term is bounded for all $\lambda \in R$ and

goes to zero faster than the second term as $|\lambda| \to \infty$. Hence there exists a constant B_0 such that

$$\int_{R^n} \left| D^\alpha\left(\frac{1}{S_\lambda(\xi)}\right)\right|^2 d\xi \leqq B_0^2|\lambda|^{-2+(2\chi(m-r-1)+n)/Nr}. \tag{23}$$

Applying Cauchy-Schwartz inequality and Plancherel's theorem, we get

$$\begin{aligned}\int_{|x|\geqq 1}\left|\mathscr{F}^{-1}\left(\frac{1}{S_\lambda(\xi)}\right)(x)\right| dx &\leqq C\left[\int_{|x|\geqq 1}|x|^{2\chi}\left|\mathscr{F}^{-1}\left(\frac{1}{S_\lambda(\xi)}\right)(x)\right|^2 dx\right]^{1/2}\\ &\leqq C'\left[\sum_{|\alpha|=\chi} a_\alpha \int_{R^n}\left|D^\alpha\left(\frac{1}{S_\lambda(\xi)}\right)\right|^2 d\xi\right]^{1/2}\end{aligned}$$

for some constants C and C'. Hence, by (23) we see that there exists a constant B_1 such that

$$\int_{|x|\geqq 1}\left|\mathscr{F}^{-1}\left(\frac{1}{S_\lambda(\xi)}\right)(x)\right| dx \leqq B_1|\lambda|^{-1+(2\chi(m-r-1)+n)/2Nr}.$$

Also, we obtain

$$\begin{aligned}\int_{|x|\leqq 1}\left|\mathscr{F}^{-1}\left(\frac{1}{S_\lambda(\xi)}\right)(x)\right| dx &\leqq \left[\int_{|x|\leqq 1} 1\cdot dx\right]^{1/2}\left[\int_{|x|\leqq 1}\left|\mathscr{F}^{-1}\left(\frac{1}{S_\lambda(\xi)}\right)(x)\right|^2 dx\right]^{1/2}\\ &\leqq C''\left[\int_{R^n}\left|\frac{1}{S_\lambda(\xi)}\right|^2 d\xi\right]^{1/2} \leqq B_2|\lambda|^{-1+n/2Nr}\end{aligned}$$

for some constant B_2. Let

$$M_\lambda = 2\max\left(B_1|\lambda|^{-1+(2\chi(m-r-1)+n)/2Nr},\ B_2|\lambda|^{-1+n/2Nr}\right).$$

Then,

$$\|\mathscr{F}^{-1}(1/S_\lambda(\xi))\|_1 \leqq M_\lambda \quad \text{for all } \lambda \in R.$$

Hence,

$$\left\|\mathscr{F}^{-1}\left(\frac{1}{S_\lambda(\xi)}\,\hat{u}(\xi)\right)\right\|_1 \leqq M_\lambda\|u\|_1 \quad \text{for all } u \in \mathscr{S}. \tag{24}$$

Since $|S_\lambda(\xi)| \geqq |\lambda| + \frac{1}{2}|P_0(\xi)|^N \geqq |\lambda| + 2$ for all $\xi \in R^n$ and $\lambda \in R$, we also have

$$\left\|\mathscr{F}^{-1}\left(\frac{1}{S_\lambda(\xi)}\,\hat{u}(\xi)\right)\right\|_2 \leqq \frac{1}{|\lambda|+2}\,\|u\|_2 \quad \text{for all } u \in \mathscr{S}. \tag{25}$$

Hence by Riesz-Thorin convexity theorem, from (24) and (25) we obtain, for $1 \leqq p \leqq 2$,

$$\left\|\mathscr{F}^{-1}\left(\frac{1}{S_\lambda(\xi)}\,\hat{u}(\xi)\right)\right\|_p \leqq \frac{M_\lambda^{2/p-1}}{(|\lambda|+2)^{2-2/p}}\,\|u\|_p \tag{26}$$

for all $u \in \mathscr{S}$. We set $\mathscr{F}^{-1}((1/S_\lambda(\xi))\hat{u}(\xi)) = R_\lambda u$, $u \in \mathscr{S}$, and let

$$\begin{aligned}\mathscr{L}u = \frac{1}{2\pi i}\int_{-\infty}^{0}(1-|\lambda|+i\lambda)^{-1/N}R_\lambda u\cdot(1+i)\,d\lambda\\ +\frac{1}{2\pi i}\int_{0}^{\infty}(1-|\lambda|+i\lambda)^{-1/N}R_\lambda u\cdot(-1+i)\,d\lambda\end{aligned} \tag{27}$$

where the Nth root is taken to be the principal value. We will show that $\mathscr{L}$ is a bounded operator on $L^p(R^n)$ if p is sufficiently close to 2 and that

$$\mathscr{L}u = \mathscr{F}^{-1}\left(\frac{1}{P_0(\xi)}\,\hat{u}(\xi)\right) \quad \text{for each } u \in \mathscr{S}.$$

Applying Minkowski's inequality for the continuous case (which is permissible since we will choose p so that the integrands in the integrals of (27) are absolutely integrable with respect to λ), we have

$$\begin{aligned}\|\mathscr{L}u\|_p &\leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} [(1-|\lambda|)^2+\lambda^2]^{-1/2N}\|R_\lambda u\|_p\,\sqrt{2}\,d\lambda \\ &\leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} [(1-|\lambda|)^2+\lambda^2]^{-1/2N}\,\frac{M_\lambda^{2/p-1}}{(|\lambda|+2)^{2-2/p}}\,\|u\|_p\,\sqrt{2}\,d\lambda.\end{aligned}$$

But

$$M_\lambda = O(|\lambda|^{-1+(2\chi(m-r-1)+n)/2Nr}) \quad \text{as } |\lambda| \to \infty,$$

so that the integral converges if

$$2-2/p+1/N-(2/p-1)(-1+(2\chi(m-r-1)+n)/2Nr) > 1$$

or if

$$0 \leqq 1/p-1/2 < r/(2\chi(m-r-1)+n). \tag{28}$$

Thus, if (28) is satisfied, there exists a constant K_p such that

$$\|\mathscr{L}u\|_p \leqq K_p\|u\|_p \quad \text{for all } u \in \mathscr{S}. \tag{29}$$

To show $\mathscr{L}u=\mathscr{F}^{-1}((1/P_0(\xi))\hat{u}(\xi))$, let $u \in \mathscr{S}$ and

$$\Gamma = \{z \in \boldsymbol{C} \mid z = 1-|\lambda|+i\lambda,\ \lambda \in R\}.$$

Then, substituting $\mathscr{F}^{-1}((1/S_\lambda(\xi))\hat{u}(\xi))$ for $R_\lambda u$ in (27), interchanging the order of integration, and recalling $S_\lambda(\xi)=1-|\lambda|+i\lambda-P_0(\xi)^N$, we obtain

$$(\mathscr{L}u)(x) = \frac{1}{(2\pi)^n}\int_{R^n} \hat{u}(\xi)e^{i\langle x,\xi\rangle}\,\frac{1}{2\pi i}\int_\Gamma \frac{z^{-1/N}}{z-P_0(\xi)^N}\,dz\,d\xi.$$

The function $z^{-1/N}$ with its principal value is analytic on the region lying on the right of Γ and also on Γ. $P_0(\xi)$ is so chosen that $P_0(\xi)^N$ lies in the interior of this region for every $\xi \in R^n$. Thus, by Cauchy's theorem

$$\frac{1}{2\pi i}\int_\Gamma \frac{z^{-1/N}}{z-P_0(\xi)^N}\,dz = \frac{1}{P_0(\xi)}$$

for every $\xi \in R^n$. Hence

$$(\mathscr{L}u)(x) = \mathscr{F}^{-1}\left(\frac{1}{P_0(\xi)}\,\hat{u}(\xi)\right)(x)$$

and so

$$\left\|\mathscr{F}^{-1}\left(\frac{1}{P_0(\xi)}\,\hat{u}(\xi)\right)\right\|_p \leqq K_p\|u\|_p \quad \text{for all } u \in \mathscr{S}$$

with p satisfying (28). Thus, $(1/P_0(\xi)) \in M_p^p$ for those p satisfying (28). Since $M_p^p = M_{p'}^{p'}$, $1/p + 1/p' = 1$, we have $(1/P_0(\xi)) \in M_p^p$ for those p satisfying

$$|1/p - 1/2| < r/(2\chi(m-r-1)+n). \tag{30}$$

By Lemma 1 and by the facts that $\mathscr{S}$ is dense in $L^p(R^n)$, $1 \leqq p < \infty$, and $\mathscr{S}$ is contained in the range of $P-(\mu_0-\lambda_0)I$, it follows $\mu_0 - \lambda_0 \in \rho(P)$ for those p satisfying (30), and Theorem 2 is proved.

COROLLARY. *Let P be a differential operator on $L^p(R^n)$, whose polynomial satisfies the conditions of Theorem* 2. *Then, $\sigma(P) = \mathscr{N}(P(\xi))$ for those p sufficiently close to* 2.

4. **Examples.** Now we will show that the growth property of $P(\xi)$ alone does not guarantee that $\sigma(P) = \mathscr{N}(P(\xi))$. To do this we first prove the following proposition.

PROPOSITION. *Let φ and ψ be functions of a single variable with continuous first derivatives on a neighborhood of a point $t_0 \geqq 0$ with $\varphi(t_0) \neq 0$. Let a be a real constant and l a nonnegative integer $< n/2$. If a function $K(t, x)$ in $n+1$ variables t and $x = (x_1, \ldots, x_n)$ has the form*

$$K(t, x) = \frac{\varphi(t)}{[|x|^2 + \psi(t)]^l} \exp\left[i\,\frac{a|x|^2}{t}\right] + K_1(t, x) \tag{31}$$

on the strip

$$\left\{(t, x) \,\middle|\, \begin{matrix} t_0 \leqq t < t_0 + 4\delta_0 \\ x \in R^n \end{matrix}\right\}$$

for some $\delta_0 > 0$ and $K_1 \in L^1(R^{n+1})$, then K cannot be the kernel of a bounded convolution operator on $L^p(R^{n+1})$ (i.e. $K \notin L_p^p$), for those p satisfying

$$1 \leqq p < (2n+2)/(n+2+2l).$$

Proof. Since $\varphi(t_0) \neq 0$ and φ is continuous on a neighborhood of t_0, we assume $\delta_0 > 0$ has been chosen so that $|\varphi(t)| \geqq \varepsilon_0 > 0$ for all $t \in [t_0, t_0 + 4\delta_0]$ and φ, φ', ψ, ψ' are all continuous on $[t_0, t_0 + 4\delta_0]$. Let

$$\begin{aligned} u_1(t) &= 1 \quad \text{if } -\delta^2 \leqq t \leqq 0, \\ &= 0 \quad \text{otherwise,} \end{aligned} \qquad \begin{aligned} u_2(x) &= 1 \quad \text{if } |x| \leqq \delta, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

where $\delta > 0$.

Let $u(t, x) = u_1(t)u_2(x)$. Let $F(t, x) = \varphi(t)/[|x|^2 + \psi(t)]^l$ and

$$E_\delta = \left\{(t, x) \,\middle|\, \begin{matrix} t_0 + \delta_0 \leqq t \leqq t_0 + 2\delta_0 \\ t\delta^{-\beta} \leqq |x| \leqq 2t\delta^{-\beta} \end{matrix}\right\}$$

where $0<\beta<1$, to be chosen more precisely later. For $(t, x) \in E_\delta$,

$$
\begin{aligned}
|(K * u)(t, x)| \geqq |F(t, x)| & \left| \int_{-\delta^2}^{0} \int_{|\sigma| \leqq \delta} \exp\left[i \frac{a|x-\sigma|^2}{t-\tau} \right] d\sigma \, d\tau \right| \\
& - \int_{-\delta^2}^{0} \int_{|\sigma| \leqq \delta} |F(t, x) - F(t-\tau, x)| d\sigma \, d\tau \\
& - \int_{-\delta^2}^{0} \int_{|\sigma| \leqq \delta} |F(t-\tau, x) - F(t-\tau, x-\sigma)| d\sigma \, d\tau \\
& - |(K_1 * u)(t, x)|.
\end{aligned} \tag{32}
$$

Since φ, ψ and their first derivatives are continuous on $[t_0, t_0+4\delta_0]$, there exist constants C_1, C_2 and $\delta_1>0$ with $\delta_0\delta_1^{-\beta} > \sqrt{M}$ such that

$$|F(t, x) - F(t-\tau, x)| \leqq C_1\delta^2/[|x|^2 - M]^l$$

and

$$|F(t-\tau, x) - F(t-\tau, x-\sigma)| \leqq C_2\delta/[|x|^2 - M]^l$$

for all $(t, x) \in E_\delta$, $-\delta^2 \leqq \tau \leqq 0$, $|\sigma| \leqq \delta$, whenever $0<\delta \leqq \delta_1$, where

$$M = \max_{t \in [t_0, t_0+4\delta_0]} |\psi(t)|.$$

We also have

$$
\begin{aligned}
\left| \int_{-\delta^2}^{0} \int_{|\sigma| \leqq \delta} \exp\left[i \frac{a|x-\sigma|^2}{t-\tau} \right] d\sigma \, d\tau \right| \geqq & \left| \int_{-\delta^2}^{0} \int_{|\sigma| \leqq \delta} \exp\left[i \frac{a|x|^2}{t} \right] d\sigma \, d\tau \right| \\
& - \int_{-\delta^2}^{0} \int_{|\sigma| \leqq \delta} \left| 2 \sin \frac{2\langle x, \sigma\rangle - |\sigma|^2}{2t} a \right| d\sigma \, d\tau \\
& - \int_{-\delta^2}^{0} \int_{|\sigma| \leqq \delta} \left| 2 \sin \frac{\tau|x-\sigma|^2}{2t(t-\tau)} a \right| d\sigma \, d\tau.
\end{aligned}
$$

Let

$$S_1(\delta) = \sup_{|\sigma| \leqq \delta;\, (t,x) \in E_\delta} \left| 2 \sin \frac{2\langle x, \sigma\rangle - |\sigma|^2}{2t} a \right|,$$

$$S_2(\delta) = \sup_{|\sigma| \leqq \delta;\, -\delta^2 \leqq \tau \leqq 0;\, (t,x) \in E_\delta} \left| 2 \sin \frac{\tau|x-\sigma|^2}{2t(t-\tau)} a \right|.$$

Then, $S_1(\delta) \to 0$ and $S_2(\delta) \to 0$ as $\delta \to 0$. Hence there exists $\delta_2>0$ such that $S_1(\delta) \leqq 1/4$ and $S_2(\delta) \leqq 1/4$ whenever $0<\delta \leqq \delta_2$. Hence

$$\left| \int_{-\delta^2}^{0} \int_{|\sigma| \leqq \delta} \exp\left[i \frac{a|x-\sigma|^2}{t-\tau} \right] d\sigma \, d\tau \right| \geqq \tfrac{1}{2}\omega_n \delta^{n+2}$$

for $(t, x) \in E_\delta$ whenever $0<\delta \leqq \delta_2$. Let $\delta_3 = \min(\delta_1, \delta_2)$. Then,

$$|(K * u)(t, x)| \geqq \frac{\varepsilon_0}{[|x|^2+M]^l} (\tfrac{1}{2}\omega_n \delta^{n+2}) - \frac{C_1\delta^2\omega_n\delta^{n+2}}{[|x|^2-M]^l} - \frac{C_2\delta\omega_n\delta^{n+2}}{[|x|^2-M]^l} - |(K_1 * u)(t, x)|$$

for all $(t, x) \in E_\delta$ whenever $0 < \delta \leqq \delta_3$. Hence,

$$\left[\int_{E_\delta} |(K * u)(t, x)|^p dx\, dt\right]^{1/p} \geqq \delta^{n+2-\beta n/p+2\beta l}[B_0 - B_1\delta^2 - B_2\delta] - \|K_1\|_1 \omega_n^{1/p}\delta^{(n+2)/p}$$

whenever $0 < \delta \leqq \delta_3$, where B_0, B_1 and B_2 are some positive constants independent of δ. Suppose K is the kernel of a bounded convolution operator on $L^p(R^{n+1})$. Then there exists a constant A_p such that

$$\left[\int_{E_\delta} |(K * u)(t, x)|^p dx\, dt\right]^{1/p} \leqq \left[\int_{R^{n+1}} |(K * u)(t, x)|^p dx\, dt\right]^{1/p}$$
$$\leqq A_p\|u\|_p = A_p\omega_n^{1/p}\delta^{(n+2)/p}.$$

Then we have

$$A_p\omega_n^{1/p}\delta^{(n+2)/p} \geqq \delta^{n+2-\beta n/p+2\beta l}[B_0 - B_1\delta^2 - B_2\delta] - \|K_1\|_1\omega_n^{1/p}\delta^{(n+2)/p} \tag{33}$$

whenever $0 < \delta \leqq \delta_3$. If $1 \leqq p < (2n+2)/(n+2+2l)$, then $(n+2)/p > n+2-n/p+2l$. Since $g(\beta) = n+2-(\beta n/p)+2\beta l$ is a continuous function of β, we can satisfy the inequality

$$(n+2)/p > n+2-\beta n/p+2\beta l \tag{34}$$

if we choose β sufficiently close to 1 with the condition $0 < \beta < 1$. Now we will arrive at a contradiction in (33) if we let $\delta \to 0$. Consequently K cannot be the kernel of a bounded convolution operator on $L^p(R^{n+1})$ for those p satisfying $1 \leqq p < (2n+2)/(n+2+2l)$.

We now consider the polynomial $P(\tau, \xi) = [\tau - |\xi|^2 - i][\tau + |\xi|^2 + i]$. It can be shown that

$$|P(\tau, \xi)|^{-1} = O([|\tau| + |\xi|]^{-1}), \qquad |\tau| + |\xi| \to \infty.$$

Thus, $P(\tau, \xi)$ satisfies the condition (1). If we "formally" compute $\mathscr{F}^{-1}(1/P(\tau, \xi))$ by first integrating with respect to τ and then applying integrations by parts to the remaining integrals, we can put $\mathscr{F}^{-1}(1/P(\tau, \xi))$ into the form

$$\mathscr{F}^{-1}\left(\frac{1}{P(\tau, \xi)}\right)(t, x) = \frac{\varphi(t)}{|x|^2 + \psi(t)} \exp\left[i\,\frac{|x|^2}{4t}\right] + K_1(t, x) \tag{35}$$

for $t \geqq 1$, where φ and ψ are functions of a single variable satisfying the conditions of the proposition and $K_1 \in L^1(R^{n+1})$ if $n \leqq 3$. The formal computation can be made rigorous by adjoining a convergence factor $\exp[-\mu^2|\xi|^2]$ to $(1/P(\tau, \xi))$ and letting μ go to zero faster than δ, where δ is the positive quantity used in the proof of the proposition. However, the computation becomes quite tedious and we omit the proof. Thus, having obtained an estimate of the form (35) for

$$\mathscr{F}^{-1}(\exp[-\mu^2|\xi|^2]/P(\tau, \xi)),$$

we will arrive at a contradiction, just as in the proof of the proposition, if we assume $(1/P(\tau, \xi)) \in M_p^p$ for $1 \leqq p < 8/7$ $(n=3, l=1)$. Since $0 \notin \mathscr{N}(P(\tau, \xi))$, it follows from Theorem 1 that $\sigma(P) = \boldsymbol{C}$ for $1 \leqq p < 8/7$, where P is the differential

operator on $L^p(R^{n+1})$ associated with $P(D_t, D)$. Thus, the growth property (1) alone does not guarantee that $\sigma(P)=\mathcal{N}(P(\xi))$.

This example in conjunction with Lemma 4 shows that the differential operator (the minimal operator) on $L^p(R^n)$ associated with $P(D)^N$ is not the same as P^N, where P is the differential operator on $L^p(R^n)$ associated with $P(D)$.

As a final example we consider the differential operator associated with the nonmultiplier example given by Littman, McCarthy and Rivière [5]. They have shown that $(1/(\tau-|\xi|^2-i)) \notin M_p^p$ for those p satisfying $|1/p-1/2|>1/(2n+2)$. From this it follows that $\sigma(P)=\boldsymbol{C}$, where P is the differential operator on $L^p(R^{n+1})$, $|1/p-1/2|>1/(2n+2)$, associated with $P(D_t, D)=D_t-\sum_{j=1}^n D_j^2$, $D_t=(1/i)\partial/\partial t$. We will show that $\sigma(P_N)=\boldsymbol{C}$, where P_N is the differential operator associated with $P(D_t, D)^{2N}$ and N is any positive integer. If $\mu \in \boldsymbol{C}$ such that μ is not in the closure of $\mathcal{N}(P(\tau, \xi)^{2N})$, then for $t>0$,

$$\mathcal{F}^{-1}(1/(P(\tau, \xi)^{2N}-\mu))(t, x) = J(t)|t|^{-n/2} \exp [i|x|^2/4t],$$

where

$$J(t) = \frac{I_0}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \frac{e^{it\tau}}{\tau^{2N}-\mu}\, d\tau \quad \text{and} \quad I_0 = \int_{R^n} \exp [i|\xi|^2]\, d\xi,$$

which exists in the generalized Riemann sense. (The method used to obtain this Fourier inverse transform is formal, but the result to be obtained can be made rigorous by adjoining a convergence factor just as in the preceding example.) We note $J(0)\neq 0$ and J is continuous for all t. Hence there exist an $\varepsilon_0>0$ and $\delta_0>0$ such that $|J(t)|\geqq \varepsilon_0$ for all $t \in [\delta_0, 5\delta_0]$. Let $t_0=\delta_0$ and

$$\varphi(t) = J(t)|t|^{-n/2}, \qquad t > 0.$$

Then, the conditions of the proposition are satisfied with $l=0$ and $K_1(t, x)\equiv 0$. Hence $(1/P(\tau, \xi)^{2N}-\mu) \notin M_p^p$ for those p satisfying $1\leqq p<(2n+2)/(n+2)$. Hence $\mu \in \sigma(P_N)$ for every $\mu \notin \mathcal{N}(P(\tau, \xi)^{2N})$ for those p with $1\leqq p<(2n+2)/(n+2)$. Since the closure of $\mathcal{N}(P(\tau, \xi)^{2N})$ is contained in $\sigma(P_N)$, it follows that $\sigma(P_N)=\boldsymbol{C}$ if $1\leqq p<(2n+2)/(n+2)$. Since $M_p^p=M_{p'}^{p'}$, $1/p+1/p'=1$, we have our result.

This is a marked contrast from the case where $P(\tau, \xi)$ has the growth property (1), in which case $\sigma(P_N)\neq \boldsymbol{C}$ (providing $\mathcal{N}(P(\tau, \xi))\neq \boldsymbol{C}$), for sufficiently large N for every p, $1\leqq p<\infty$.

Added in proof. We have been informed by M. Schechter that he has obtained a more general result than our Theorem 2. In particular, he can replace the condition (2) of Theorem 2 by the simple condition that $\mathcal{N}(P(\xi))\neq \boldsymbol{C}$. His results will appear soon.

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